Killara
HIGH SCHOOL

2010

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Task 5

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question

Total marks – 38

- Attempt questions 1 – 4

Question 1 (9 marks)

a) Suppose that $(3+2x)^{15} = \sum_{k=0}^{15} t_k x^k$

(i) use binomial theorem to write down an expression for t_k. **1**

(ii) Show that $\dfrac{t_{k+1}}{t_k} = \dfrac{30-2k}{3k+3}$. **2**

(iii) Hence find the greatest coefficient in the expansion of $(3+2x)^{15}$. **2**

b) (i) Write down the binomial expansion of $(1+x)^n$ in ascending powers of x.

Hence, show that ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n$. **2**

(ii) Find how many groups of 1 or more digits can be formed from the digits **2**

0, 1, 2, 3, 4, 5, 6, 7, 8, 9 where repetition is not allowed?

End of Question 1

Question 2 (9 marks)

a) A factory machining car parts finds that 98% are machined correctly. From a sample of 40 car parts, calculate to 3 decimal places the probability that

(i) exactly 38 of the parts are correctly machined. **2**

(ii) less than three parts are incorrectly machined. **2**

b) A motorway pay station has five tollgates, three of which are automatically operated and two of which are manually operated. Drivers with the exact money can use any one of the five gates, but drivers requiring change have to use one of the two manually operated gates.

A Ford driver, a Holden driver and a Toyota driver use the motorway every day.

(i) On one day the Ford driver requires change but the other two drivers have the exact money. **2**

Find the number of ways in which the three drivers can go through the pay station so that they all use different gates.

(ii) On another day, all three drivers have the exact money. **3**

Find the probability that the three drivers can go through the pay station so that exactly one goes through a manually operated gate, and each uses a separate gate.

End of Question 2

Question 3 (11 marks)

a) In the expansion of $\left(x^2 + \dfrac{1}{2x}\right)^{14}$ in powers of x, find the coefficient of x^{13} as a rational number. **2**

b) (i) Using the definition of ${}^nC_k$, prove **3**

$$ {}^nC_k + {}^nC_{k-1} = {}^{n+1}C_k . $$

(ii) Explain the meaning of this result with regards to Pascal's triangle. **1**

c) Consider the geometric series

$$ S = 1 + (1+x) + (1+x)^2 + (1+x)^3 + \ldots\ldots + (1+x)^n . $$

(i) Show that $S = \dfrac{(1+x)^{n+1} - 1}{x}$. **1**

(ii) Hence, show that

$$ S = {}^{n+1}C_1 + {}^{n+1}C_2\, x + {}^{n+1}C_3\, x^2 + \ldots\ldots + {}^{n+1}C_{r+1}\, x^r + {}^{n+1}C_{n+1}\, x^n . $$ **1**

(iii) By considering terms of x^r from each expression for *S*, prove that **3**

$$ {}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \ldots\ldots + {}^rC_r = {}^{n+1}C_{r+1} . $$

End of Question 3

Question 4 (9 marks)

a) Using a suitable example, explain why the result **3**

$$^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\ {}^nC_1 + {}^mC_{r-2}\ {}^nC_2 + {}^mC_{r-3}\ {}^nC_3 + \ldots + {}^mC_1\ {}^nC_{r-1} + {}^nC_r,$$

(where $m, n \geq r$) is true.

b) Let $F(x) = 1 + 2\left({}^nC_1\right)x + 3\left({}^nC_2\right)x^2 + \ldots.. + (n+1)\left({}^nC_n\right)x^n$.

(i) By integrating both sides of this equation with respect to x, show that **3**

$$F(x) = \frac{d}{dx}\left(x(1+x)^n\right)$$

(ii) Hence, or otherwise, show that **3**

$$1^2 + 2^2\left({}^nC_1\right)x + 3^2\left({}^nC_2\right)x^2 + \ldots.. + (n+1)^2\left({}^nC_n\right)x^n = F(x) + xF'(x)$$

End of paper

KILLARA HIGH

2010 AUG.

Mathematics Extension 1 Marking Scheme.

a) $(3+2x)^{15} = \sum_{k=0}^{15} t_k x^k$

$= t_0 x^0 + t_1 x^1 + t_2 x^2 + \dots + t_{15} x^{15}$

(i) $t_k = \binom{15}{k} 3^{15-k} 2^k$ — 1 mark.

(ii) $t_{k+1} = \binom{15}{k+1} 3^{15-(k+1)} 2^{k+1}$

$= \binom{15}{k+1} 3^{14-k} 2^{k+1}$

$\frac{t_{k+1}}{t_k} = \frac{\binom{15}{k+1} 3^{14-k} 2^{k+1}}{\binom{15}{k} 3^{15-k} 2^k}$

$= \frac{15!}{(k+1)!(14-k)!} \times \frac{k!(15-k)!}{15!} \cdot \frac{2}{3}$ ✓ — 1 mark for the correct expansion of $\binom{15}{k+1}$ and $\binom{15}{k}$.

$= \frac{15-k}{k+1} \cdot \frac{2}{3}$

$= \frac{30-2k}{3k+3}$ ✓ — 1 mark for correct solution.

(iii) The greatest Coefficient occurs when

$t_{k+1} \leq t_k$.

$\therefore \quad 30 - 2k \leq 3k + 3$.

$27 \leq 5k$.

$k \geqslant \frac{27}{5} = 5\frac{2}{5}$.

$\therefore \quad k = 6$

$\therefore$ Greatest Coefficient is

$$\binom{15}{6} 3^{15-6} 2^6 = \underline{\underline{\binom{15}{6} 3^9 2^6}}.$$

Sets $\frac{t_{k+1}}{t_k} \leq 1$.

1 mark for the Correct value of k.

1 mark Correct Coefficient.

b) (i). $(1+x)^n$

$$= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n$$ 1 mark.

Let $x = 1$.

$$\therefore (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{r} + \dots + \binom{n}{n}.$$ 1 mark.

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{r} + \dots + \binom{n}{n}.$$

(ii). We need groups of one or more from the 10 digits $0, 1, 2, \dots, 9$.

from Q(i),

$$2^n = 1 + \sum_{r=1}^{n} \binom{n}{r}.$$

$$\therefore \sum_{r=1}^{n} \binom{n}{r} = 2^n - 1.$$

1 mark.

$\therefore \sum_{r=1}^{10} \binom{10}{r} = 2^{10} - 1 = \underline{\underline{1023}}.$ 1 mark.

Question 2

a) (i) Let p = probability of correctly machined parts = 0.98.

q = probability of incorrectly machined parts = 0.02.

X = no: of correctly machined parts.

$$P(X=r) = \binom{40}{r}(0.98)^r (0.02)^{40-r}$$

(i) $P(X=38) = \binom{40}{38}(0.98)^{38}(0.02)^2$ 1 mark

1 mark.

$= 0.145$

(ii) $P(X \geq 38)$

$= 0.1448 + \binom{40}{39}(0.98)^{39}\, 0.02 + \binom{40}{40}(0.98)^{40}$ 1 mark.

1 mark.

$= \underline{\underline{0.954}}.$

b) (i) The ford driver can choose 2 gates. ✓ 1 mark

This means, the rest of the two drivers can choose any of the 4 gates which can be done in $^4P_3$ ways.

$\therefore$ Total no: of ways $= 2 \times {^4P_3} = 24$ ✓ 1 mark.

(ii) No: of ways in which you can choose the driver that goes through the manual gate $= {}^3C_1$.

No: of ways in which this particular driver can go through the manual gate $= 2$.

This leaves 3 automatic gates and 2 drivers.

No: of arrangements $= {}^3P_2$.

$\therefore$ Total no: of ways in which this can be done $= {}^3C_1 \times 2 \times {}^3P_2 = 36$.

$\therefore$ Probability $= \dfrac{36}{{}^5P_3} = \dfrac{3}{5}$.

1 mark for ${}^3C_1 \times 2$ which is the no: of ways in which manual gates will be used.

1 mark for multiplying with ${}^3P_2$.

1 mark for probability.

Question 3.

a) $\left(x^2 + \frac{1}{2x}\right)^{14}$

The general term is $\binom{14}{k}(x^2)^{14-k}\left(\frac{1}{2x}\right)^k$.

$\binom{14}{k}\left(\frac{1}{2}\right)^k x^{2(14-k)-k} = \dots x^{13}$.

$\therefore 2(14-k) - k = 13$.

$28 - 2k - k = 13 \qquad \therefore k = 5$ ✓ 1 mark

$\therefore$ Coefficient of $x^{13} = \binom{14}{5}\left(\frac{1}{2}\right)^5$

$= \dfrac{1001}{16}$ ✓ 1 mark.

b) (i) $\binom{n}{k} + \binom{n}{k-1}$

$$= \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!}$$ ✓ 1 mark.

$$= \frac{n!}{(k-1)!\,(n-k)!}\left[\frac{1}{k} + \frac{1}{n-k+1}\right]$$

$$= \frac{n!}{(k-1)!\,(n-k)!}\left[\frac{n-\cancel{k}+1+\cancel{k}}{k(n-k+1)}\right]$$

$$= \frac{n!}{(k-1)!\,(n-k)!} \times \frac{(n+1)}{k(n-k+1)}$$ ✓ 1 mark.

$$= \frac{(n+1)!}{k!\,(n-k+1)!}$$

$$= \binom{n+1}{k}$$ ✓ 1 mark.

(ii) The sum of the $(k-1)^{th}$ and k^{th} numbers in the n^{th} row of Pascal's triangle equals the k^{th} term of $(n+1)^{th}$ row. 1 mark.

c) $S = 1 + (1+x) + (1+x)^2 + \dots + (1+x)^n$.

$a = 1, \quad r = (1+x)$. GP with n terms

(i) $$S = 1\left[\frac{(1+x)^{n+1} - 1}{1+x-1}\right]$$

$$= \frac{(1+x)^{n+1} - 1}{x}.$$

1 mark.

(ii) $$S = \frac{\left[\binom{n+1}{0} + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \dots + \binom{n+1}{n+1}x^{n+1} - 1\right]}{x}.$$

$$= \frac{\cancel{1} + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \dots + \binom{n+1}{n+1}x^{n+1} \cancel{-1}}{x}.$$

$$= \binom{n+1}{1} + \binom{n+1}{2}x + \binom{n+1}{3}x^2 + \dots + \binom{n+1}{n+1}x^n$$

1 mark.

(iii) From (ii), the coefficient of x^r is $\binom{n+1}{r+1}$ ——— (1) ✓ 1 mark.

Consider the series

$$1 + (1+x) + (1+x)^2 + \dots + \underbrace{(1+x)^r + (1+x)^{r+1} + \dots + (1+x)^{n-1}} + \underbrace{(1+x)^n}:$$

∴ The underlined terms contains x^r.

Their coefficients are

$$\binom{r}{r} + \binom{r+1}{r} + \dots + \binom{n-1}{r} + \binom{n}{r}. \text{ — (2)}$$ ✓ 1 mark.

$$\therefore \binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \dots \binom{n-1}{r} + \binom{n}{r}.$$ ✓ 1 mark.

Using (1) and (2).

Question 4.

a) Suppose you have 'm' Maths books and 'n' Science books.

You want to choose 'r' books.

This is represented by $\binom{m+n}{r}$. ✓ | 1 mark.

The r books can be selected as follows:

All 'r' books 'm' maths books can be done in $\binom{m}{r}$ ways.

or $(r-1)$ maths books and 1 Science books can be chosen in $\binom{m}{r-1}\cdot\binom{n}{1}$ ways

or $(r-2)$ maths books and 2 Science books can be chosen in $\binom{m}{r-2}\cdot\binom{n}{2}$ ways.

$\vdots$

or 1 math book and $(r-1)$ Science books can be chosen in $\binom{m}{1}\binom{n}{r-1}$ ways

or all r Science books can be chosen in $\binom{n}{r}$ ways.

explains the procedure 1 mark. (All possible ways this can be done).

These selections are mutually exclusive and

$$\therefore \binom{m+n}{r} = \binom{m}{r} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r-2}\binom{n}{2} + \dots + \binom{m}{1}\binom{n}{r-1} + \binom{n}{r}$$ ✓

1 mark.

(b)

(i) $F(x) = 1 + 2\binom{n}{1}x + 3\binom{n}{2}x^2 + \cdots + (n+1)\binom{n}{n}x^n$.

$$\int F(x)\,dx = x + 2\binom{n}{1}\frac{x^2}{2} + 3\binom{n}{2}\frac{x^3}{3} + \cdots + (n+1)\binom{n}{n}\frac{x^{n+1}}{n+1} + c. \checkmark$$ 1 mark.

$$= x + \binom{n}{1}x^2 + \binom{n}{2}x^3 + \cdots + \binom{n}{n}x^{n+1} + c$$

$$= x\left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right] + c$$

$$= x(1+x)^n + c. \checkmark$$ 1 mark.

$$\therefore F(x) = \frac{d}{dx}\left[x(1+x)^n + c\right]$$

$$= \frac{d}{dx}\left[x(1+x)^n\right] + \frac{d}{dx}(c)$$

$$= \frac{d}{dx}\left[x(1+x)^n\right] + 0 \checkmark$$ 1 mark.

$$= \frac{d}{dx}\left[x(1+x)^n\right]$$

(ii) $$x F(x) = x + 2\binom{n}{1}x^2 + 3\binom{n}{2}x^3 + \cdots + (n+1)\binom{n}{n}x^{n+1} \checkmark$$ 1 mark.

$$\frac{d}{dx}\left(xF(x)\right) = 1 + 2^2\binom{n}{1}x + 3^2\binom{n}{2}x^2 + \cdots + (n+1)^2\binom{n}{n}x^n \quad \text{— (1)} \checkmark$$ 1 mark.

Using chain rule,

$$\frac{d}{dx}\left(xF(x)\right) = xF'(x) + F(x) \quad \text{— (2)}.$$

From (1) & (2)

$$1 + 2^2\binom{n}{1}x + 3^2\binom{n}{2}x^2 + \cdots + (n+1)^2\binom{n}{n}x^n = \underline{\underline{xF'(x) + F(x)}}. \checkmark$$ 1 mark.