Write or stick your student number here

SPECTEMUR AGENDO

**2025** **TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

# Mathematics Extension 1

**General instructions**

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Student Number at the top of this page and at the top of each writing booklet

**Total marks: 70**

**Section I – 10 marks (pages 2–7)**

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II – 60 marks (pages 8–13)**

- Attempt Questions 11–15
- Allow about 1 hour and 45 minutes for this section

## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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**1** The polynomial equation $2x^3 - 3x^2 - 4x + 6 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

What is the value of $\alpha\beta\gamma$?

A. $-6$

B. $-3$

C. $-2$

D. $\frac{3}{2}$

**2** The polynomial $P(x) = x^4 + ax^2 + b$ leaves a remainder of 3 when divided by $(x - 2)$.

What is the remainder when $P(x)$ is divided by $(x + 2)$?

A. $-3$

B. $-1$

C. 3

D. 7

**3** In a barrel there are 50 marbles of various colours. Of these, 4 are green, 19 are blue, 11 are yellow, 13 are purple and 3 are orange.

What is the least number of marbles that can be selected from the barrel to ensure that 7 of the selected marbles are of the same colour?

A. 26

B. 30

C. 31

D. 35

**4** Which of the following is the primitive of $\dfrac{1}{\sqrt{4-9x^2}}$ ?

A. $\dfrac{1}{6}\sin^{-1}\left(\dfrac{2x}{3}\right)+c$

B. $\dfrac{1}{3}\sin^{-1}\left(\dfrac{2x}{3}\right)+c$

C. $\dfrac{1}{6}\sin^{-1}\left(\dfrac{3x}{2}\right)+c$

D. $\dfrac{1}{3}\sin^{-1}\left(\dfrac{3x}{2}\right)+c$

**5** A curve is defined by the parametric equations below.

$$x=\sin\theta$$
$$y=\cos^2\theta-3$$

Which of the following represents the Cartesian equation of the curve?

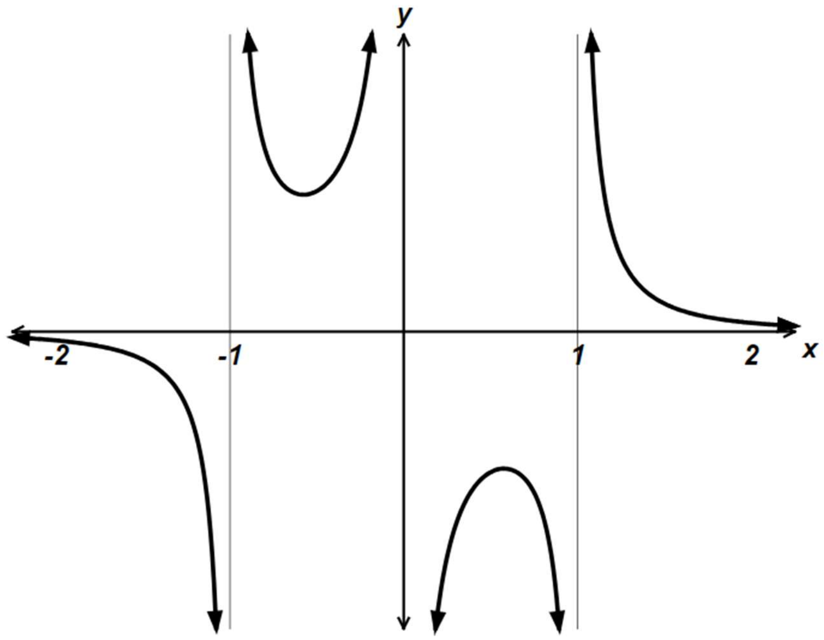
A. $y=x^2-3$

B. $y=-x-2$

C. $y=-x^2-2$

D. $y=-\ 2x^2-2$

**6** The graph below shows $y = \dfrac{1}{f(x)}$.




Which of the equations below could represent $y = f(x)$?

A. $f(x) = x^2 - 1$

B. $f(x) = x(x^2 - 1)$

C. $f(x) = x^2(x^2 - 1)$

D. $f(x) = x^2(x^2 - 1)^2$

**7** A particle's displacement is given by $x = 2\sin(\pi t) + 1$.

Which of the following statements is true?

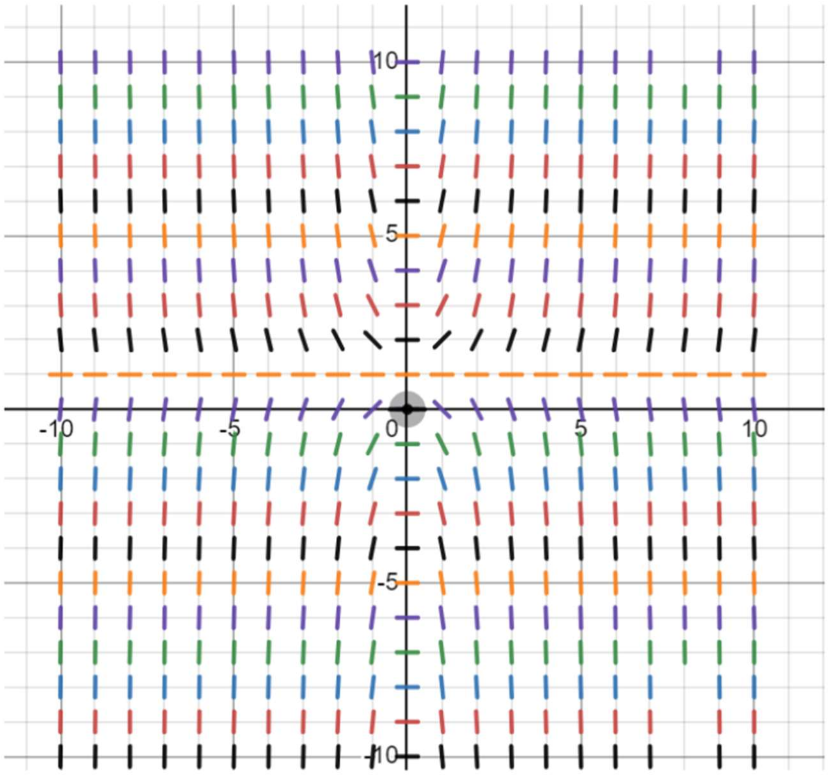
A. The maximum velocity of the particle occurs at $t = 2$ when $x = 1$.

B. The maximum velocity of the particle occurs at $t = 2$ when $x = 2\pi$.

C. The maximum velocity of the particle occurs at $t = 1.5$ when $x = -1$.

D. The maximum velocity of the particle occurs at $t = 0.5$ when $x = 3$.

**8** The diagram shows the direction field of a differential equation.




Which of the following differential equations could have the given direction field?

A. $\dfrac{dy}{dx} = x^2 + y^2$

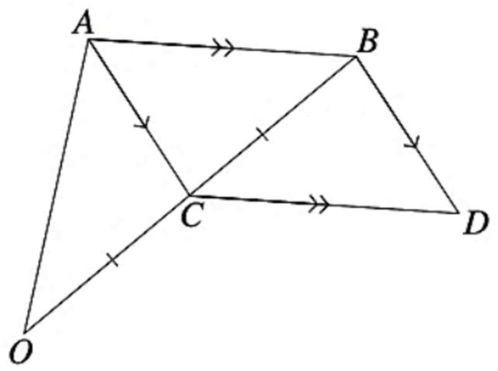
B. $\dfrac{dy}{dx} = x^2 - y^2$

C. $\dfrac{dy}{dx} = y^2 - x^2$

D. $\dfrac{dy}{dx} = xy - x$

**9** In the diagram, the position vectors of the points $A$ and $B$ are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, where $O$ is the origin.

Point $C$ is the midpoint of $OB$ and point $D$ is such that $ABDC$ is a parallelogram.




Which of the following is the position vector of $D$?

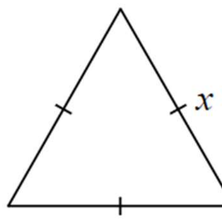
A. $\frac{3}{2}\underset{\sim}{b} + \underset{\sim}{a}$

B. $\frac{3}{2}\underset{\sim}{b} - \underset{\sim}{a}$

C. $\frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

D. $\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}$

**10** The diagram shows an equilateral triangle with side length $x$ cm and area $A$ cm$^2$.




The area of the triangle is increasing at a rate of 6 cm$^2$ per second.

At what rate is the side length $x$ expanding when the triangle has a side length of 4 cm?

A. $\dfrac{2}{\sqrt{3}}$ cm/s

B. $\sqrt{3}$ cm/s

C. 6 cm/s

D. $12\sqrt{3}$ cm/s

## Section II

**60 marks**
**Attempt Questions 11–15**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (12 marks) Use the Question 11 Writing Booklet

(a) (i) Express $2\cos x - \sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° \leq \alpha \leq 360°$. **2**

(ii) Hence find the range of the function $g(x) = (2\cos x - \sin x)^2$. **1**

(b) Solve $-\dfrac{1}{x+1} \leq x + 3$. **3**

(c) A group of six girls and two boys are to be seated around a round table.

(i) In how many different ways can the eight people be seated? **1**

(ii) What is the probability the two boys won't be seated next to each other? **2**

(d) Use the substitution $x = 2\cos u$ to evaluate $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4 - x^2}}\, dx$. **3**

**End of Question 11**

**Question 12** (12 marks) Use the Question 12 Writing Booklet

(a) (i) By letting $t = \tan\theta$, or otherwise, show that $\operatorname{cosec}(2\theta) - \cot(2\theta) = \tan\theta$. **2**

(ii) Hence find the exact value of $\tan\dfrac{\pi}{8}$. **1**

(b) Sketch the graph of $y = \pi - \cos^{-1}(2x)$, clearly indicating the endpoints and any intercepts with the coordinate axes. **2**

(c) Given that $y = e^{2x} + e^{-2x}$, determine the values of constants $a$ and $b$ that satisfy the differential equation $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = 5e^{2x} + e^{-2x}$. **2**

(d) The population of a particular town is growing at a rate proportional to the excess of its current size $N$ above 3000 individuals.

That is, $N$ satisfies the equation $\dfrac{dN}{dt} = k(N - 3000)$.

At the start of 2020 the population was 25 000, and at the start of 2025 the population was 29 400.

(i) Show that $N = 3000 + Ae^{kt}$ satisfies this model of population growth. **1**

(ii) Hence show that $A$ = 22 000 and $k = \dfrac{1}{5}\ln\left(\dfrac{6}{5}\right)$. **2**

(iii) In what year will the population first exceed 35 000? **1**

(iv) What will be the rate of population growth when the population reaches 35 000? **1**

**End of Question 12**

**Question 13** (12 marks) Use the Question 13 Writing Booklet

(a) (i) Find $\dfrac{d}{dx}(x\tan^{-1}x)$. **1**

(ii) Hence find $\displaystyle\int \tan^{-1}x\,dx$. **2**

(b) A particle is fired from a point on the ground, taken to be the origin, with speed of projection 80 ms$^{-1}$ at an angle $\alpha$ above the horizontal.

The particle needs to clear a fence 5 metres high at a horizontal distance 400 metres from the point of projection.

The position vector of the particle is $\underset{\sim}{r}(t)$, where $t$ is the time after projection. The acceleration due to gravity is $g = 10$ ms$^{-2}$.

(i) Show that $\underset{\sim}{r}(t) = (80t\cos\alpha)\underset{\sim}{i} + (80t\sin\alpha - 5t^2)\underset{\sim}{j}$. **3**

(ii) Hence find the possible values of $\alpha$ for the particle to clear the fence, correct to the nearest minute. **3**

(c) Use mathematical induction to prove that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers $n$. **3**

**End of Question 13**

**Question 14** (12 marks) Use the Question 14 Writing Booklet

(a) A Maths club has $2n$ members, half of which are male and half female. **2**

A Pi Day committee is made up of 3 members of this club.

By considering all different combinations for this committee, show that

$$n\,{}^{n}\mathbf{C}_2 + {}^{n}\mathbf{C}_3 = \frac{1}{2}\,{}^{2n}\mathbf{C}_3 \text{ , where } n \geq 3.$$

(b) In the expansion of $(3+2x)^n$, the coefficients of $x^2$ and $x$ are in the ratio 3:1. **3**

Find the value of $n$.

(c) Solve the differential equation $(x^2+1)\dfrac{dy}{dx} = 6xy$, given that $y = 2$ when $x = 1$. **3**
Give your answer as the function $y$ in terms of $x$.

(d) An acute-angled triangle $XYZ$ has an area of 40 square units. **4**

The vector $\overrightarrow{YX} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ and $\overrightarrow{YZ} = \begin{pmatrix} p \\ q \end{pmatrix}$.

Given $|\overrightarrow{YZ}| = 8\sqrt{5}$, find the possible values of $p$ and $q$.

**End of Question 14**

**Question 15** (12 marks) Use the Question 15 Writing Booklet

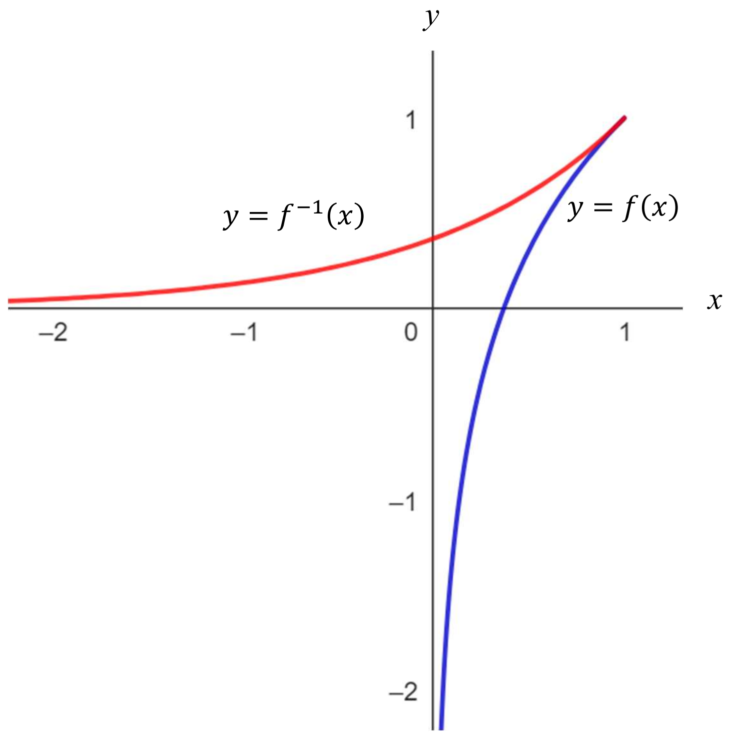
(a) (i) Show that $\sin(x-y)\sin(x+y)=\sin^2 x-\sin^2 y$. **2**

(ii) Hence or otherwise, solve $\sin^2 3x-\sin^2 x=\sin 4x$ for $[0, \pi]$. **2**

(b) The function $f(x)=1+\ln x$ is defined in the domain $(0, 1]$.

(i) Show that $\dfrac{d}{dx}(x\ln x)=1+\ln x$. **1**

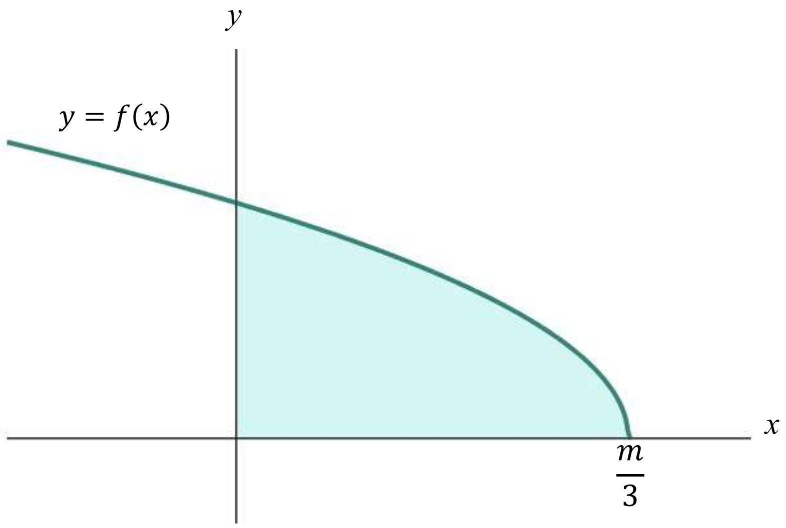
(ii) The diagram shows the graphs of the function $y=f(x)$ and the inverse function $y=f^{-1}(x)$ **3**




Find in simplest exact form the area of the region in the first quadrant bounded by the curves $y=f(x)$, $y=f^{-1}(x)$ and the coordinate axes.

**Question 15 continues on page 13**

(c) Let $f(x) = \sqrt{m - 3x}$ for $x \le \frac{m}{3}$. The graph of $y = f(x)$ is shown. **4**




The shaded area, enclosed by the graph of $y = f(x)$, the $x$-axis and the $y$-axis, is rotated about the $y$-axis.

Find the value of $m$ such that the volume of the solid formed is $\frac{5000\pi}{27}$ cubic units.

**End of paper**

## Section I

### Multiple choice answer key

| Question | Answer |
|---|---|
| 1 | B |
| 2 | C |
| 3 | A |
| 4 | D |
| 5 | C |
| 6 | B |
| 7 | A |
| 8 | D |
| 9 | B |
| 10 | B |

**1** The polynomial equation $2x^3 - 3x^2 - 4x + 6 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

What is the value of $\alpha\beta\gamma$?

A. –6

B. –3

C. –2

D. $\frac{3}{2}$

***Solution:***

$$\begin{aligned}\alpha\beta\gamma &= -\frac{d}{a}\\ &= -\frac{6}{2}\\ &= -3\end{aligned}$$

→ ***B***

**2** The polynomial $P(x) = x^4 + ax^2 + b$ leaves a remainder of 3 when divided by $(x - 2)$.

What is the remainder when $P(x)$ is divided by $(x + 2)$?

A. –3

B. –1

C. 3

D. 7

***Solution:***

$P(x)$ is even, so $P(-2) = P(2)$

Hence remainder = 3

→ ***C***

**3** In a barrel there are 50 marbles of various colours. Of these, 4 are green, 19 are blue, 11 are yellow, 13 are purple and 3 are orange.

What is the least number of marbles that can be selected from the barrel to ensure that 7 of the selected marbles are of the same colour?

A. 26

B. 30

C. 31

D. 35

***Solution:***

Can have maximum of 4 green and 3 orange, then 6 each of blue, yellow or purple without having 7 of any one colour. Selecting one extra marble will ensure 7 of the same colour.

$4 + 3 + 6 + 6 + 6 + 1 = 26$

$\rightarrow$ ***A***

**4** Which of the following is the primitive of $\dfrac{1}{\sqrt{4-9x^2}}$ ?

A. $\dfrac{1}{6}\sin^{-1}\left(\dfrac{2x}{3}\right)+c$

B. $\dfrac{1}{3}\sin^{-1}\left(\dfrac{2x}{3}\right)+c$

C. $\dfrac{1}{6}\sin^{-1}\left(\dfrac{3x}{2}\right)+c$

D. $\dfrac{1}{3}\sin^{-1}\left(\dfrac{3x}{2}\right)+c$

***Solution:***

Using reference sheet, $\dfrac{1}{\sqrt{4-9x^2}}\,dx=\dfrac{1}{3}\sin^{-1}\left(\dfrac{3x}{2}\right)+c$

$\rightarrow$ ***D***

**5** A curve is defined by the parametric equations below.

$$x = \sin\theta$$
$$y = \cos^2\theta - 3$$

Which of the following represents the Cartesian equation of the curve?

A. $y = x^2 - 3$

B. $y = -x - 2$

C. $y = -x^2 - 2$

D. $y = -\ 2x^2 - 2$

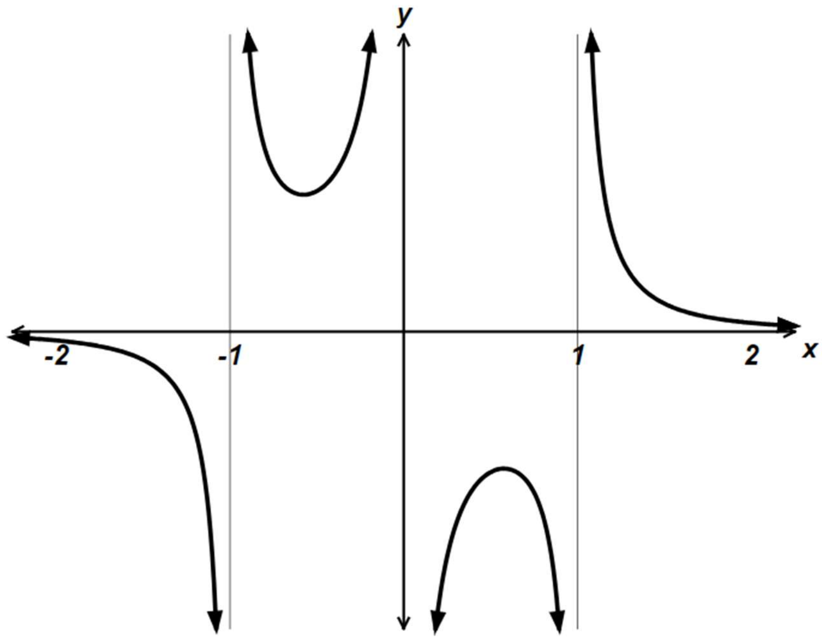
***Solution:***

$$x = \sin\theta$$
$$x^2 = \sin^2\theta$$

$$y = (1 - \sin^2\theta) - 3$$
$$y = -\sin^2\theta - 2$$
$$y = -x^2 - 2$$

$\rightarrow$ ***C***

**6** The graph below shows $y = \dfrac{1}{f(x)}$.




Which of the equations below could represent $y = f(x)$?

A. $f(x) = x^2 - 1$

B. $f(x) = x(x^2 - 1)$

C. $f(x) = x^2(x^2 - 1)$

D. $f(x) = x^2(x^2 - 1)^2$

***Solution:***

$f(x)$ has roots at $x = -1,\ 0,\ 1$

Change of sign at each root indicates they each have odd multiplicity

$$\begin{aligned} f(x) &= x(x-1)(x+1) \\ &= x(x^2 - 1) \end{aligned}$$

$\rightarrow$ ***B***

**7** A particle's displacement is given by $x = 2\sin(\pi t) + 1$.

Which of the following statements is true?

A. The maximum velocity of the particle occurs at $t = 2$ when $x = 1$.

B. The maximum velocity of the particle occurs at $t = 2$ when $x = 2\pi$.

C. The maximum velocity of the particle occurs at $t = 1.5$ when $x = -1$.

D. The maximum velocity of the particle occurs at $t = 0.5$ when $x = 3$.

***Solution:***

$$x = 2\sin(\pi t) + 1$$
$$\dot{x} = 2\pi\cos(\pi t)$$

Maximum velocity occurs when $\cos(\pi t) = 1$

$$\pi t = 0,\ 2\pi$$
$$t = 0,\ 2$$

When $t = 0$, $x = 2\sin(0) + 1$

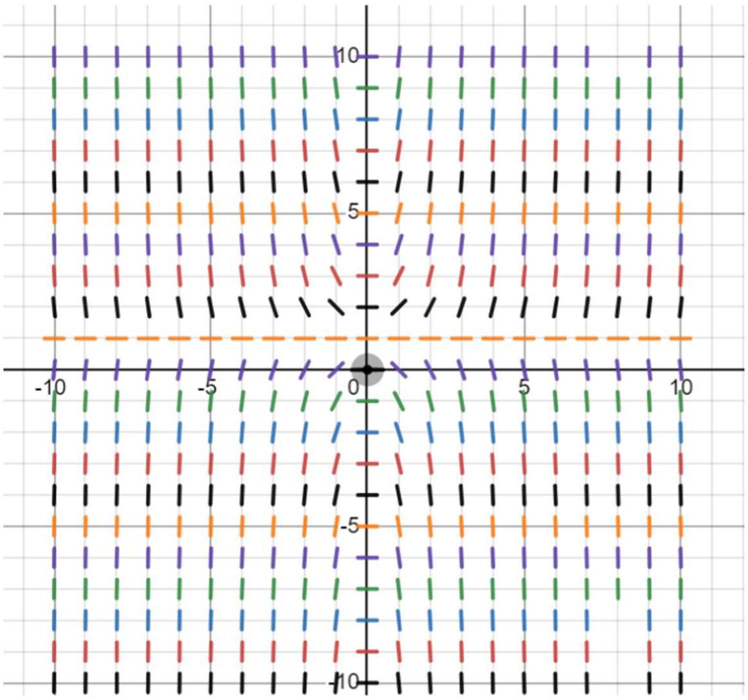
$x = 1$

When $t = 2$, $x = 2\sin(2\pi) + 1$

$x = 1$

$\rightarrow$ ***A***

**8** The diagram shows the direction field of a differential equation.




Which of the following differential equations could have the given direction field?

A. $\dfrac{dy}{dx} = x^2 + y^2$

B. $\dfrac{dy}{dx} = x^2 - y^2$

C. $\dfrac{dy}{dx} = y^2 - x^2$

D. $\dfrac{dy}{dx} = xy - x$

***Solution:***

When $y = 1$, $\dfrac{dy}{dx} = 0$

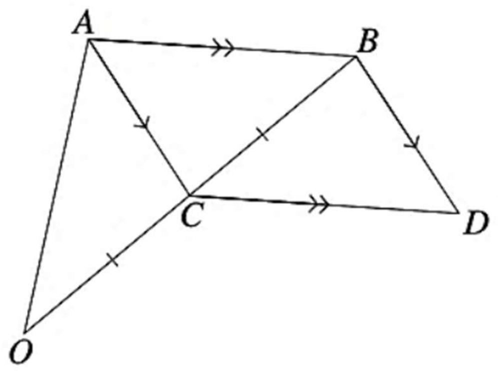
Consider option D: $\dfrac{dy}{dx} = x(y - 1)$

This is the only differential equation with a constant solution of $y = 1$

$\rightarrow$ ***D***

**9** In the diagram, the position vectors of the points $A$ and $B$ are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, where $O$ is the origin.

Point $C$ is the midpoint of $OB$ and point $D$ is such that $ABDC$ is a parallelogram.




Which of the following is the position vector of $D$?

A. $\frac{3}{2}\underset{\sim}{b} + \underset{\sim}{a}$

B. $\frac{3}{2}\underset{\sim}{b} - \underset{\sim}{a}$

C. $\frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

D. $\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}$

***Solution:***

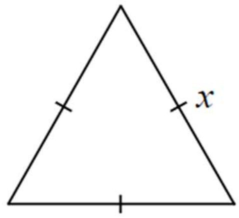
$$\overrightarrow{OC} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\underset{\sim}{b}$$

$$\overrightarrow{CD} = \overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$$

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \frac{1}{2}\underset{\sim}{b} + \underset{\sim}{b} - \underset{\sim}{a} = \frac{3}{2}\underset{\sim}{b} - \underset{\sim}{a}$$

→ ***B***

**10** The diagram shows an equilateral triangle with side length $x$ cm and area $A$ cm$^2$.




The area of the triangle is increasing at a rate of 6 cm$^2$ per second.

At what rate is the side length $x$ expanding when the triangle has a side length of 4 cm?

A. $\frac{2}{\sqrt{3}}$ cm/s

B. $\sqrt{3}$ cm/s

C. 6 cm/s

D. $12\sqrt{3}$ cm/s

***Solution:***

$$A = \frac{1}{2} \times x^2 \times \sin\frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{4} x^2$$

$$\frac{dA}{dx} = \frac{\sqrt{3}}{2} x$$
$$= \frac{\sqrt{3}}{2} \times 4$$
$$= 2\sqrt{3}$$

$$\frac{dx}{dt} = \frac{dx}{dA} \times \frac{dA}{dt}$$
$$= \frac{1}{2\sqrt{3}} \times 6$$
$$= \sqrt{3}$$

$\rightarrow$ ***B***

**Question 11**

(a) (i) Express $2\cos x - \sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$ and $0° \leq \alpha \leq 360°$. **2**

***Sample answer:***

$$
\begin{aligned}
2\cos x - \sin x &= R\cos(x+\alpha) \\
&= R(\cos x\cos\alpha - \sin x\sin\alpha)
\end{aligned}
$$

$$
\begin{aligned}
R\cos\alpha &= 2 \\
R\sin\alpha &= 1
\end{aligned}
$$

$$
\begin{aligned}
R^2 &= 2^2 + 1^2 \\
R &= \sqrt{5}
\end{aligned}
$$

$$
\begin{aligned}
\tan\alpha &= \frac{1}{2} \\
\alpha &= 26°34'
\end{aligned}
$$

$$\therefore\ 2\cos x - \sin x = \sqrt{5}\cos\left(x + 26°34'\right)$$

| Criteria | Marks |
|---|---|
| Provides correct solution | 2 |
| Finds the value of $R$ or $\alpha$, or equivalent merit | 1 |

(ii) Hence find the range of the function $g(x) = (2\cos x - \sin x)^2$. **1**

***Sample answer:***

$$
\begin{aligned}
g(x) &= \left[\sqrt{5}\cos\left(x + 26°34'\right)\right]^2 \\
&= 5\cos^2\left(x + 26°34'\right)
\end{aligned}
$$

Range: $0 \leq y \leq 5$

| Criteria | Marks |
|---|---|
| Provides correct answer | 1 |

(b) Solve $-\dfrac{1}{x+1} \leq x+3$. **3**

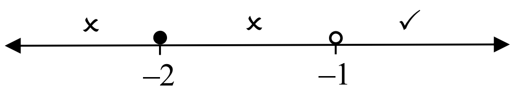
***Sample answer:***

Critical values:

$$x \neq -1$$

$$
\begin{aligned}
-\frac{1}{x+1} &= x+3 \\
-1 &= x^2+4x+3 \\
0 &= x^2+4x+4 \\
0 &= (x+2)^2 \\
x &= -2
\end{aligned}
$$

Test values:




Hence $x = -2$ or $x > -1$

| Criteria | Marks |
|---|---|
| Provides correct solution | 3 |
| Identifies the correct critical values, or equivalent merit | 2 |
| Excludes $x = -1$ OR finds $x = -2$ as a critical value, or equivalent merit | 1 |

(c) A group of six girls and two boys are to be seated around a round table.

(i) In how many different ways can the eight people be seated? **1**

***Sample answer:***

$7! = 5040$

| Criteria | Marks |
|---|---|
| Provides correct answer | 1 |

(ii) What is the probability the two boys won't be seated next to each other? **2**

***Sample answer:***

Number of arrangements with boys seated together $= 2 \times 6!$
$= 1440$

Number of arrangements boys not seated together $= 5040 - 1440$
$= 3600$

$$P(\text{boys not seated together}) = \frac{3\,600}{5\,040} = = \frac{5}{7}$$

| Criteria | Marks |
|---|---|
| Provides correct solution | 2 |
| Finds the number of arrangements with the boys seated next to each other, or equivalent merit | 1 |

(d) Use the substitution $x = 2\cos u$ to evaluate $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\, dx$. **3**

***Sample answer:***

$$x = 2\cos u \qquad\qquad x = 0 \qquad\qquad x = 1$$

$$\frac{dx}{du} = -2\sin u \qquad\qquad 2\cos u = 0 \qquad\qquad 2\cos u = 1$$

$$dx = -2\sin u\; du \qquad\qquad u = \frac{\pi}{2} \qquad\qquad u = \frac{\pi}{3}$$

$$\begin{aligned}
\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\, dx &= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \frac{4\cos^2 u}{\sqrt{4-4\cos^2 u}} \cdot (-2\sin u)\; du \\
&= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{8\cos^2 u \sin u}{2\sin u}\, du \\
&= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 4\cos^2 u\; du \\
&= 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 + \cos(2u)\; du \\
&= 2\left[u + \frac{1}{2}\sin(2u)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= 2\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - \left(\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right)\right] \\
&= 2\left(\frac{\pi}{2} + 0 - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) \\
&= \frac{\pi}{3} - \frac{\sqrt{3}}{2}
\end{aligned}$$

| **Criteria** | **Marks** |
|---|---|
| Provides correct solution | 3 |
| Obtains correctly simplified integrand in terms of $\cos u$, or equivalent merit | 2 |
| Finds an expression for $dx$ in terms of $u$ OR correctly adjust bounds of integration, or equivalent merit | 1 |

**Question 12**

(a) (i) By letting $t = \tan\theta$, or otherwise, show that $\operatorname{cosec}(2\theta) - \cot(2\theta) = \tan\theta$. **2**

***Sample answer:***

Let $t = \tan\theta$

$$\begin{aligned}\operatorname{cosec}(2\theta) - \cot(2\theta) &= \frac{1+t^2}{2t} - \frac{1-t^2}{2t} \\ &= \frac{1+t^2-1+t^2}{2t} \\ &= \frac{2t^2}{2t} \\ &= t \\ &= \tan\theta\end{aligned}$$

| Criteria | Marks |
|---|---|
| Provides correct solution | 2 |
| Correctly expresses the LHS in terms of $t$, or equivalent merit | 1 |

(ii) Hence find the exact value of $\tan\frac{\pi}{8}$. **1**

***Sample answer:***

$$\begin{aligned}\tan\frac{\pi}{8} &= \operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4} \\ &= \sqrt{2} - 1\end{aligned}$$

| Criteria | Marks |
|---|---|
| Provides correct answer | 1 |

(b) Sketch the graph of $y = \pi - \cos^{-1}(2x)$, clearly indicating the endpoints and any intercepts with the coordinate axes. **2**

***Sample answer:***

$y = \pi - \cos^{-1}(2x)$ $\rightarrow$ horizontal dilation
vertical reflection across $x$ axis
vertical translation up $\pi$ units

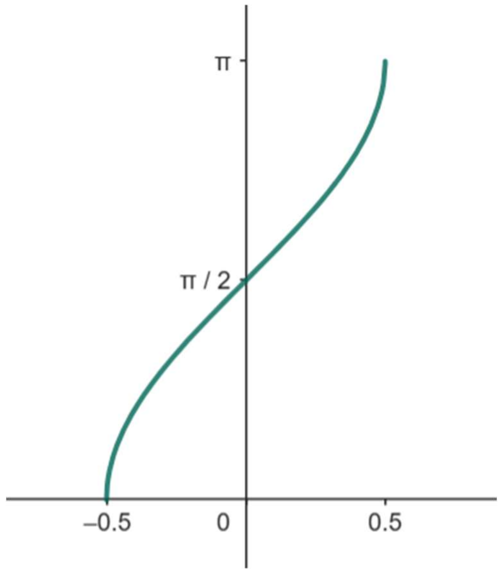
Domain: $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

Range: $0 \le \cos^{-1}(2x) \le \pi$

$-\pi \le -\cos^{-1}(2x) \le 0$

$0 \le \pi - \cos^{-1}(2x) \le \pi$




| Criteria | Marks |
|---|---|
| Provides correct graph | 2 |
| Correctly identifies domain OR range of the function OR correct shape | 1 |

(c) Given that $y = e^{2x} + e^{-2x}$, determine the values of constants $a$ and $b$ that satisfy the differential equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 5e^{2x} + e^{-2x}$. **2**

***Sample answer:***

$$y = e^{2x} + e^{-2x}$$

$$\frac{dy}{dx} = 2e^{2x} - 2e^{-2x}$$

$$\frac{d^2y}{dx^2} = 4e^{2x} + 4e^{-2x}$$

$$\begin{aligned}\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by &= 4e^{2x} + 4e^{-2x} + a(2e^{2x} - 2e^{-2x}) + b(e^{2x} + e^{-2x}) \\ &= (4 + 2a + b)e^{2x} + (4 - 2a + b)e^{-2x}\end{aligned}$$

$$\begin{aligned}4 + 2a + b &= 5 \\ 2a + b &= 1\end{aligned}$$

$$\begin{aligned}4 - 2a + b &= 1 \\ -2a + b &= -3\end{aligned}$$

$$a = 1,\ b = -1$$

| Criteria | Marks |
|---|---|
| Provides correct solution | 2 |
| Correctly finds first and second derivatives, or equivalent merit | 1 |

(d) The population of a particular town is growing at a rate proportional to the excess of its current size $N$ above 3000 individuals.

That is, $N$ satisfies the equation $\frac{dN}{dt} = k(N - 3000)$.

At the start of 2020 the population was 25 000, and at the start of 2025 the population was 29 400.

(i) Show that $N = 3000 + Ae^{kt}$ satisfies this model of population growth. **1**

***Sample answer:***

$$N = 3000 + Ae^{kt}$$

$$\begin{aligned}\frac{dN}{dt} &= kAe^{kt} \\ &= k(3000 + Ae^{kt} - 3000) \\ &= k(N - 3000)\end{aligned}$$

| **Criteria** | **Marks** |
|---|---|
| Provides correct solution | 1 |

(ii) Hence show that $A = 22\ 000$ and $k = \frac{1}{5}\ln\left(\frac{6}{5}\right)$. **2**

***Sample answer:***

When $t = 0$, $N = 25\ 000$

$$\begin{aligned}25000 &= 3000 + Ae^0\\ A &= 22000\end{aligned}$$

When $t = 5$, $N = 29\ 400$

$$\begin{aligned}29400 &= 3000 + 22000\,e^{5k}\\ 26400 &= 22000\,e^{5k}\\ e^{5k} &= \frac{6}{5}\\ 5k &= \ln\left(\frac{6}{5}\right)\\ k &= \frac{1}{5}\ln\left(\frac{6}{5}\right)\end{aligned}$$

| Criteria | Marks |
|---|---|
| Provides correct solution | 2 |
| Correctly finds the value of $A$, or equivalent merit | 1 |

(iii) In what year will the population first exceed 35 000? **1**

***Sample answer:***

$$\begin{aligned}3000 + 22000\,e^{kt} &> 35000\\ 22000\,e^{kt} &> 32000\\ e^{kt} &> \frac{16}{11}\\ kt &> \ln\left(\frac{16}{11}\right)\\ t &> \frac{5\ln\left(\frac{16}{11}\right)}{\ln\left(\frac{6}{5}\right)}\\ t &> 10.27\end{aligned}$$

Since $t = 10$ at the start of 2030, the population will first exceed 35 000 during 2030.

| Criteria | Marks |
|---|---|
| Provides correct answer | 1 |

(iv) What will be the rate of population growth when the population reaches 35 000? **1**

***Sample answer:***

$$\begin{aligned}\frac{dN}{dt} &= k(N-3000)\\ &= \frac{1}{5}\ln\left(\frac{6}{5}\right)\cdot(35000-3000)\\ &= 1166.857\ \ldots\\ &\approx 1167\end{aligned}$$

| **Criteria** | **Marks** |
|---|---|
| Provides correct answer | 1 |

**Question 13**

(a) (i) Find $\frac{d}{dx}(x \tan^{-1} x)$. **1**

***Sample answer:***

$$\frac{d}{dx}(x \tan^{-1} x) = \tan^{-1} x \times 1 + x \times \frac{1}{1+x^2}$$
$$= \tan^{-1} x + \frac{x}{1+x^2}$$

| Criteria | Marks |
|---|---|
| Provides correct answer | 1 |

(ii) Hence find $\int \tan^{-1} x \, dx$. **2**

***Sample answer:***

$$\int \tan^{-1} x \, dx = x\tan^{-1} x - \int \frac{x}{1+x^2} \, dx$$
$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} \, dx$$
$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$$

| Criteria | Marks |
|---|---|
| Provides correct solution | 2 |
| Uses the result from (i) to find an expression for the required integral | 1 |

(b) A particle is fired from a point on the ground, taken to be the origin, with speed of projection 80 ms$^{-1}$ at an angle $\alpha$ above the horizontal.

The particle needs to clear a fence 5 metres high at a horizontal distance 400 metres from the point of projection.

The position vector of the particle is $\underset{\sim}{r}(t)$, where $t$ is the time after projection. The acceleration due to gravity is $g = 10$ ms$^{-2}$.

(i) Show that $\underset{\sim}{r}(t) = (80t\cos\alpha)\underset{\sim}{i} + (80t\sin\alpha - 5t^2)\underset{\sim}{j}$. **3**

***Sample answer:***

Horizontal components:

$\ddot{x} = 0$

$\dot{x} = \int 0 \, dt$

$= c$

When $t = 0$, $\dot{x} = 80\cos\alpha$

$\therefore \dot{x} = 80\cos\alpha$

$x = \int 80\cos\alpha \, dt$

$= 80t\cos\alpha + c$

When $t = 0$, $x = 0$

$\therefore x = 80t\cos\alpha$

Vertical components:

$\ddot{y} = -10$

$\dot{y} = \int -10 \, dt$

$= -10t + c$

When $t = 0$, $\dot{y} = 80\sin\alpha$

$\therefore \dot{y} = -10t + 80\sin\alpha$

$y = \int -10t + 80\sin\alpha \, dt$

$= -5t^2 + 80t\sin\alpha + c$

When $t = 0$, $y = 0$

$\therefore y = -5t^2 + 80t\sin\alpha$

Hence $\underset{\sim}{r}(t) = (80t\cos\alpha)\underset{\sim}{i} + (80t\sin\alpha - 5t^2)\underset{\sim}{j}$

| **Criteria** | **Marks** |
|---|---|
| Provides correct solution | 3 |
| Correctly finds an expression for the horizontal OR vertical component of $\underset{\sim}{r}$, or equivalent merit | 2 |
| Makes some progress towards finding an expression for the horizontal OR vertical component of $\underset{\sim}{r}$, or equivalent merit | 1 |

(ii) Hence find the possible values of $\alpha$ for the particle to clear the fence, correct to the nearest minute. **3**

***Sample answer:***

When $x = 400$, we need $y > 5$

$$400 = 80t\cos\alpha$$
$$t = 5\sec\alpha$$

$$-5(5\sec\alpha)^2 + 80(5\sec\alpha)\sin\alpha > 5$$
$$-125\sec^2\alpha + 400\tan\alpha - 5 > 0$$
$$-125(1+\tan^2\alpha) + 400\tan\alpha - 5 > 0$$
$$-125\tan^2\alpha + 400\tan\alpha - 130 > 0$$
$$25\tan^2\alpha - 80\tan\alpha + 26 < 0$$

Critical values:

$$\tan\alpha = \frac{80 \pm \sqrt{(-80)^2 - 4(25)(26)}}{2(25)}$$
$$= \frac{80 \pm \sqrt{3800}}{50}$$
$$\alpha = 70°33' \text{ or } 20°10'$$

Hence $20°10' < \alpha < 70°33'$

| Criteria | Marks |
|---|---|
| Provides correct solution | 3 |
| Finds the required quadratic equation in terms of tan $\alpha$ which will give the critical values | 2 |
| Finds the required expression for $t$ in terms of $\alpha$, or equivalent merit | 1 |

(c) Use mathematical induction to prove that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers $n$. **3**

***Sample answer:***

Step 1: Prove the statement to be true for $n = 1$

$$\begin{aligned} 9^3 - 4^1 &= 729 - 4 \\ &= 725 \\ &= 5 \times 145 \end{aligned}$$

$\therefore$ true for $n = 1$

Step 2: Assume the statement to be true for some positive integer $n = k$

i.e. assume $9^{k+2} - 4^k = 5M \quad \text{for } M \in \mathbb{Z}$

$$9^{k+2} = 5M + 4^k$$

Step 3: Prove the statement to be true for $n = k + 1$, if it is true for $n = k$

$$\begin{aligned} 9^{k+3} - 4^{k+1} &= 9 \times 9^{k+2} - 4 \times 4^k \\ &= 9(5M + 4^k) - 4 \times 4^k \qquad \text{using assumption} \\ &= 45M + 9 \times 4^k - 4 \times 4^k \\ &= 45M + 5 \times 4^k \\ &= 5(9M + 4^k) \\ &= 5J \quad \text{for } J \in \mathbb{Z} \end{aligned}$$

$\therefore$ true for $n = k + 1$, if true for $n = k$

Since true for $n = 1$, the statement is true for all $n \geq 1$ by mathematical induction.

| **Criteria** | **Marks** |
|---|---|
| Provides correct proof | 3 |
| Proves the inductive step, or equivalent merit | 2 |
| Establishes the base case, or equivalent merit | 1 |

**Question 14**

(a) A Maths club has $2n$ members, half of which are male and half female. **2**

A Pi Day committee is made up of 3 members of this club.

By considering all different combinations for this committee, show that

$$n\ {}^{n}\mathbf{C}_2 + {}^{n}\mathbf{C}_3 = \frac{1}{2}\ {}^{2n}\mathbf{C}_3 \text{ , where } n \geq 3.$$

***Sample answer:***

Total number of committees possible $= {}^{2n}\mathbf{C}_3$

Committees with 3 males, 0 females $= {}^{n}\mathbf{C}_3$

Committees with 2 males, 1 female $= {}^{n}\mathbf{C}_2 \times {}^{n}\mathbf{C}_1$

Committees with 1 male, 2 females $= {}^{n}\mathbf{C}_1 \times {}^{n}\mathbf{C}_2$

Committees with 0 males, 3 females $= {}^{n}\mathbf{C}_3$

Hence ${}^{n}\mathbf{C}_3 + {}^{n}\mathbf{C}_2 \times {}^{n}\mathbf{C}_1 + {}^{n}\mathbf{C}_1 \times {}^{n}\mathbf{C}_2 + {}^{n}\mathbf{C}_3 = {}^{2n}\mathbf{C}_3$

$$2\ {}^{n}\mathbf{C}_3 + 2\ {}^{n}\mathbf{C}_1 \times {}^{n}\mathbf{C}_2 = {}^{2n}\mathbf{C}_3$$

But ${}^{n}\mathbf{C}_1 = n$

So $2\ {}^{n}\mathbf{C}_3 + 2n\ {}^{n}\mathbf{C}_2 = {}^{2n}\mathbf{C}_3$

$${}^{n}\mathbf{C}_3 + n\ {}^{n}\mathbf{C}_2 = \frac{1}{2}\ {}^{2n}\mathbf{C}_3$$

| **Criteria** | **Marks** |
|---|---|
| Provides correct solution | 2 |
| Identifies total number of committees possible with different numbers of males, or equivalent merit | 1 |

(b) In the expansion of $(3+2x)^n$, the coefficients of $x^2$ and $x$ are in the ratio 3:1. **3**

Find the value of $n$.

***Sample answer:***

$$(3+2x)^n = {}^n\mathbf{C}_0\, 3^n + {}^n\mathbf{C}_1\, 3^{n-1}(2x) + {}^n\mathbf{C}_2\, 3^{n-2}(2x)^2 + \ldots$$

Coefficient of $x^2 = {}^n\mathbf{C}_2 \times 3^{n-2} \times 2^2$

Coefficient of $x = {}^n\mathbf{C}_1 \times 3^{n-1} \times 2$

$$\begin{aligned}
{}^n\mathbf{C}_2 \times 3^{n-2} \times 2^2 &= 3\left({}^n\mathbf{C}_1 \times 3^{n-1} \times 2\right) \\
{}^n\mathbf{C}_2 \times 3^{n-2} \times 2^2 &= {}^n\mathbf{C}_1 \times 3^n \times 2 \\
{}^n\mathbf{C}_2 \times 2 &= {}^n\mathbf{C}_1 \times 3^2 \\
\frac{n!}{2!(n-2)!} \times 2 &= n \times 9 \\
n(n-1) &= 9n \\
n^2 - n &= 9n \\
n^2 - 10n &= 0 \\
n(n-10) &= 0 \\
n &= 0 \text{ or } 10
\end{aligned}$$

But $n > 0 \ \therefore n = 10$

| Criteria | Marks |
|---|---|
| Provides correct solution | 3 |
| Obtains a correct equation in terms of $n$, with factorials but without ${}^n\mathbf{C}_k$ notation, or equivalent merit | 2 |
| Obtains a correct expression for one coefficient, or equivalent merit | 1 |

(c) Solve the differential equation $(x^2 + 1)\dfrac{dy}{dx} = 6xy$, given that $y = 2$ when $x = 1$. **3**

Give your answer as the function $y$ in terms of $x$.

***Sample answer:***

$$(x^2 + 1)\frac{dy}{dx} = 6xy$$

$$\frac{1}{y}\,dy = \frac{6x}{x^2+1}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{6x}{x^2+1}\,dx$$

$$\int \frac{1}{y}\,dy = 3\int \frac{2x}{x^2+1}\,dx$$

$$\ln y = 3\ln(x^2+1) + c$$

$$y = e^{3\ln(x^2+1)+c}$$

$$y = Ae^{\ln(x^2+1)^3}$$

$$y = A(x^2+1)^3$$

When $x = 1$, $y = 2$

$$2 = A(1+1)^3$$

$$A = \frac{1}{4}$$

$$\therefore\ y = \frac{1}{4}(x^2+1)^3$$

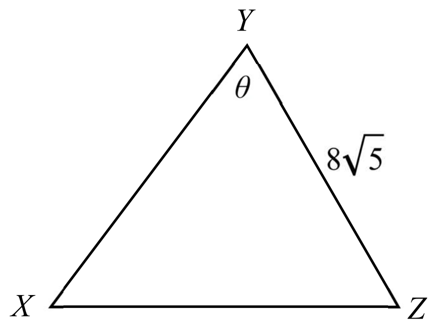
| **Criteria** | **Marks** |
|---|---|
| Provides correct solution | 3 |
| Correctly integrates to find an expression for $y$, without finding the constant of integration, or equivalent merit | 2 |
| Separates the variables in the differential equation, or equivalent merit | 1 |

(d) An acute-angled triangle $XYZ$ has an area of 40 square units. **4**

The vector $\overrightarrow{YX} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ and $\overrightarrow{YZ} = \begin{pmatrix} p \\ q \end{pmatrix}$.

Given $|\overrightarrow{YZ}| = 8\sqrt{5}$, find the possible values of $p$ and $q$.

***Sample answer:***




$$|\overrightarrow{YX}| = \sqrt{6^2 + 2^2}$$
$$= 2\sqrt{10}$$

$$Area = \frac{1}{2}ab\sin C$$
$$40 = \frac{1}{2} \times 8\sqrt{5} \times 2\sqrt{10} \times \sin\theta$$
$$40 = 40\sqrt{2}\ \sin\theta$$
$$\sin\theta = \frac{1}{\sqrt{2}}$$
$$\theta = \frac{\pi}{4}$$

Consider $\overrightarrow{YX} \cdot \overrightarrow{YZ}$

$$6p + 2q = 2\sqrt{10} \times 8\sqrt{5} \times \cos\frac{\pi}{4}$$
$$6p + 2q = 80\sqrt{2} \times \frac{1}{\sqrt{2}}$$
$$6p + 2q = 80$$
$$q = 40 - 3p \qquad ①$$

Consider $|\overrightarrow{YZ}|$

$$\sqrt{p^2 + q^2} = 8\sqrt{5}$$
$$p^2 + q^2 = 320 \qquad ②$$

Substitute ① into ②

$$
\begin{aligned}
p^2 + (40 - 3p)^2 &= 320 \\
p^2 + 1600 - 240p + 9p^2 &= 320 \\
10p^2 - 240p + 1280 &= 0 \\
p^2 - 24p + 128 &= 0 \\
(p - 16)(p - 8) &= 0 \\
p &= 16 \text{ or } 8
\end{aligned}
$$

When $p = 16$, $q = 40 - 3(16)$
$= -8$

When $p = 8$, $q = 40 - 3(8)$
$= 16$

Hence either $p = 16$ and $q = -8$ or $p = 8$ and $q = 16$

| **Criteria** | **Marks** |
|---|---|
| Provides correct solution | 4 |
| Obtains a pair of simultaneous equations relating $p$ and $q$ and correctly eliminates $p$ or $q$, or equivalent merit | 3 |
| Obtains one equation relating $p$ and $q$, or equivalent merit | 2 |
| Obtains a correct value for $\theta$, or equivalent merit | 1 |

**Question 15**

(a) (i) Show that $\sin(x-y)\sin(x+y)=\sin^2 x-\sin^2 y$. **2**

***Sample answer:***

$$\begin{aligned}\sin(x-y)\sin(x+y)&=(\sin x\cos y-\sin y\cos x)(\sin x\cos y+\sin y\cos x)\\&=\sin^2 x\cos^2 y-\sin^2 y\cos^2 x\\&=\sin^2 x(1-\sin^2 y)-\sin^2 y(1-\sin^2 x)\\&=\sin^2 x-\sin^2 x\sin^2 y-\sin^2 y+\sin^2 x\sin^2 y\\&=\sin^2 x-\sin^2 y\end{aligned}$$

| **Criteria** | **Marks** |
|---|---|
| Provides correct solution | 2 |
| Makes significant progress towards solution e.g. eliminates compound angles AND expands brackets | 1 |

(ii) Hence or otherwise, solve $\sin^2 3x-\sin^2 x=\sin 4x$ for $[0, \pi]$. **2**

***Sample answer:***

$$\begin{aligned}\sin^2 3x-\sin^2 x&=\sin 4x\\ \sin(3x-x)\sin(3x+x)&=\sin 4x\\ \sin 2x\sin 4x&=\sin 4x\\ \sin 2x\sin 4x-\sin 4x&=0\\ \sin 4x(\sin 2x-1)&=0\end{aligned}$$

$$0\le x\le\pi$$
$$0\le 2x\le 2\pi$$
$$0\le 4x\le 4\pi$$

$\sin 4x=0$

$4x=0, \pi, 2\pi, 3\pi, 4\pi$

$x=0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

$\sin 2x=1$

$2x=\frac{\pi}{2}$

$x=\frac{\pi}{4}$

Hence $x=0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

| **Criteria** | **Marks** |
|---|---|
| Provides correct solution | 2 |
| Obtains the equation $\sin 4x(\sin 2x-1)=0$, or equivalent merit | 1 |

(b) The function $f(x) = 1 + \ln x$ is defined in the domain (0, 1].

(i) Show that $\dfrac{d}{dx}(x \ln x) = 1 + \ln x$. **1**

***Sample answer:***

Using the product rule, $\dfrac{d}{dx}(uv) = vu' + uv'$

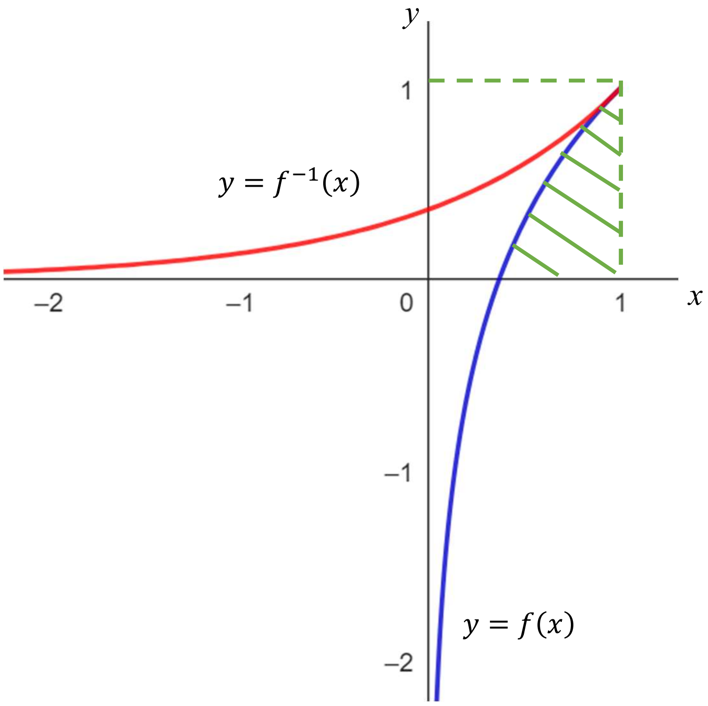
Let $u = x$, $v = \ln x$

$$\frac{d}{dx}(x\ln x) = \ln x \times 1 + x \times \frac{1}{x}$$

$$= \ln x + 1$$

| **Criteria** | **Marks** |
|---|---|
| Provides correct solution, demonstrating use of the product rule | 1 |

(ii) The diagram shows the graphs of the function $y = f(x)$ and the inverse function $y = f^{-1}(x)$ Find in simplest exact form the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = f^{-1}(x)$ and the coordinate axes. **3**




***Sample answer:***

For $x$ intercept of $y = f(x)$:

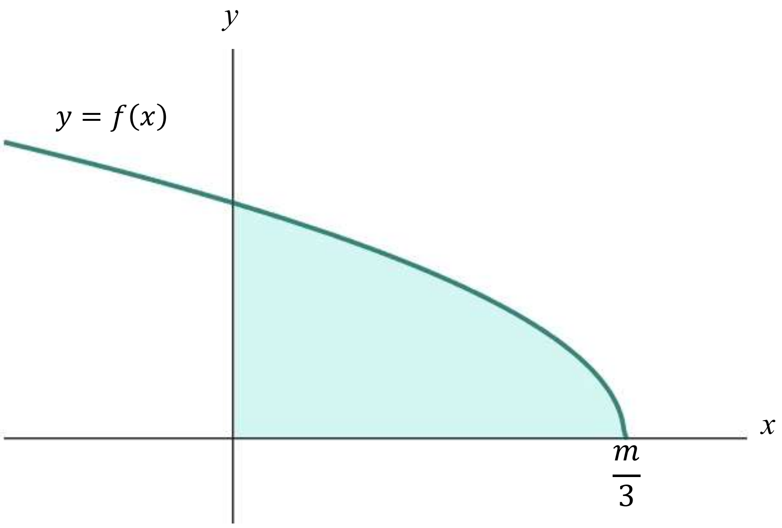
$$
\begin{aligned}
1 + \ln x &= 0 \\
\ln x &= -1 \\
x &= \frac{1}{e}
\end{aligned}
$$

Required area = area of square – 2 x shaded area

$$
\begin{aligned}
Area &= 1 \times 1 - 2\int_{\frac{1}{e}}^{1} 1 + \ln x \, dx \\
&= 1 - 2\left[x \; lnx\right]_{\frac{1}{e}}^{1} \\
&= 1 - 2\left[\ln(1) - \frac{1}{e}\ln\left(\frac{1}{e}\right)\right] \\
&= 1 - 2\left(0 + \frac{1}{e}\right) \\
&= 1 - \frac{2}{e}
\end{aligned}
$$

| Criteria | Marks |
|---|---|
| Provides correct solution | 3 |
| Obtains a correct integral expression for the required area | 2 |
| Finds the $x$ intercept of $y = f(x)$, or equivalent merit | 1 |

(c) Let $f(x) = \sqrt{m - 3x}$ for $x \leq \frac{m}{3}$. The graph of $y = f(x)$ is shown. **4**




The shaded area, enclosed by the graph of $y = f(x)$, the $x$-axis and the $y$-axis, is rotated about the $y$-axis.

Find the value of $m$ such that the volume of the solid formed is $\frac{5000\pi}{27}$ cubic units.

***Sample answer:***

$$y = \sqrt{m - 3x}$$
$$y^2 = m - 3x$$
$$3x = m - y^2$$
$$x = \frac{1}{3}(m - y^2)$$
$$x^2 = \frac{1}{9}(m - y^2)^2$$
$$x^2 = \frac{1}{9}(m^2 - 2my^2 + y^4)$$

$$V = \pi \int_a^b x^2 \, dy$$

$$V = \frac{\pi}{9} \int_0^{\sqrt{m}} m^2 - 2my^2 + y^4 \, dy$$

$$\frac{5000\pi}{27} = \frac{\pi}{9} \left[ m^2 y - \frac{2my^3}{3} + \frac{y^5}{5} \right]_0^{\sqrt{m}}$$

$$\frac{5000}{3} = m^2\left(\sqrt{m}\right) - \frac{2m}{3}\left(\sqrt{m}\right)^3 + \frac{1}{5}\left(\sqrt{m}\right)^5$$

$$\frac{5000}{3} = m^{\frac{5}{2}} - \frac{2}{3} m^{\frac{5}{2}} + \frac{1}{5} m^{\frac{5}{2}}$$

$$\frac{5000}{3} = \frac{8}{15} m^{\frac{5}{2}}$$

$$3125 = m^{\frac{5}{2}}$$

$$m = 3125^{\frac{2}{5}}$$

$$m = 25$$

| **Criteria** | **Marks** |
|---|---|
| Provides correct solution | 4 |
| Obtains correct equation to be solved for $m$, or equivalent merit | 3 |
| Obtains correct integral expression for the required volume, or equivalent merit | 2 |
| Obtains correct expression for $x^2$ OR correct bounds of integration, or equivalent merit | 1 |