Name: ______________________________________

Class: _________________

2019

Higher School Certificate

Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–10)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

Write your answers on the multiple choice answer sheet provided.

1. The solution to the equation $2^x + 2^{x+1} = 8$ is:

   (A) $x = 1$

   (B) $x = 3 - \log_2 3$

   (C) $x = \log_2 5$

   (D) $x = \frac{1}{4}$

2. A polynomial equation has roots α, β and γ, where

   $\alpha + \beta + \gamma = 0;\ \ \alpha\beta\gamma = 1;\ \ \alpha\beta + \alpha\gamma + \beta\gamma = 1$

   Which polynomial equation has roots α, β and γ

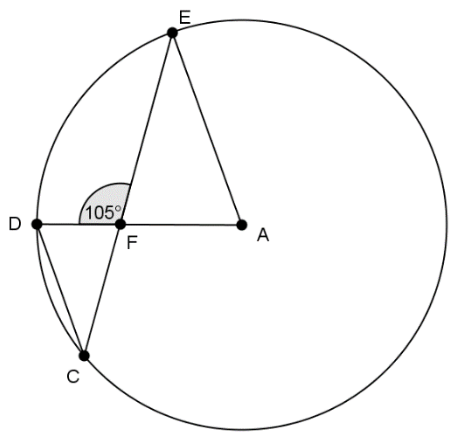
   (A) $x^3 + x - 1 = 0$

   (B) $x^3 + x + 1 = 0$

   (C) $x^3 - x + 1 = 0$

   (D) $x^3 - x - 1 = 0$

3. *D, C,* and *E* are points on a circle with centre *A.*
   DC is parallel to *AE. AD* intersect *CE* at *F.* $\angle$DFE = 105°



Not to scale

The value of $\angle FAE$ is

(A) 75°

(B) 52.5°

(C) 70°

(D) 105°

4. Consider the function $f(x) = e^{x+2}$ and its inverse function $f^{-1}(x)$. What is the value of $f^{-1}(e^2)$?

(A) $f^{-1}(x) = e^{-(e^2)-2}$

(B) $f^{-1}(x) = e^{(e^2)+2}$

(C) $f^{-1}(x) = 0$

(D) $f^{-1}(x) = 4$

5. The value of $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{x+1}-2}$ is:

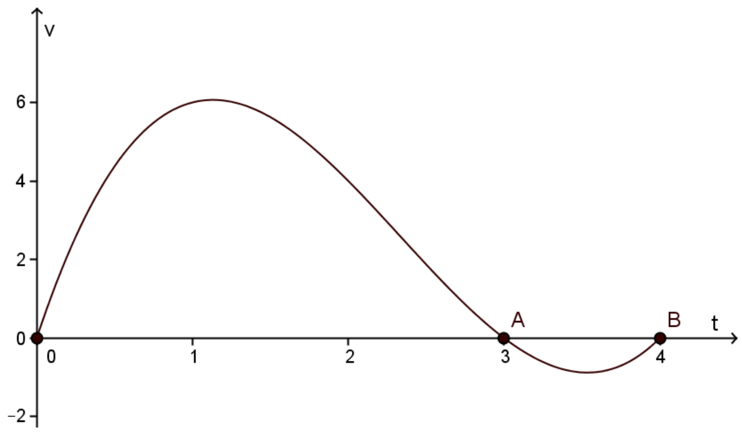
(A) 4

(B) 0

(C) Not defined

(D) 1

6. The graph describes the velocity *v* of a particle at time *t*.



Which of the following statements is true?

(A) The particle is at the origin at A and B.

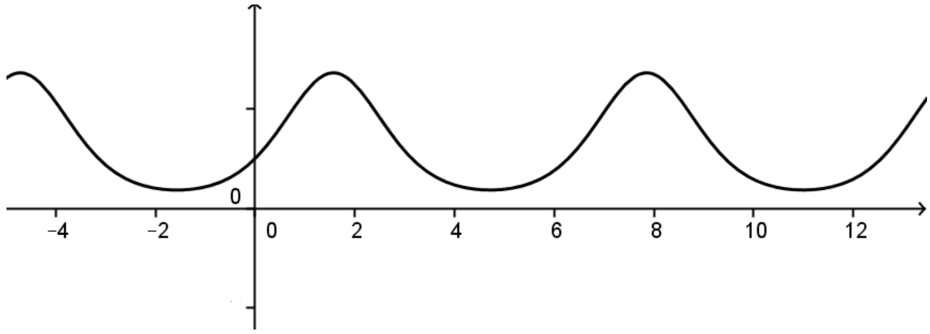
(B) The particle moves in a positive direction and changes direction after 3 seconds.

(C) The particle has a negative acceleration throughout the last second of its trajectory.

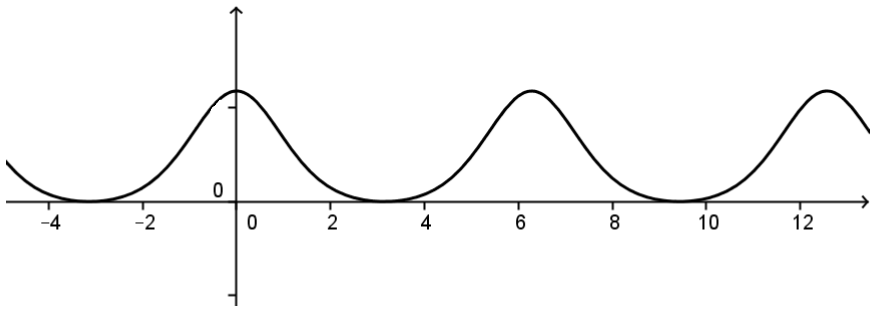
(D) The particle returns to the origin after 4 seconds and comes to rest.

7. A particle is moving along a straight line so that initially its displacement is $x = 1$, its velocity is $v = 2$, and its acceleration is $a = 4$.
Which is a possible equation describing the motion of the particle?
(A) $v = x^2 + 2x + 4$
(B) $v^2 = 4(x^2 - 2)$
(C) $v = 2 + 4\ln x$
(D) $v = 2\sin(x - 1) + 2$

8. The following curve is the graph of the function $y = e^{\sin x}$



The graph is translated vertically and horizontally so that the resulting graph describes an even function which touches the x-axis, as shown:



A possible equation for the resulting graph is:

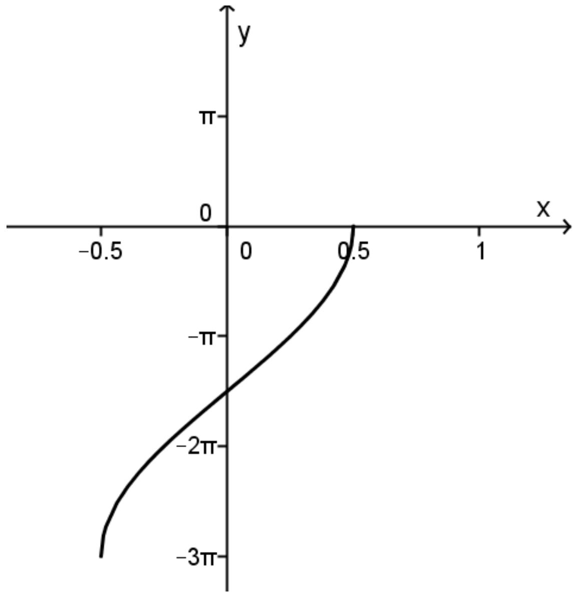
(A) $y = e^{\sin\left(x - \frac{\pi}{2}\right)} - e$

(B) $y = e^{\sin\left(x + \frac{3\pi}{2}\right)} - \frac{1}{e}$

(C) $y = e^{\sin\left(x + \frac{\pi}{2}\right)} - \frac{1}{e}$

(D) $y = e^{\sin\left(x + \frac{3\pi}{2}\right)} - e$

9. The following curve is the graph of which of the following function?



(A) $y = -3\sin^{-1}\dfrac{x}{2}$

(B) $y = -3\cos^{-1} 2x$

(C) $y = 3\cos^{-1} 2x$

(D) $y = \dfrac{1}{2}\cos^{-1} 3x$

10. Let $f(x) = ax^m$ and $g(x) = bx^n$, where *a*, *b*, *m* and *n* are positive integers. For both *f* and *g*, the domain is all real *x*.

If $f'(x)$ is a primitive of $g(x)$, then which one of the following must be true?

(A) $\dfrac{m}{n}$ is an integer

(B) $\dfrac{n}{m}$ is an integer

(C) $\dfrac{a}{b}$ is an integer

(D) $\dfrac{b}{a}$ is an integer

Section II begins on the next page

Section II

60 marks

Attempt Questions 11–14

Allow about 1 hour and 45 minutes for this section

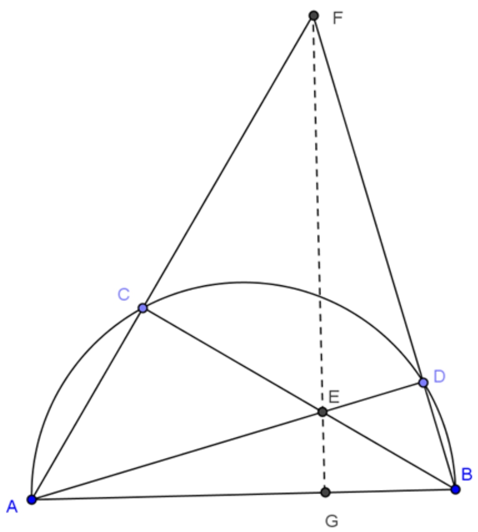
Start each question on a new page. Extra paper is available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) **Start a new page**.

(a) Find the exact value of $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x dx$ **2**

(b)



The points D, C lie on a semicircle with AB as diameter.
AC, BD produced intersect at F; AD, BC intersect at E.

(i) Prove that $CEDF$ is a cyclic quadrilateral. **1**

(ii) Prove that FE is perpendicular to AB. **2**

(c) Let α, β and γ, be the three angles of a given triangle.

(i) Show that $\tan(\alpha+\beta) = -\tan\gamma$ **1**

(ii) Hence, or otherwise, prove that: $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \times \tan\beta \times \tan\gamma$ **2**

(d) (i) Show that the equation $\ln(x) = \cos x$ has a solution between $x = 1$ and $x = 2$ **1**

(ii) Using one application of Newton's Method, with $x = 1.5$ as your initial value, find a better estimate for this solution. Present your answer accurate to three decimal places. **2**

(e) The velocity, *v*, measured in centimetres per second, of a particle moving in simple harmonic motion along the *x* axis, is given by $v^2 = 16x - 4x^2 + 20$

(i) Show that $\ddot{x} = -4(x-2)$ **2**

(ii) Find the maximum speed of the particle. **2**

Question 12 (15 marks) **Start a new page**.

(a) Find a general solution to the equation $\sin 3x = \cos 2x$ where x is measured in radians. **2**

(b) Air is escaping from a spherical balloon at the rate of 2 cm^3 per second. How fast is the surface area of the balloon shrinking when the radius is 2cm? **3**

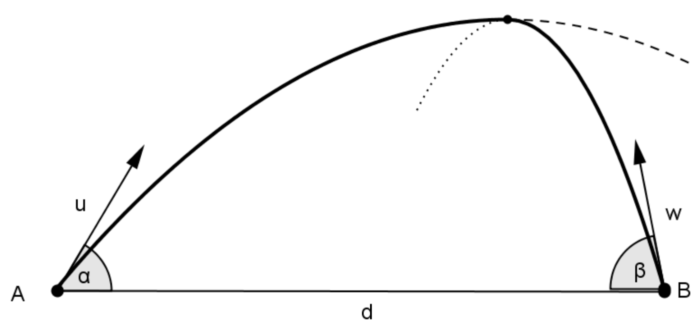
(c) If there exists a non-zero constant term in the expansion of $\left(x^2 - \frac{1}{\sqrt{x}}\right)^n$, show that n is a multiple of 5. **2**

(d) From a group of 6 men and 7 women, a committee of 5 people is to be formed.

(i) What is the probability that in the committee there is at least one man and at least one woman? **2**

(ii) Three particular women and two particular men are chosen to be on the committee and the committee members are seated at random around a circular table. What is the probability that the two men are not seated next to each other? **2**

(e) Points A and B are located d metres apart on a horizontal plane. A projectile is fired from A towards B with initial velocity u m/s at angle α to the horizontal. At the same time, another projectile is fired from B towards A with initial velocity w ms^{-1} d at angle β to the horizontal, as shown in the diagram. The projectiles collide when they both reach their maximum height.



The equations of motion of a projectile fired from the origin with initial velocity V m/s at angle θ to the horizontal are: $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$ (Do NOT prove this)

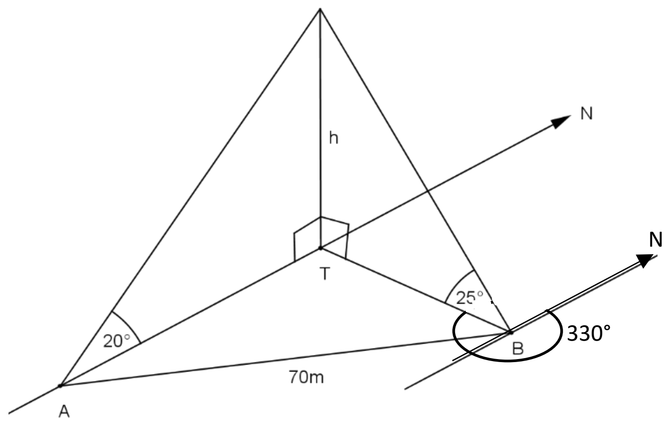
(i) Show that the projectile fired from A reaches its maximum height at time $t = \frac{usin\alpha}{g}$ **1**

(ii) Show that $u\sin\alpha = w\sin\beta$ **1**

(iii) The distance between A and B, is given by: $d = \frac{uw}{g}\sin(\alpha+\beta)$ **2**

Question 13 (15 marks) **Start a new page**.

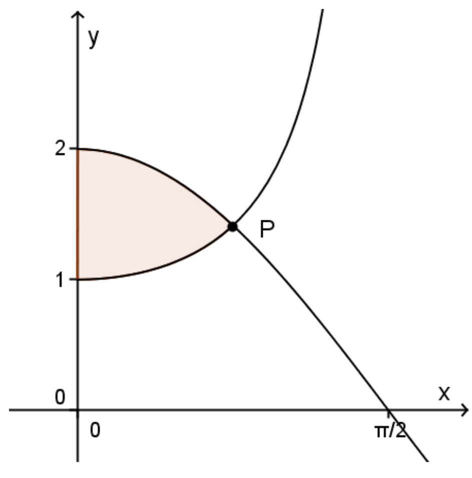
(a) John stands at point *A* and sees a tower due north. The angle of elevation from *A to* the top of the tower is 20°. He then walks in a straight line 70 m to point *B* and notices the tower is now positioned at a bearing of 330° from him. The angle of elevation from point *B* to the top of the tower is now 25°. **4**



Copy the diagram on to your answer sheet showing points *A, B* and *T* and relevant lengths and bearings.

Calculate the height of the tower, correct to 1 decimal place.

(b) *P* is the point of intersection between $x = 0$ and $x = \frac{\pi}{2}$ of the graphs of $y = \sec x$ and $y = 2\cos x$, as shown.



(i) Verify that the x-coordinate of *P* is $\frac{\pi}{4}$ **1**

(ii) The shaded region makes a revolution about the x-axis. Show that the volume of the resulting solid is $\frac{\pi^2}{2}$ cubic units. **3**

(c) Use mathematical induction to prove that for all positive integers $n \geq 1$: **3**

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = 1 + (n-1) \times 2^n$$

(d) Four fair dice are rolled. Any die showing 6 is left alone, while the remaining dice are rolled again.

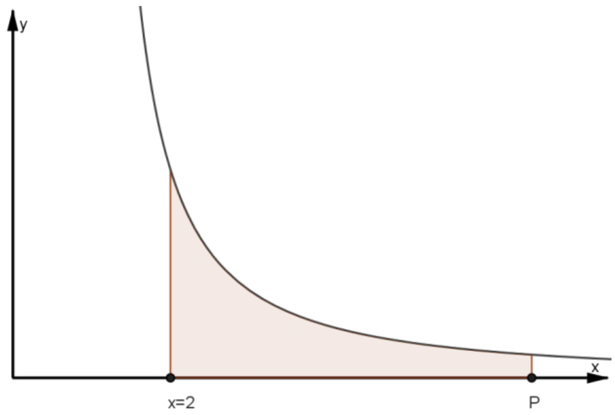
(i) Find the probability (correct to 2 decimal places) that after the first roll of the dice, exactly one of the four dice is showing 6. **1**

(ii) Find the probability (correct to 2 decimal places) that after the second roll of the dice exactly two of the four dice are showing 6. **3**

Question 14 continues over the page

Question 14 (15 marks) **Start a new page**.

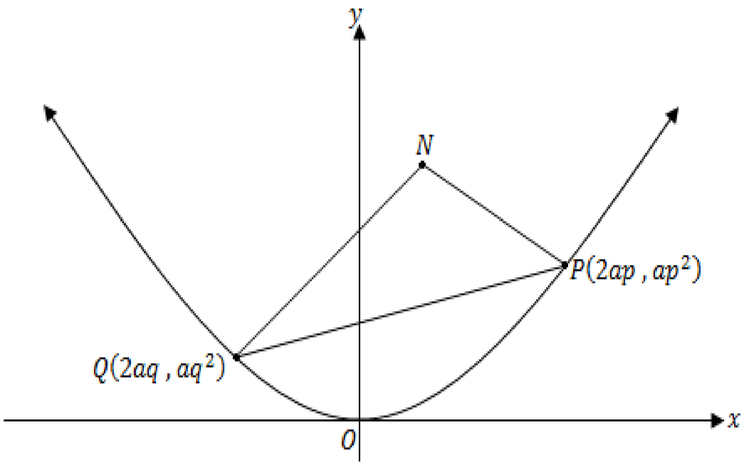
(a) (i) Find $\int \frac{1}{x\ln x}dx$ using the substitution $u = \ln x$ **1**



The curve above is the graph of the function $y = \frac{1}{x\ln x}$

(ii) The area shaded is bounded by the curve, the x axis and the lines $x = 2$ and $x = p$ is 1 square unit. Using part (i), or otherwise, show that $p = 2^e$. **3**

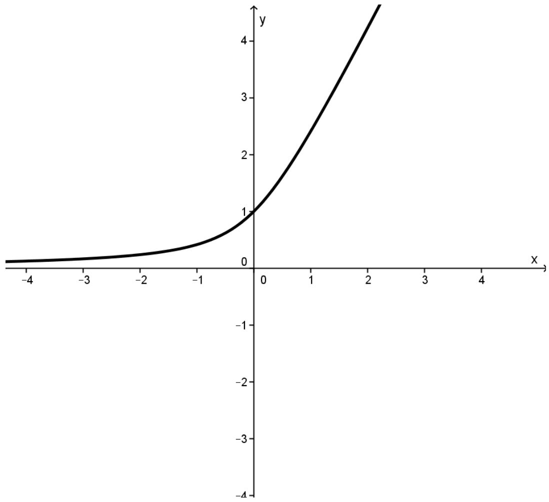
(b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two variable points on a parabola $x^2 = 4ay$.



(i) If the variable chord PQ is always parallel to the line $y = x$, show that $p + q = 2$. **2**

(ii) The normals at P and Q meet at N. Prove that the locus of *N* is a straight line. **3**
[You may assume that the gradient of the tangent at P is p.]

(c) Consider the function $f(x) = x + \sqrt{x^2 + 1}$



(i) Copy the graph of *f*(*x*) and sketch the graph of $f^{-1}(x)$, the inverse function of $f(x)$, on the same axes. **1**

(ii) Show that $f^{-1}(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$ **2**

(iii) By comparing the graphs of $f(x)$ and $f^{-1}(x)$, or otherwise, show that: **3**

$$\int_0^1 \left(x + \sqrt{x^2 + 1}\right) dx = \frac{1}{2}\left(1 + \sqrt{2} + \ln(1 + \sqrt{2})\right)$$

END OF EXAMINATION

Name: ______________________________________

Class: _________________

2019

Higher School Certificate

Trial Examination

Mathematics Extension 1

SOLUTIONS

Section I B, A, C, C, A, B, C, C, B

Write your answers on the multiple choice answer sheet provided.

1.

$$2^x + 2^x \times 2 = 8$$
$$2^x \times 3 = 8$$
$$2^x = \frac{8}{3}$$
$$x = \log_2\left(\frac{8}{3}\right)$$
$$x = \log_2 8 - \log_2 3$$
$$\underline{x = 3 - \log_2 3}$$

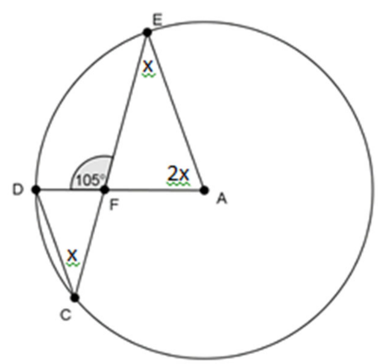
2. A polynomial equation has roots α, β and γ, where

$\alpha + \beta + \gamma =$ - B $\qquad \alpha\beta + \alpha\gamma + \beta\gamma =$ C; $\alpha\beta\gamma =$-D

$$B = 0$$
$$C = 1$$
$$D = -1$$
$$\underline{x^3 + x - 1 = 0}$$

3.

$$3x = 105$$
$$x = 35$$
$$\angle FAE = 2x = 70$$



Not to scale

4. $y = e^{x+2}$

$$x = e^{y+2}$$
$$y = \ln(x) - 2$$
$$f^{-1}(x) = \ln(x) - 2$$
$$f^{-1}(e^2) = \ln(e^2) - 2 = 0$$

5.

$$\lim_{x\to3}\frac{x-3}{\sqrt{x+1}-2}$$

$$\lim_{x\to3}\frac{x-3}{\sqrt{x+1}-2}\times\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$$

$$\lim_{x\to3}\frac{(x-3)\left(\sqrt{x+1}+2\right)}{x+1-4}$$

$$\lim_{x\to3}\frac{(x-3)\left(\sqrt{x+1}+2\right)}{x-3}$$

$$\lim_{x\to3}\frac{(x-3)\left(\sqrt{x+1}+2\right)}{x-3}$$

$$\lim_{x\to3}\left(\sqrt{x+1}+2\right)$$

$$\lim_{x\to3}\left(\sqrt{4}+2\right)=4$$

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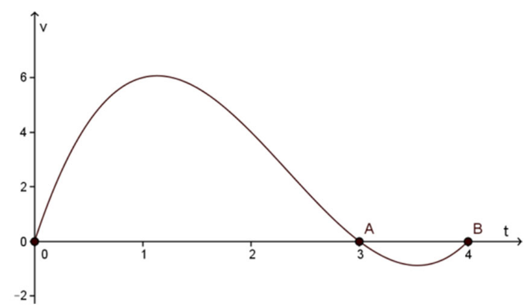
6. The particle moves in a positive direction and changes direction after 3 seconds.



7. $v = 2\sin(x-1)+2 \quad v^2 = 4\,[\sin(x-1)+1]^2$

At x=1 V=2

Use $a = \frac{d}{dx}\frac{v^2}{2} = \frac{d}{dx}2\,[\sin(x-1)+1]^2$

$a = 4[\sin(x-1)+1]\cos(x-1)$

At x=1 a=4

So, D

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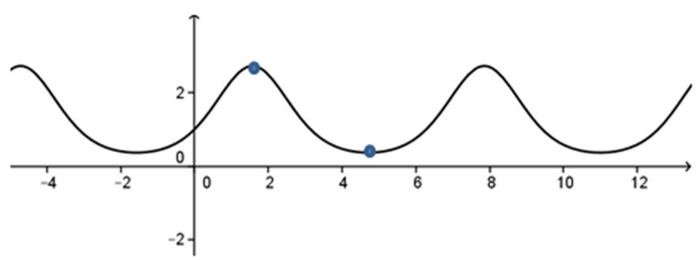
8. The following curve is the graph of the function $y = e^{\sin x}$

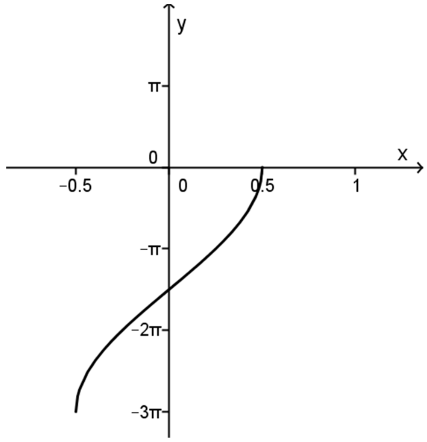


$$-1\le \sin x\le 1$$

Maximum point is at $y=e$ $\left(\frac{\pi}{2},e\right)$ Minimum point is at $y=e^{-1}$ $\left(\frac{3\pi}{2},\frac{1}{e}\right)$

The graph is translated vertically $\frac{1}{e}$ and horizontally $\frac{\pi}{2}$ So C

9. The following curve is the graph of the function:



At $x = 0 \;\; y = -\dfrac{3\pi}{2}$ So $y = -3\cos^{-1} 2x$ So, C

10.

Given $f'(x) = \int g(x)\,dx$,

$$amx^{m-1} = \frac{b}{n+1}x^{n+1}$$

$m - 1 = n + 1 \ldots (1)$ (equate powers)

$am = \dfrac{b}{n+1} \ldots (2)$ (equate coefficients)

Solving simultaneous equations:

$$m(n+1) = \frac{b}{a}$$

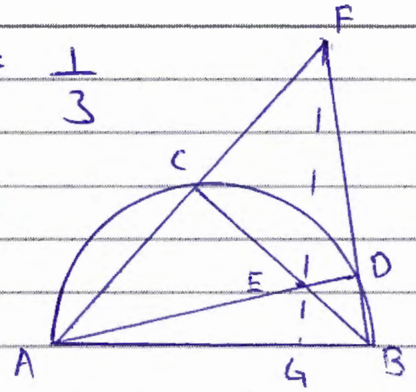
Since $m, n+1$ are both integers (i.e. $\in Z^+$),

$\Rightarrow m(n+1) \in Z^+$

$\therefore \dfrac{b}{a} = m(n+1) \in Z^+$

$\Rightarrow D$

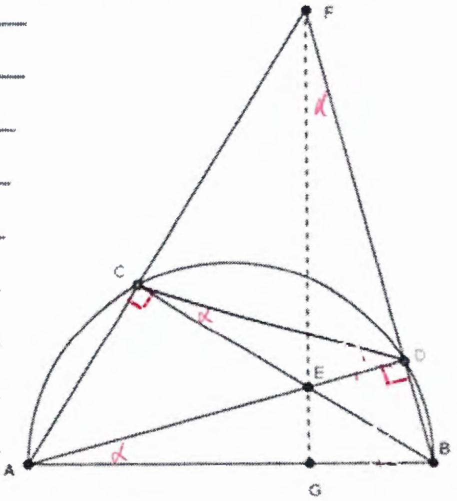
MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
a) $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx$		① for Integrating
$= \left[\frac{\sin^3 x}{3} \right]_0^{\frac{\pi}{2}}$		① for correct substitution and evaluation provided the expression wasn't made simpler.
$= \frac{1}{3}$		
b) 		
i $\angle ACB = \angle ADB = 90°$ (angles in a semi-circle)		
$\angle FCB + \angle ACB = 180°$ (angle sum of a straight angle)		
$\therefore \angle FCB = 180 - 90$ $= 90°$		
$\angle FDA + \angle BDA = 180$ (angle sum of a straight angle) $\therefore \angle FDA = 180 - 90$ $= 90°$		
$\therefore \angle FCB + \angle FDA = 180°$ $\therefore$ CEDF is a cyclic quadrilateral.		①

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments

ii



Construct CD

let $\angle DAB = \alpha$

$\angle DAB = \angle DCB$ (angles at the circumference standing on the same arc DB)
$= \alpha$

$\angle DCB = \angle DFE$ (angles at the circumference standing on the same arc, cyclic quad $\angle$FDE)
$= \alpha$

$\angle FGB = 180 - \angle FBG - \alpha$ (angle sum of ΔFGB)

But $\angle FBG = 180 - \alpha - 90$ (angle sum of ΔDAB)

$\therefore \angle FGB = 180 - (180 - \alpha - 90) - \alpha$
$= 90$

$\therefore FE \perp AB$

Mark Allocations:

① if students were able to use 'angles at the circumference standing on the same arc' within the cyclic quadrilateral $\angle$EDF proven in part i)

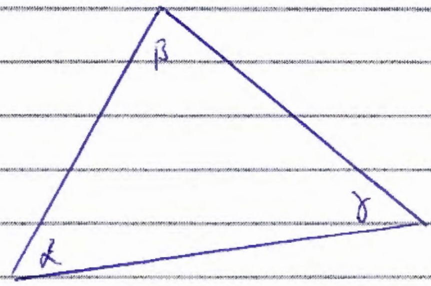
Marker's Comments:

Note: Anyone who just states cyclic quadrilaterals without proving them will get 0.

Other methods include:

* Similar Triangles
* Cyclic quadrilaterals
* Altitudes of a triangle are concurrent.

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments
c) 		
i $\alpha+\beta+\gamma = 180$ (angle sum of a triangle)		Must have all or part of this line in your proof.
$\therefore \alpha+\beta = 180-\gamma$		
$\therefore \tan(\alpha+\beta) = \tan(180-\gamma)$		
$= -\tan\gamma$ (tan is negative in 2nd quadrant)		
ii $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = -\tan\gamma$		① for expanding $\tan(\alpha+\beta)$
$\therefore \tan\alpha+\tan\beta = \tan\alpha\tan\beta\tan\gamma - \tan\gamma$		① finishing the proof off.
$\therefore \tan\alpha+\tan\beta+\tan\gamma = \tan\alpha\tan\beta\tan\gamma$		
d) let $f(x) = \ln x - \cos x$		
$f(1) = 0 - \cos 1$		Must show the value for both $f(1)$ and $f(2)$!
$= -0.54$		
< 0		
$f(2) = \ln 2 - \cos 2$		
$= 1.11$		
> 0		
$\therefore$ There lies a solution between $x=1$ and $x=2$		

MATHEMATICS Extension 1 : Question........

Suggested Solutions	Marks	Marker's Comments
ii) $f'(x) = \frac{1}{x} + \sin x$		1
$\therefore x_2 = 1.5 - \frac{f(1.5)}{f'(1.5)}$		
$= 1.299$		1
e) i $v^2 = 16x - 4x^2 + 20$		
$\frac{1}{2}v^2 = 8x - 2x^2 + 10$		1 for multiplying half
$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8 - 4x$		1 for differentiating
$= -4(x-2)$		
$\therefore \ddot{x} = -4(x-2)$		
ii) $\ddot{x} = 0 \rightarrow x = 2$.		1 $x=2$
$\therefore v^2 = 16(2) - 4(2)^2 + 20$		
$= 32 - 16 + 20$		
$= 36$		
$\therefore v = \pm 6$ cm/s		
$\therefore$ Max Speed $= 6$ cm/s.		1

Extension 1 Question 12	TRIAL	Term 3 2019	JRAHS

(a)

$$\sin 3x = \cos 2x$$

$$\cos\left(\frac{\pi}{2} - 3x\right) = \cos 2x$$

1 For expressing the given equation in a form that will enable one to obtain the general solution.

$$\frac{\pi}{2} - 3x = 2\pi n \pm 2x, n \in Z$$

$$3x \pm 2x = \frac{\pi}{2} - 2\pi n$$

$$x = \frac{\pi}{10} - \frac{2\pi n}{5} \quad or \quad x = \frac{\pi}{2} - 2\pi n$$

1 For both correct solutions

(i)

$$\cos 2x = \cos\left(\frac{\pi}{2} - 3x)\right) \Rightarrow x = \frac{2\pi n}{5} + \frac{\pi}{10} \quad or \ x = \frac{\pi}{2} - 2\pi n$$

(ii)

$$\sin 3x = \sin\left(\frac{\pi}{2} - 2x\right) \qquad \Rightarrow x = \frac{n\pi \ + \ (-1)^n \frac{\pi}{2}}{3 \ + \ (-1)^n 2}$$

(iii)

$$\sin\left(\frac{\pi}{2} - 2x\right) = \sin 3\text{x} \qquad \Rightarrow x = \frac{\frac{\pi}{2} \ - n\pi}{2 \ + \ (-1)^n 3}$$

(iv)

$$\sin(2x + x) = \cos 2x$$

$$\Rightarrow \quad 4 \ \sin^3 x - 2\sin^2 x - 3\sin x + 1 = 0$$

$$\Rightarrow \qquad \sin x = 1 \quad or \quad \sin x = \frac{-1 \pm \sqrt{5}}{4}$$

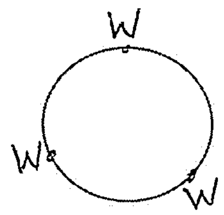
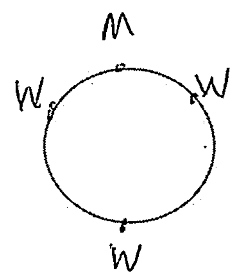
This step is needed for the 1st mark

$\Rightarrow \quad x = \frac{\pi}{2}$ or $x = \frac{\pi}{10}$ $(or\ 18^0)$ or $\frac{3\pi}{10}$ $(or\ \ 54^0)$ [acute angles]

So formulate the general solutions!!!!!

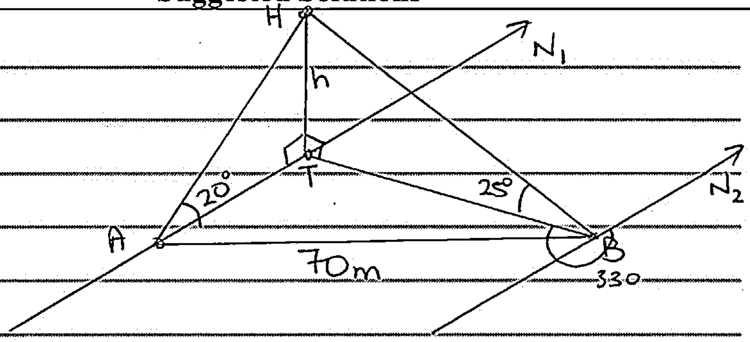
(b) $$V = \frac{4}{3}\pi r^3$$ $$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$		
$$\Rightarrow -2 = 4\pi r^2 \times \frac{dr}{dt}$$ $$\therefore \frac{dr}{dt} = -\frac{1}{8\pi}$$	1	For recognising that the rate of volume change is negative AND showing use of the related rates concept.
But $A = 4\pi r^2$ $$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$ $$= 8\pi r \times \left(-\frac{1}{8\pi}\right)$$ $$= -r$$		
When $r = 2$, $\frac{dA}{dt} = -2$	1	Find the rate of change of the area.
Therefore the balloon is shrinking at 2 cm^2 /s	1	For answering the question i.e. rate at which the balloon shrinks
Note: $$V = \frac{4}{3}\pi r^3 = \frac{1}{3}A\,r$$ $$\frac{dV}{dr} \neq \frac{1}{3}A$$		Maximum 2 marks if negative rate AND/OR concluding statement omitted.

(c) $$\left(x^2-\frac{1}{\sqrt{x}}\right)^n=\sum_{k=0}^{n}\binom{n}{k}(x^2)^k\left(-\frac{1}{\sqrt{x}}\right)^{n-k}$$ $$T_{k+1}=\binom{n}{k}(x^2)^k\left(-\frac{1}{\sqrt{x}}\right)^{n-k}$$ $$=\binom{n}{k}x^{2k}(-1)^{n-k}x^{-\frac{1}{2}(n-k)}$$ For non-zero constant $\frac{5}{2}k-\frac{1}{2}n=0$ Which gives $n=5k$ But $k\in Z$, and hence n is a multiple of 5 Note: If $T_{k+1}=\binom{n}{k}(x^2)^{n-k}\left(-\frac{1}{\sqrt{x}}\right)^k$, then $n=\frac{5k}{4}=5\left(\frac{k}{4}\right)$, which still implies that n is a multiple of 5	 1 1	 For the compact form of the binomial expansion or the general term, **including** the consideration for the negative in the 2nd term of the binomial. For correctly equating the power of the term in x to zero.
(d) (i) Sample space with no restrictions: $\binom{13}{5}=1287$ $P(1M,1W)=1-(P(all\ M)+P(all\ W))$ $=1-\frac{\binom{6}{5}+\binom{7}{5}}{\binom{13}{5}}$ $=\frac{140}{143}$ OR (4M, 1W), (3M, 2W), (2M, 3W), (1M, 4W) $P=\frac{\binom{6}{4}\times\binom{7}{1}+\binom{6}{3}\times\binom{7}{2}+\binom{6}{2}\times\binom{7}{3}+\binom{6}{1}\times\binom{7}{4}}{\binom{13}{5}}$ $=\frac{1260}{1287}$ $=\frac{140}{143}$	 1 1	 Sample space - no restrictions Correct probability

(ii) With no restrictions, 5 people in a circle can Be arranged in 4! Ways.  Men can be together in 2! Ways Women can be arranged in 3! Ways $\therefore P(men\ NOT\ together) = 1 - P(men\ together)$ $= 1 - \frac{2! \times 3!}{4!}$ $= 1 - \frac{1}{2}$ $= \frac{1}{2}$ OR  Fix Man: So 2 spots for Him. Woman in 3! Ways So $P(not\ together) = \frac{3! \times 2}{4!} = \frac{1}{2}$	1 1	For sample space Correct probability
(e) (i) $y = ut\sin\alpha - \frac{1}{2}gt^2$ $\dot{y} = u\sin\alpha - gt$ Bu at maximum height $\dot{y} = 0$ Hence $u\sin\alpha - gt = 0$, from which $t = \frac{u\sin\alpha}{g}$	1	For using y and $\dot{y}$ to get time of flight
(ii) By similar arguments to (i), the time for the second particle to reach maximum height is $t = \frac{w\sin\beta}{g}$ As both particles are fired simultaneously, their flight times to collide are equal. Hence $\frac{u\sin\alpha}{g} = \frac{w\sin\beta}{g}$ From which $u\sin\alpha = w\sin\beta$	1	**NOTE:** reaching the same maximum height does not necessarily mean that their flight times are equal. Equating their vertical heights is another way to get the result i.e. $y_A = y_B$

(iii)	As distances are involved, there is no need to consider direction Now $d = x_A + x_B$ $= ut_A \cos\ \alpha + wt_B \cos\ \beta$ $= u\left(\frac{w \sin\beta}{g}\right)\cos\alpha + w\left(\frac{u\sin\alpha}{g}\right)\cos\beta$ since $t_A = t_B$, $\frac{u\sin\alpha}{g} = \frac{w\sin\beta}{g}$ $\therefore d = \frac{uw}{g}(\sin\beta\cos\alpha + \sin\alpha\cos\beta)$ $= \frac{uw}{g}\sin(\alpha+\beta)$	1 1	This step must be shown to get the 1st mark The reason for the transformation of the distance equation to include sines and cosines **NOTE**: This is a SHOW question and as the algebra is trivial, all working/reasons MUST be included

MATHEMATICS Extension 1 : Question 13

Suggested Solutions	Marks	Marker's Comments
a) 		
$\angle N_2BT = 360° - 330°$ (angles about a point) $= 30°$		
$\angle ATB = 30°$ (alternate angles, parallel lines) In $\triangle ATH$, $\tan 20 = \frac{h}{AT}$, $AT = \frac{h}{\tan 20}$ In $\triangle BTH$, $\tan 25 = \frac{h}{BT}$, $BT = \frac{h}{\tan 25}$	1	
$70^2 = \left(\frac{h}{\tan 20}\right)^2 + \left(\frac{h}{\tan 25}\right)^2 - 2\left(\frac{h}{\tan 20}\right)\left(\frac{h}{\tan 25}\right)\cos 30$	1	
$\therefore h^2 = \dfrac{70^2}{\frac{1}{\tan^2 20} + \frac{1}{\tan^2 25} - \frac{2\cos 30}{\tan 20 \tan 25}}$		
$= 2522.748669\ldots$	1	remember to use degrees!!! on calculator.
$\therefore h = 50.226971\ldots$		
$= 50.2$ m (1dp)	1	
b) i) $\sec x = 2\cos x$ $1 = 2\cos^2 x$ $\cos^2 x = \frac{1}{2}$ $\cos x = \pm\frac{1}{\sqrt{2}}$	1	
$x = \frac{\pi}{4}$ for $0 \le x \le \frac{\pi}{2}$		
OR when $x = \frac{\pi}{4}$, $\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$ and $2\cos\left(\frac{\pi}{4}\right) = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$		
$\therefore \sec\frac{\pi}{4} = 2\cos\frac{\pi}{4}$ $\therefore$ x-coordinate of P is $\frac{\pi}{4}$	1	

MATHEMATICS Extension 1 : Question..13...

Suggested Solutions	Marks	Marker's Comments
ii) $V = \pi \int_0^{\frac{\pi}{4}} (4\cos^2 x - \sec^2 x)\,dx$	1	
$= \pi \int_0^{\frac{\pi}{4}} (2\cos 2x + 2 - \sec^2 x)\,dx$	1	
as $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$		
$= \pi \left[\sin 2x + 2x - \tan x\right]_0^{\frac{\pi}{4}}$	1	
$= \pi\left[\sin\frac{\pi}{2} + \frac{\pi}{2} - \tan\frac{\pi}{4} - 0\right]$		
$= \pi\left(1 + \frac{\pi}{2} - 1\right)$		
$= \frac{\pi^2}{2}\ u^3$		
c) $1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \ldots + n\times 2^{n-1} = 1 + (n-1)\times 2^n$		
Prove true for $n=1$		
$LHS = 1\times 2^0$		
$= 1$		
$RHS = 1 + (1-1)\times 2^1$	1	
$= 1+0$		
$= 1$		
$\therefore LHS = RHS$		
$\therefore$ true for $n=1$		1 for:
Assume true for $n=k$, $k\in\mathbb{Z}^+$		- $k\in\mathbb{Z}^+$
ie. $1\times 2^0 + 2\times 2^1 + \ldots + k\times 2^{k-1} = 1 + (k-1)\times 2^k$		- statements of $n=k$, $n=k+1$
Prove true for $n=k+1$		- conclusion
ie. $1\times 2^0 + 2\times 2^1 + \ldots + k\times 2^{k-1} + (k+1)\times 2^k$		-by assumption
$= 1 + k\times 2^{k+1}$		
$LHS = 1\times 2^0 + 2\times 2^1 + \ldots + k\times 2^{k-1} + (k+1)\times 2^k$		
$= 1 + (k-1)\times 2^k + (k+1)\times 2^k$		
(by assumption)		
$= 1 + 2^k(k-1+k+1)$	1	
$= 1 + 2^k(2k)$		
$= 1 + 2^{k+1}\times k$		
$= RHS$		
$\therefore$ true for $n=k+1$		
$\therefore$ the statement is true by the process of mathematical induction		

MATHEMATICS Extension 1 : Question 13

Suggested Solutions	Marks	Marker's Comments
d) i) $P(\text{one } 6) = {}^4C_1 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^3$		
$= \frac{125}{324}$		
$= 0.38580\ldots$		
$= 0.39$ (2dp)	1	
ii) $P(\text{two } 6 \text{ after two rolls})$		
$= P(\text{1st roll no 6's}) \times P(\text{2nd roll two 6's})$		$1 \Rightarrow$ one case
$+ P(\text{1st roll one 6}) \times P(\text{2nd roll one 6})$		$2 \Rightarrow$ two cases
$+ P(\text{1st roll two 6's}) \times P(\text{2nd roll no 6})$		$3 \Rightarrow$ all cases plus final solution correct.
$= \left(\frac{5}{6}\right)^4 \times {}^4C_2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2$		
$+ {}^4C_1 \times \left(\frac{1}{6}\right) \times \left(\frac{5}{6}\right)^3 \times {}^3C_1 \times \left(\frac{1}{6}\right) \times \left(\frac{5}{6}\right)^2$		
$+ {}^4C_2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 \times \left(\frac{5}{6}\right)^2$		
$= 0.055816329\ldots + \frac{3125}{23328}$		
$+ \frac{625}{7776}$		
$= 0.270151034\ldots$		
$= 0.27$ (2dp)		

Question 14 (15 marks)

(a)

(i) Let $u = \ln x$ $\qquad \frac{du}{dx} = \frac{1}{x} \qquad xdu = dx$

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{xu}xdu = \int \frac{1}{u}du = \ln u + C$$

$$\int \frac{1}{x\ln x}dx = \ln(\ln x) + C$$

- Ignore absolute value on the log
- Must use substitution for the mark
- Students must make sure they do not have multiple variables, ie. u and x in the integral.
- Some students still forgetting the +C

1

(ii) $1 = \int_2^P \frac{1}{x\ln x}dx$

$1 = [\ln(\ln x)]_2^P$

$1 = \ln(\ln P) - \ln(\ln 2)$

$1 = \ln\left(\frac{\ln P}{\ln 2}\right)$

$1 = \ln(\log_2 P)$ (change of base)

$e = \log_2 P$

$P = 2^e$ as required

OR

$\frac{\ln P}{\ln 2} = e$

$\ln P = e\ln 2$

$e^{e\ln} = P$

$(e^{\ln})^e = P$

$2^e = P$

1st mark for applying log laws

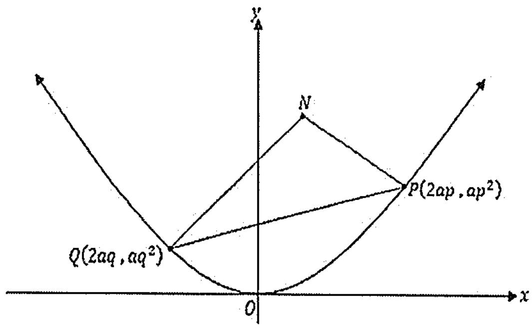
2[nd] mark

3[rd] mark

3

(b)

(i) Gradient of the line $y = x$ is 1



$$gradientPQ = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$gradientPQ = \frac{a(p^2 - q^2)}{2a(p-q)}$$

$$gradientPQ = \frac{a(p-q)(p+q)}{2a(p-q)}$$

$$1 = \frac{p+q}{2}$$

$p + q = 2$ as required

2

(ii) Equation of normals: $y - ap^2 = -\frac{1}{p}(x - 2ap)$; $y - aq^2 = -\frac{1}{q}(x - 2aq)$ [1[st] mark]

Solve simultaneously. $y = ap^2 - \frac{1}{p}(x - 2ap)$; $y = aq^2 - \frac{1}{q}(x - 2aq)$

$ap^2 - \frac{1}{p}(x - 2ap) = aq^2 - \frac{1}{q}(x - 2aq) \qquad / \times pq$

$ap^3q - q(x - 2ap) = aq^3p - p(x - 2aq)$

3

$$ap^3q - aq^3p = q(x-2ap) - p(x-2aq)$$

$$apq\left(p^2 - q^2\right) = x(q-p)$$

$$apq\left(p-q\right)(p+q) = -x(p-q)$$

$-apq(p+q) = x$ use: $p+q=2$

$-2apq = x$ [2nd mark for using $p+q=2$]

substitute in $y = ap^2 - \frac{1}{p}(x-2ap)$ to find y value

$$y = ap^2 - \frac{1}{p}(-2apq - 2ap)$$

$y = ap^2 + 2aq + 2a$ use: $p = 2-q$

$$y = ap(2-q) + 2a(2-p) + 2a$$

$$y = 2ap - apq + 4a - 2ap + 2a$$

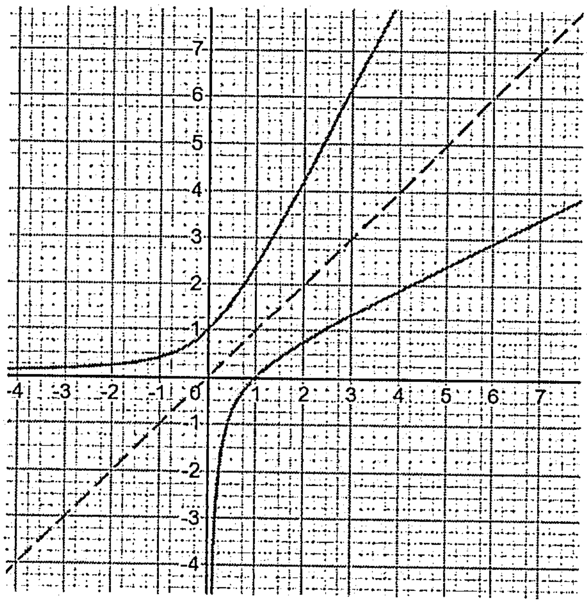
$y = -apq + 6a$ use: $-apq = \frac{x}{2}$

[3rd mark for eliminating p and q and obtaining straight line in form y=mx+b]

$y = \frac{x}{2} + 6a$ as required

(c)

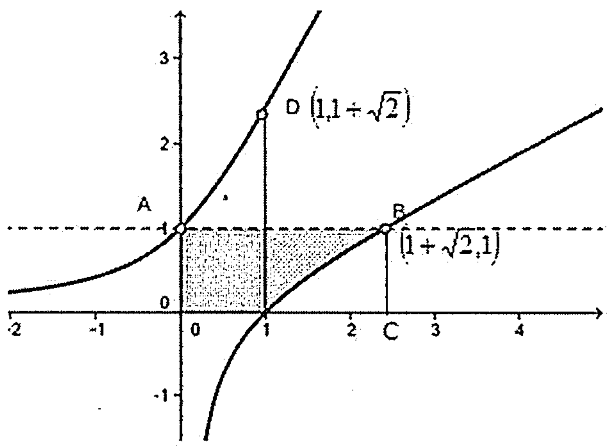
$f(x) = x + \sqrt{x^2+1}$



When asked to sketch the inverse of a function students must ensure that the two graphs are symmetrical. 1

Therefore:

- x and y axes must have the **same scale**
- y = x should also be drawn to show the line of symmetry
- intercepts must be clearly labelled

	(i) $y = x+\sqrt{x^2+1}$ $x = y+\sqrt{y^2+1}$ $x-y = \sqrt{y^2+1}$ $(x-y)^2 = y^2+1$ $x^2-2xy+y^2 = y^2+1$ $x^2-2xy = 1$ $x^2-1 = 2xy$ $\frac{x^2-1}{2x} = y$ $\frac{1}{2}\left(\frac{x^2-1}{x}\right) = y$ $f^{-1}(x) = \frac{1}{2}\left(x-\frac{1}{x}\right)$	2
	(ii)  [1st mark for explaining how to use graph to evaluate integral] By inspection of the two graphs: $\int_0^1\left(x+\sqrt{x^2+1}\right)dx = \text{Area of rectangle ABCO} - \frac{1}{2}\int_1^{1+\sqrt{2}}\left(x-\frac{1}{x}\right)dx$ $A = 1+\sqrt{2} - \frac{1}{2}\left[\frac{x^2}{2}-\ln x\right]_1^{1+\sqrt{2}}$ $A = 1+\sqrt{2} - \frac{1}{2}\left[\frac{(1+\sqrt{2})^2}{2}-\ln(1+\sqrt{2})-\left(\frac{1}{2}\right)\right]$ [2nd mark for substitution into correct expression]	3

$$A = 1+\sqrt{2} - \left[\frac{(1+\sqrt{2})^2}{4} - \frac{\ln(1+\sqrt{2})}{2} - \frac{1}{4}\right]$$

$$A = 1+\sqrt{2} - \left[\frac{3+2\sqrt{2}-1}{4} - \frac{\ln(1+\sqrt{2})}{2}\right]$$

$$A = \frac{4+4\sqrt{2}}{4} - \frac{2+2\sqrt{2}}{4} + \frac{\ln(1+\sqrt{2})}{2}$$

$$A = \frac{4+4\sqrt{2}-2-2\sqrt{2}}{4} + \frac{\ln(1+\sqrt{2})}{2}$$

$$A = \frac{2+2\sqrt{2}}{4} + \frac{\ln(1+\sqrt{2})}{2}$$

$A = \dfrac{1+\sqrt{2}}{2} + \dfrac{\ln(1+\sqrt{2})}{2} = \dfrac{1}{2}\left[1+\sqrt{2}+\ln(1+\sqrt{2})\right]$ as required [3rd mark for final expression]