Newcastle Grammar School	**Student Number**

2022 HSC TRIAL EXAMINATION

Year 12 **Extension 1 Mathematics**

Examiner: AF

READING TIME – 10 MINUTES

WORKING TIME – 2 HOURS

General Instructions

- Write using black pen.
- Read the instructions carefully – you may be required to answer the questions in the space provided, or use the writing booklets provided, or a combination of the two.
- If you use booklets, start each section in a separate writing booklet.
- Write your student number clearly on each page.
- NESA-approved calculators may be used, unless stated otherwise.
- All diagrams should be drawn in pencil.
- Do not remove this question paper from the examination room.

Section	Guidance	Marks Available	Your Score
SECTION I	• Type of Questions – Multiple Choice • Attempt Questions 1-10 • Allow about 15 minutes for this section	10	
SECTION II	• Type of Questions – Extended Response • Attempt Questions 11-14 • Allow about 1 hr 45 mins minutes for this section	60	
	TOTALS	70	

FINAL MARK	/ 70	%

Your examination paper begins overleaf.

SECTION I

Questions 1-10 (10 marks)

On the **multiple-choice answer sheet provided**, shade in the best answer for each question.

1 Which of the following polynomials has a factor of $(x+1)$?

(A) $P(x)=x^3-5x^2+4$

(B) $P(x)=x^3-5x^2+6$

(C) $P(x)=x^3-x^2$

(D) $P(x)=x^4+x^2$

2 Given $f(x)=\dfrac{1}{\sqrt{x}-3}$, what are the domain and range of $y=f^{-1}(x)$?

(A) $Domain: x\in(-\infty,\frac{-1}{3})\cup(\frac{-1}{3},\infty)$
$Range: y\in[0,9)\cup(9,\infty)$

(B) $Domain: x\in[0,9)\cup(9,\infty)$
$Range: y\in(-\infty,\frac{-1}{3})\cup(\frac{-1}{3},\infty)$

(C) $Domain: x\in(-\infty,\frac{-1}{3}]\cup(0,\infty)$
$Range: y\in[0,9)\cup(9,\infty)$

(D) $Domain: x\in[0,9)\cup(9,\infty)$
$Range: y\in(-\infty,\frac{-1}{3})\cup(0,\infty)$

3 Given that $\tan\theta=\frac{1}{3}$, what is the exact value of $\tan(\theta+\frac{\pi}{3})$?

(A) $\dfrac{1-3\sqrt{3}}{3+\sqrt{3}}$

(B) $\dfrac{1+3\sqrt{3}}{3-\sqrt{3}}$

(C) $\dfrac{\sqrt{3}-3}{3\sqrt{3}+1}$

(D) $\dfrac{\sqrt{3}+3}{3\sqrt{3}-1}$

4 Which of the following integrals is obtained when the substitution $u=(\ln x)^2$

is applied to: $\int_e^{e^2} \frac{(\ln x)^3}{x}\,dx$?

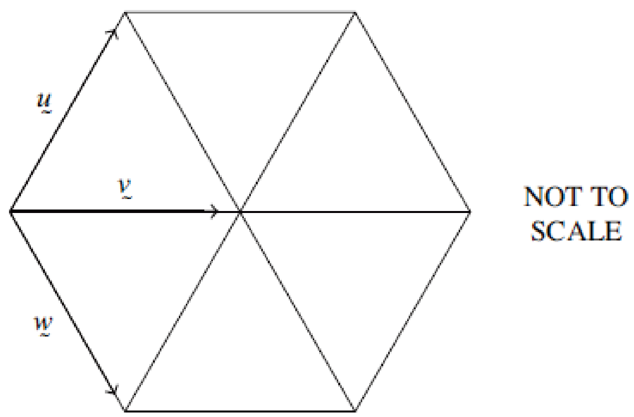
(A) $\frac{1}{2}\int_1^4 u\,du$

(B) $2\int_1^4 u\,du$

(C) $2\int_1^4 u^{\frac{3}{2}}\,du$

(D) $\int_1^4 u^6\,du$

5 Six equilateral triangles form a hexagon with side lengths of 4cm. The vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are shown in the diagram.



Which of the following is the value of $\underset{\sim}{u}\cdot(\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w})$?

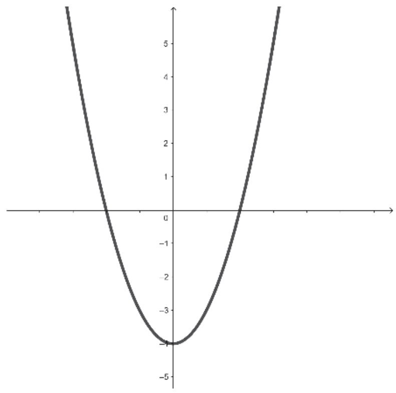
(A) 8

(B) 16

(C) 32

(D) 48

6 The graph of $y = f(x)$ is shown here:



How many real solutions does the equation $[f(x)]^2 = x$ have?

(A) 1

(B) 2

(C) 3

(D) 4

7 The random variable X has a binomial distribution where $E(X) = 3$ and $Var(X) = \dfrac{9}{4}$.

$P(X = 2)$ is equal to:

(A) $\dfrac{3}{16} \times \left(\dfrac{3}{4}\right)^9$

(B) $66 \times \left(\dfrac{3}{4}\right)^{12}$

(C) $\dfrac{1}{64} \times \left(\dfrac{1}{4}\right)^{10}$

(D) $\dfrac{33}{8} \times \left(\dfrac{3}{4}\right)^{10}$

8 What is the value of $\displaystyle\int_0^{\pi} 4\cos^4 x - \cos^2 2x \, dx$?

(A) $\dfrac{\pi}{4}$

(B) $\dfrac{\pi}{2}$

(C) π

(D) 2π

9 Out of a group of twelve friends, nine wear glasses and three do not wear glasses. Five of the friends go out to dinner together. In how many ways will there be more friends who wear glasses than friends who don't wear glasses in the group who go out for dinner?

(A) 792

(B) 756

(C) 220

(D) 126

10 A bag contains ten tickets numbered 1, 2, 3……,10.

Joanna chooses a ticket at random, records the number and **replaces** the ticket in the bag. The process is repeated two more times so that a total of three tickets are drawn.

What is the probability that the number on the first ticket drawn is higher than the second AND the second is higher than the third?

(A) $\frac{3}{25}$

(B) $\frac{1}{6}$

(C) $\frac{18}{25}$

(D) $\frac{4}{5}$

END OF SECTION 1

SECTION II

Question 11 (15 marks) **START A NEW BOOKLET**

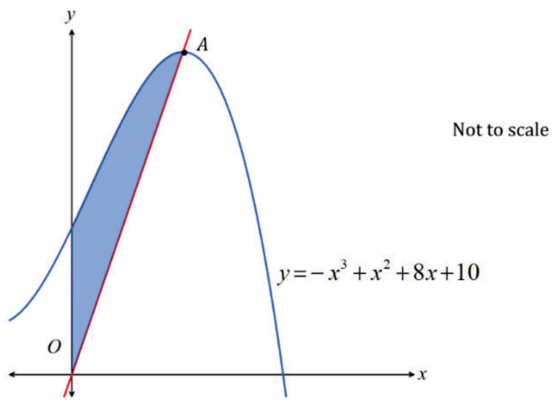
a) Find the exact value of $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ 3

b) At a certain university, students receive one of the following grades for each subject: High Distinction (HD), Distinction (D), Credit (C), Pass (P) or Fail (F). 2

What is the minimum number of students required in a subject so that six students are guaranteed to receive the same grade? (Justify your answer with appropriate calculations and/or reasoning.)

c) Find the angle between the vectors $\underset{\sim}{a}=-2\underset{\sim}{i}-6\underset{\sim}{j}$ and $\underset{\sim}{b}=4\underset{\sim}{i}-2\underset{\sim}{j}$, correct to the nearest degree. 2

d) A curve with the equation $y=-x^3+x^2+8x+10$ has a maximum turning point at A. The shaded region below is bounded by the curve, the y-axis, and the line from O to A, where O is the origin.



i) Show that the x-coordinate of A is 2. 2

ii) Find the exact area of the region. 3

e) Evaluate $\displaystyle\int_{-\frac{\pi}{8}}^{\pi} \sin^2(2x)\,dx$ 3

END OF QUESTION 11

<u>Question 12</u> (15 marks) **START A NEW BOOKLET**

a) Consider the expansion of $(2x-p)^9$. The coefficient of the seventh term is $672\,000$. Find the exact value of p. 2

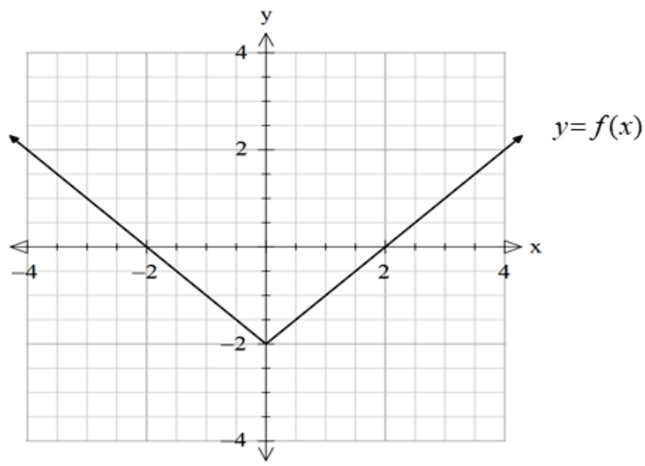
b) The polynomial $P(x)=8x^4-38x^3+9x^2+ax+b$ has a double root at $x=3$. 3
Find the values of a and b where a and b are real numbers.

c) If $\overrightarrow{OA}=\begin{pmatrix}14\\6\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}2\\-2\end{pmatrix}$, find the value of k for which the vector $\begin{pmatrix}k\\-3\end{pmatrix}$ is parallel to $\overrightarrow{AB}$. 2

d) Solve $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)+\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=1$ for $-\pi\le x\le\pi$ 3

e) It is given that $\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{c}+\underset{\sim}{d}$ and $\underset{\sim}{b}+\underset{\sim}{c}$ is perpendicular to $\underset{\sim}{a}+\underset{\sim}{d}$, 3
where $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c}$ and $\underset{\sim}{d}$ are non-zero vectors. Show that $\underset{\sim}{b}-\underset{\sim}{d}$ is perpendicular to $\underset{\sim}{c}-\underset{\sim}{a}$.

f) The sketch of the graph of $y=f(x)$ is shown below.



Consider the equation $|f(x)|=k$ where k is any real constant.

Define function $N(k)=$ the number of real solutions to the equation $|f(x)|=k$.

Sketch the graph of the function $N(k)$ on the axes provided (**separate answer sheet**). 2

END OF QUESTION 12

<u>Question 13</u> (16 marks) **START A NEW BOOKLET**

a) i) Show that $\dfrac{1}{(n+1)!}-\dfrac{n+1}{(n+2)!}=\dfrac{1}{(n+2)!}$ 1

ii) Use mathematical induction to show that, for all integers $n \geq 1$, 3

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\dots\dots\dots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$$

b) Consider the function $f(x)=\log_4 x+\log_5 x \quad (x>0)$

i) Explain why the function has an inverse function. Use calculations and / or reasoning to support your answer. 2

ii) Show that the inverse function is given by $f^{-1}(x)=e^{x\left(\frac{\ln 4\times\ln 5}{\ln 20}\right)}$ 2

iii) Hence, or otherwise, solve $\log_4 x+\log_5 x=2$. Give your answer correct to three decimal places. 1

c) An efficiency management consultant analysis finds that 89% of trains run on time.
On a particular day there are 1000 train trips.
Let X be the binomial random variable representing the number of on-time trains.

i) Find $E(X)$ 1

ii) Show that X has a standard deviation of approximately 10 1

iii) Justify the feasibility of using a normal approximation for this distribution, and use a normal approximation to find the probability that fewer than 90% of the trains run on time. 2

d) Given that $y=\tan^{-1}(2x)$ is a solution to the differential equation:

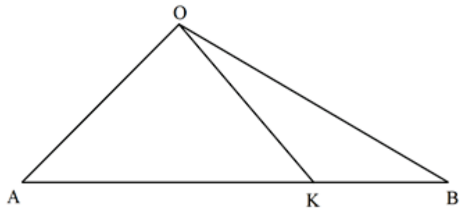
$$(a+bx^2)\frac{d^2y}{dx^2}+c\frac{dy}{dx}=\frac{8(2-\tan(y))}{\sec^2(y)}$$ 3

find the values of a, b and c where a, b and $c \in \mathbb{R}$.

END OF QUESTION 13

<u>Question 14</u> (14 marks) **START A NEW BOOKLET**

a) In triangle OAB, K divides AB in the ratio $\lambda : \mu$ (that is $AK : KB = \lambda : \mu$)



i) Show that $\overrightarrow{AK} = \left[\dfrac{\lambda}{\lambda+\mu}\right]\left[\overrightarrow{OB} - \overrightarrow{OA}\right]$ 1

ii) Hence, prove that $\overrightarrow{OK} = \left[\dfrac{1}{\lambda+\mu}\right]\left[\lambda\overrightarrow{OB} + \mu\overrightarrow{OA}\right]$ 2

iii) Use the result in part ii) to find the position vector of a point that divides the line connecting $A(2,5)$ and $B(12,20)$ in the ratio $2:3$ 2

b) Solve the differential equation $\dfrac{dy}{dx} = x\sqrt{(1-x^2)(1-y^2)}$, given that $y(1) = 0$.

Express y as a function of x. 3

c) Consider the function with the rule $f(x) = \dfrac{x}{\sqrt{2-x}}$

i) Explain why $y = f(x)$ is on/above the x axis between $x = 0$ and $x = 1$. 1

ii) Find the exact area of the region enclosed by the graph $y = f(x)$, the x axis and the line $x = 1$. (You will need to use the substitution $u = 2 - x$.) 2

iii) The region described in part i) is rotated about the x-axis to form a solid of revolution. Find the volume of this solid of revolution. (You will need to use division of polynomials to re-write the integrand.) 3

END OF QUESTION 14

END OF PAPER

NGS NEWCASTLE GRAMMAR SCHOOL **Newcastle Grammar School**	SOLUTIONS **Student Number** ☐☐☐☐☐☐☐☐

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SECTION I

Questions 1-10 (10 marks)

On the **multiple-choice answer sheet provided**, shade in the best answer for each question.

1 Which of the following polynomials has a factor of $(x+1)$?

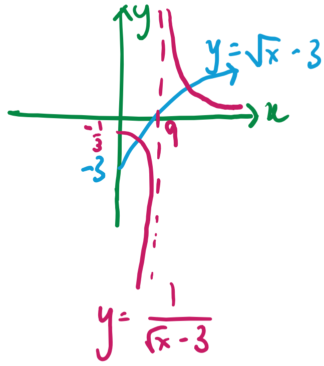
(A) $P(x)=x^3-5x^2+4$ $P(-1) = -1-5+4 \neq 0$

(B) $P(x)=x^3-5x^2+6$ $P(-1) = -1-5+6 = 0$

(C) $P(x)=x^3-x^2$

(D) $P(x)=x^4+x^2$

2 Given $f(x)=\dfrac{1}{\sqrt{x}-3}$, what are the domain and range of $y=f^{-1}(x)$?



$f(x)$: Domain : $x \in [0,9) \cup (9,\infty)$

Range : $y \in (-\infty, -\frac{1}{3}] \cup (0,\infty)$

(A) $Domain: x\in(-\infty,\frac{-1}{3})\cup(\frac{-1}{3},\infty)$

$Range: y\in[0,9)\cup(9,\infty)$

(B) $Domain: x\in[0,9)\cup(9,\infty)$

$Range: y\in(-\infty,\frac{-1}{3})\cup(\frac{-1}{3},\infty)$

(C) $Domain: x\in(-\infty,\frac{-1}{3}]\cup(0,\infty)$

$Range: y\in[0,9)\cup(9,\infty)$

(D) $Domain: x\in[0,9)\cup(9,\infty)$

$Range: y\in(-\infty,\frac{-1}{3})\cup(0,\infty)$

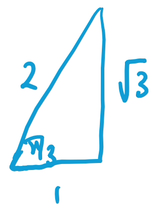
3 Given that $\tan\theta = \frac{1}{3}$, what is the exact value of $\tan(\theta + \frac{\pi}{3})$?

(A) $\frac{1-3\sqrt{3}}{3+\sqrt{3}}$

(B) $\frac{1+3\sqrt{3}}{3-\sqrt{3}}$

(C) $\frac{\sqrt{3}-3}{3\sqrt{3}+1}$

(D) $\frac{\sqrt{3}+3}{3\sqrt{3}-1}$



$$\tan(\theta + \pi/3)$$

$$= \frac{\tan\theta + \tan \pi/3}{1 - \tan\theta \tan \pi/3}$$

$$= \frac{\frac{1}{3} + \sqrt{3}}{1 - \frac{1}{3} \times \sqrt{3}}$$

$$= \frac{\frac{1+3\sqrt{3}}{\cancel{3}}}{\frac{3-\sqrt{3}}{\cancel{3}}} = \frac{1+3\sqrt{3}}{3-\sqrt{3}}$$

4 Which of the following integrals is obtained when the substitution $u = (\ln x)^2$ is applied to: $\int_e^{e^2} \frac{(\ln x)^3}{x} dx$?

(A) $\frac{1}{2}\int_1^4 u \, du$

(B) $2\int_1^4 u \, du$

(C) $2\int_1^4 u^{\frac{3}{2}} \, du$

(D) $\int_1^4 u^6 \, du$

$$du = \frac{2\ln x}{x} dx$$

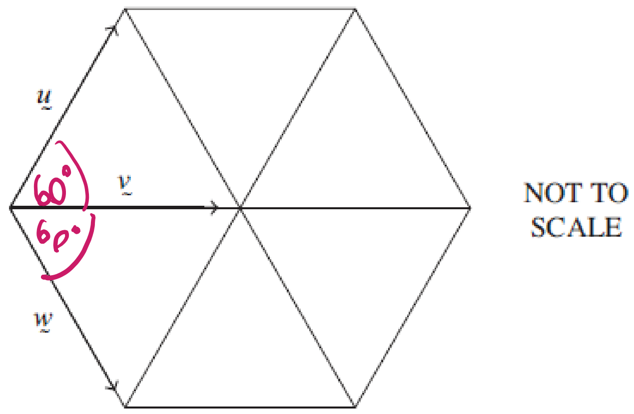
@ $x = e$ $u = (\ln e)^2 = 1$

$x = e^2$ $u = (\ln e^2)^2 = 4$

$$= \frac{1}{2}\int_1^4 (\ln x)^2 \times \frac{2\ln x}{x} dx$$

$$= \frac{1}{2}\int_1^4 u \, du$$

5 Six equilateral triangles form a hexagon with side lengths of 4cm. The vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are shown in the diagram.



Which of the following is the value of $\underset{\sim}{u} \bullet (\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w})$?

(A) 8

(B) 16

(C) 32

(D) 48

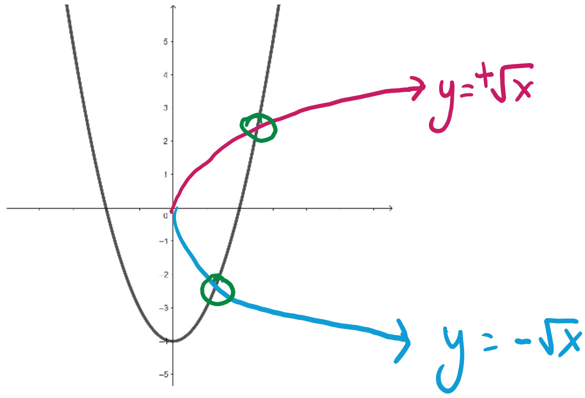
$= \underset{\sim}{u} \cdot \underset{\sim}{u} + \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{w}$

$= |\underset{\sim}{u}|^2 + |\underset{\sim}{u}||\underset{\sim}{v}| \cos 60^\circ + |\underset{\sim}{u}||\underset{\sim}{w}| \cos 120^\circ$

$= 4^2 + 4^2 \cos 60^\circ - 4^2 \cos 60^\circ$

$= 16$

6 The graph of $y = f(x)$ is shown here:



How many real solutions does the equation $[f(x)]^2 = x$ have?

$f(x) = \pm\sqrt{x}$

(A) 1

(B) 2

(C) 3

(D) 4

7 The random variable X has a binomial distribution where $E(X) = 3$ and $Var(X) = \frac{9}{4}$.

$P(X = 2)$ is equal to:

(A) $\frac{3}{16} \times \left(\frac{3}{4}\right)^9$

(B) $66 \times \left(\frac{3}{4}\right)^{12}$

(C) $\frac{1}{64} \times \left(\frac{1}{4}\right)^{10}$

(D) $\frac{33}{8} \times \left(\frac{3}{4}\right)^{10}$

$np = 3$ — (1)

$np(1-p) = \frac{9}{4}$ — (2)

$\frac{(2)}{(1)}: 1 - p = \frac{\frac{9}{4}}{3} = \frac{9}{12} = \frac{3}{4}$

$\therefore p = \frac{1}{4}$ $\therefore n = \frac{3}{\frac{1}{4}} = 12$

$(p+q)^n$

$\left(\frac{1}{4} + \frac{3}{4}\right)^{12}$

$P(X=2)$

$= {}^{12}C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{10}$

$= \underbrace{66 \times \frac{1}{16}}_{\frac{33}{8}} \times \left(\frac{3}{4}\right)^{10}$

8 What is the value of $\int_0^{\pi} 4\cos^4 x - \cos^2 2x \, dx$?

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) π (D) 2π

Consider $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$\cos^4 x = \left[\frac{1}{2}(\cos 2x + 1)\right]^2 = \frac{1}{4}\left[\cos^2 2x + 2\cos 2x + 1\right]$

$\therefore I = \int_0^{\pi} \cancel{4} \times \frac{1}{\cancel{4}}(\cancel{\cos^2 2x} + 2\cos 2x + 1) - \cancel{\cos^2 2x} \, dx$

$= \int_0^{\pi} 2\cos 2x + 1 \, dx = \left[\frac{\cancel{2}\sin 2x}{\cancel{2}} + x\right]_0^{\pi} = (\sin 2\pi + \pi) - (\sin 0 + 0)$

$= \pi$

9 Out of a group of twelve friends, nine wear glasses and three do not wear glasses. Five of the friends go out to dinner together. In how many ways will there be more friends who wear glasses than friends who don't wear glasses in the group who go out for dinner?

(A) 792

(B) 756

(C) 220

(D) 126

Case 1: All five wear glasses

$\boxed{{}^9C_5 \times {}^3C_0}$

Case 2: Four wear glasses

$\boxed{{}^9C_4 \times {}^3C_1}$

Case 3: Three wear glasses

$\boxed{{}^9C_3 \times {}^3C_2}$

10 A bag contains ten tickets numbered 1, 2, 3……,10.

Joanna chooses a ticket at random, records the number and **replaces** the ticket in the bag. The process is repeated two more times so that a total of three tickets are drawn.

What is the probability that the number on the first ticket drawn is higher than the second AND the second is higher than the third?

↳ Need three distinct numbers $(A, B, C : A \neq B \neq C)$

(A) $\frac{3}{25}$

(B) $\frac{1}{6}$

(C) $\frac{18}{25}$

(D) $\frac{4}{5}$

Total selections (no restrictions) $= 10 \times 10 \times 10 = 1000$

Nº selections 3 distinct nºs $= 10 \times 9 \times 8 = 720$

Nº arrangements of these $= 3! = 6$

Nº Arrangements with $A > B > C = 1$

$\therefore$ Probability $= \frac{1}{6} \times \frac{720}{1000} = \frac{12}{100}$

$= \frac{3}{25}$

SECTION II

Question 11

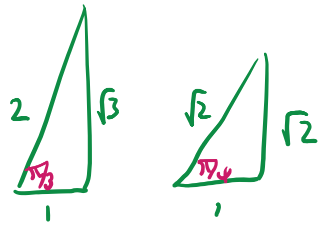
a) Find the exact value of $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ 3

$$= \left[\sin^{-1}\frac{x}{2}\right]_{\sqrt{2}}^{\sqrt{3}} \quad 1$$

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{\sqrt{2}}{2}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{\sqrt{2}} \quad 1$$

$$= \pi/3 - \pi/4 = \boxed{\frac{\pi}{12}} \quad 1$$



b) At a certain university, students receive one of the following grades for each subject: High Distinction (HD), Distinction (D), Credit (C), Pass (P) or Fail (F). 2

What is the minimum number of students required in a subject so that six students are guaranteed to receive the same grade? (Justify your answer with appropriate calculations and/or reasoning.)

There are 5 categories of grades. By the $\boxed{\text{pigeonhole principle,}}$

the minimum number $= \underbrace{5\times5+1}_{\text{process } 1} = \underbrace{\boxed{26 \text{ students}}}_{\text{correct values } 1}$

c) Find the angle between the vectors $\underset{\sim}{a} = -2\underset{\sim}{i} - 6\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$, correct to the nearest degree. 2

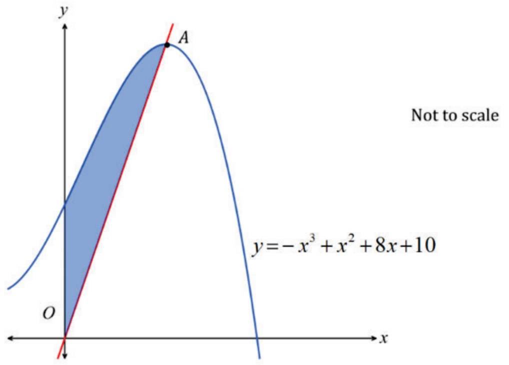
$$\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos\theta$$

$$\cos\theta = \frac{\overbrace{-2\times4 + -6\times-2}^{\frac{1}{2}}}{\underbrace{\sqrt{2^2+6^2}\times\sqrt{4^2+2^2}}_{\frac{1}{2}}} = \frac{4}{\sqrt{40}\times\sqrt{20}} = \frac{1}{\sqrt{50}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{50}}\right) = 81^\circ 52' 11.63''$$

$$\boxed{\theta \approx 82^\circ \quad (\text{nd})} \quad 1$$

d) A curve with the equation $y = -x^3 + x^2 + 8x + 10$ has a maximum turning point at A. The shaded region below is bounded by the curve, the y-axis, and the line from O to A, where O is the origin.



i) Show that the x-coordinate of A is 2. 2

ii) Find the exact area of the region. 3

i) x-coord of A @ $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} < 0$

$\frac{dy}{dx} = -3x^2 + 2x + 8$ $\frac{1}{2}$

for $\frac{dy}{dx} = 0$: $3x^2 - 2x - 8 = 0$

$(3x+4)(x-2) = 0$

$x = -4/3, 2$

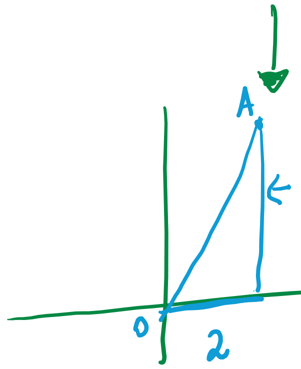
3 4
1 -2

Solve 1

[subst only = $\frac{1}{2}$]

$\underbrace{\text{Clearly } x > 0 \text{ (diagram)}}_{\text{justify choice of sol'n}}$ $\frac{1}{2}$ $\therefore$ $\boxed{x\text{-coord of A is 2 \#}}$

ii) Area $= \int_0^2 -x^3 + x^2 + 8x + 10\,dx$ - Area Δ



$-2^3 + 2^2 + 8(2) + 10$

$= 22$

$$\text{Area} = \int_0^2 \left(-x^3 + x^2 + 8x + 10\right) dx - \underbrace{\left(\tfrac{1}{2} \times 2 \times 22\right)}_{\frac{1}{2}}$$

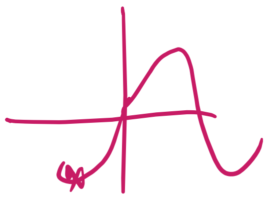
$$= \overbrace{\left[-\frac{x^4}{4} + \frac{x^3}{3} + 4x^2 + 10x \right]_0^2}^{1} - 11$$

$$\frac{1}{2} \left\{ = \left[\left(-\frac{2^4}{4} + \frac{2^3}{3} + \frac{4(2)^2}{2} + 10(2) \right) - \left(0 + 0 + 0 + 0 \right) \right] - 11 \right.$$

$$= \boxed{\frac{38}{3} \, u^2} \quad 1$$

e) Evaluate $\displaystyle\int_{-\frac{\pi}{8}}^{\pi} \sin^2(2x)\, dx$ 3

$$= \frac{1}{2} \int_{-\frac{\pi}{8}}^{\pi} 1 - \cos 4x \; dx \quad 1$$



$$= \frac{1}{2} \left[x - \frac{\sin 4x}{4} \right]_{-\pi/8}^{\pi} \quad \frac{1}{2}$$

$$= \frac{1}{2} \left[\left(\pi - \frac{\sin 4\pi}{4} \right) - \left(-\frac{\pi}{8} - \frac{\sin\left(-\frac{\pi}{2}\right)}{4} \right) \right] \quad \frac{1}{2}$$

$$= \frac{1}{2}\left[\left(\pi - 0\right) - \left(-\frac{\pi}{8} - \left(-\frac{1}{4}\right)\right)\right] \quad \frac{1}{2}$$

$$= \boxed{\frac{9\pi}{16} - \frac{1}{8}} \quad \text{OR} \quad \boxed{\frac{9\pi - 2}{16}} \quad \frac{1}{2}$$

Question 12

a) Consider the expansion of $(2x - p)^9$. The coefficient of the seventh term is $672\,000$. Find the exact value of p. 2

$$T_7 = {}^9C_6 (2x)^3 (-p)^6 \quad 1$$

$$\text{Coeff}_7 = {}^9C_6 (2)^3 (-p)^6 = 672000$$

$$672p^6 = 672000$$

$$\underbrace{p = \sqrt[6]{1000} = (10^3)^{\frac{1}{6}} = \boxed{\sqrt{10}}}_{1}$$

b) The polynomial $P(x)=8x^4-38x^3+9x^2+ax+b$ has a double root at $x=3$. Find the values of a and b where a and b are real numbers. 3

$P(3)=0$ and $P'(3)=0$

$P'(x)=32x^3-114x^2+18x+a$ 1

$P'(3)=0$: $32(3)^3-114(3)^2+18(3)+a=0$

$-108+a=0$

$\boxed{a=108}$ 1

$P(3)=0$: $8(3)^4-38(3)^3+9(3)^2+108(3)+b=0$

$27+b=0$

$\boxed{b=-27}$ 1

e) If $\overrightarrow{OA}=\begin{pmatrix}14\\6\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}2\\-2\end{pmatrix}$, find the value of k for which the vector $\begin{pmatrix}k\\-3\end{pmatrix}$ is parallel to $\overrightarrow{AB}$. 2

$\overrightarrow{AB}=\begin{pmatrix}2-14\\-2-6\end{pmatrix}=\begin{pmatrix}-12\\-8\end{pmatrix}$ $\frac{1}{2}$

Method One:

For $\overrightarrow{AB}\parallel\begin{pmatrix}k\\-3\end{pmatrix}$: $\begin{pmatrix}-12\\-8\end{pmatrix}=\lambda\begin{pmatrix}k\\-3\end{pmatrix}$ $\frac{1}{2}$

$-8=-3\lambda \Rightarrow \lambda=\frac{8}{3}$

$\therefore -12=\frac{8k}{3} \Rightarrow k=\frac{-12\times3}{8}$

1 $\boxed{k=-\frac{9}{2}}$

Method Two:

$\begin{pmatrix}-12\\-8\end{pmatrix}\parallel\begin{pmatrix}k\\-3\end{pmatrix}$

$\frac{k}{-12}=\frac{-3}{-8}$ $\frac{1}{2}$

$k=\frac{-3\times-12}{-8}=\boxed{-\frac{9}{2}}$ 1

d) Solve $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)+\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=1$ for $-\pi \le x \le \pi$ 3

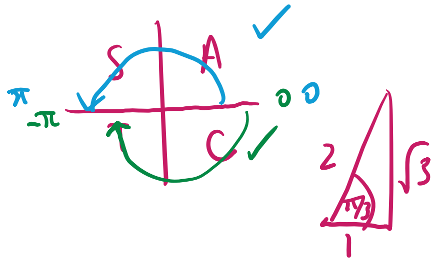
From reference sheet: $\cos A \cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$

$\sqrt{2}\left[\cos\left(x-\frac{\pi}{4}\right)+\cos\left(x+\frac{\pi}{4}\right)\right]=1$

$\sqrt{2}\left[2\cos x \cos\frac{\pi}{4}\right]=1$ 1

$2\cancel{\sqrt{2}} \times \cos x \times \frac{1}{\cancel{\sqrt{2}}} = 1$

$\cos x = \frac{1}{2}$ 1



$x = 0-\pi/3,\ 0+\pi/3$

$\underbrace{\boxed{x = -\pi/3,\ \pi/3}}_{1}$

e) It is given that $\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{c}+\underset{\sim}{d}$ and $\underset{\sim}{b}+\underset{\sim}{c}$ is perpendicular to $\underset{\sim}{a}+\underset{\sim}{d}$, where $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c}$ and $\underset{\sim}{d}$ are non-zero vectors. Show that $\underset{\sim}{b}-\underset{\sim}{d}$ is perpendicular to $\underset{\sim}{c}-\underset{\sim}{a}$. 3

$(\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{c}+\underset{\sim}{d})=0$ and $(\underset{\sim}{b}+\underset{\sim}{c})\cdot(\underset{\sim}{a}+\underset{\sim}{d})=0$

$\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{a}\cdot\underset{\sim}{d}+\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{d}=0$ $\frac{1}{2}$

$\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{d}=-\underset{\sim}{a}\cdot\underset{\sim}{d}-\underset{\sim}{b}\cdot\underset{\sim}{c}$ — (1)

$\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{d}+\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{d}=0$ $\frac{1}{2}$

$\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{d}=-\underset{\sim}{a}\cdot\underset{\sim}{b}-\underset{\sim}{c}\cdot\underset{\sim}{d}$ — (2)

Equating (1) & (2): $-\underset{\sim}{a}\cdot\underset{\sim}{d}-\underset{\sim}{b}\cdot\underset{\sim}{c}=-\underset{\sim}{a}\cdot\underset{\sim}{b}-\underset{\sim}{c}\cdot\underset{\sim}{d}$

1 $\underset{\sim}{a}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{d}+\underset{\sim}{c}\cdot\underset{\sim}{d}-\underset{\sim}{b}\cdot\underset{\sim}{c}=0$

$\underset{\sim}{a}\cdot(\underset{\sim}{b}-\underset{\sim}{d})+\underset{\sim}{c}\cdot(\underset{\sim}{d}-\underset{\sim}{b})=0$

1

$\underset{\sim}{a}\cdot(\underset{\sim}{b}-\underset{\sim}{d})-\underset{\sim}{c}\cdot(\underset{\sim}{b}-\underset{\sim}{d})=0$

$(\underset{\sim}{a}-\underset{\sim}{c})\cdot(\underset{\sim}{b}-\underset{\sim}{d})=0$

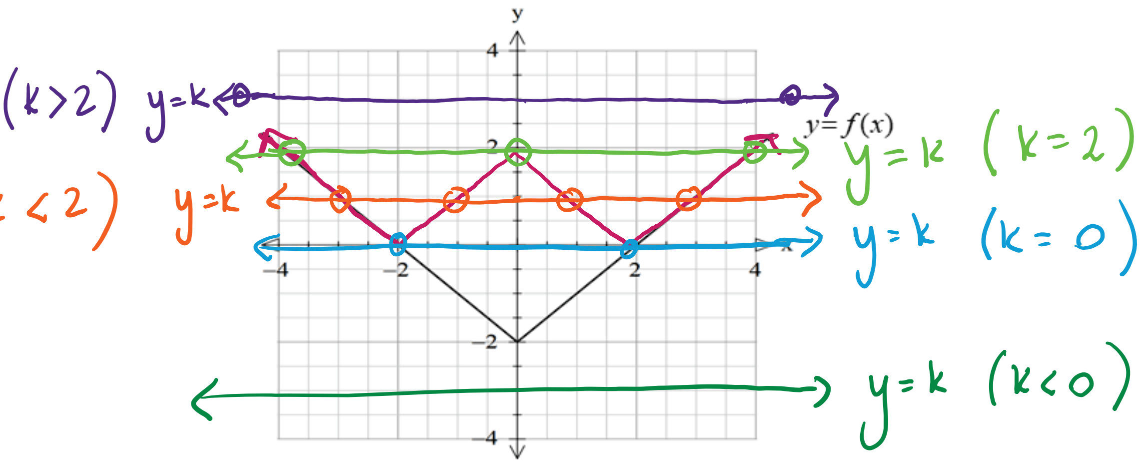
$-(\underset{\sim}{c}-\underset{\sim}{a})\cdot(\underset{\sim}{b}-\underset{\sim}{d})=0$

$(\underset{\sim}{c}-\underset{\sim}{a})\cdot(\underset{\sim}{b}-\underset{\sim}{d})=0$

$\therefore$ $\boxed{\underset{\sim}{b}-\underset{\sim}{d} \text{ is perpendicular to } \underset{\sim}{c}-\underset{\sim}{a}}$ #

(dot product equals zero)

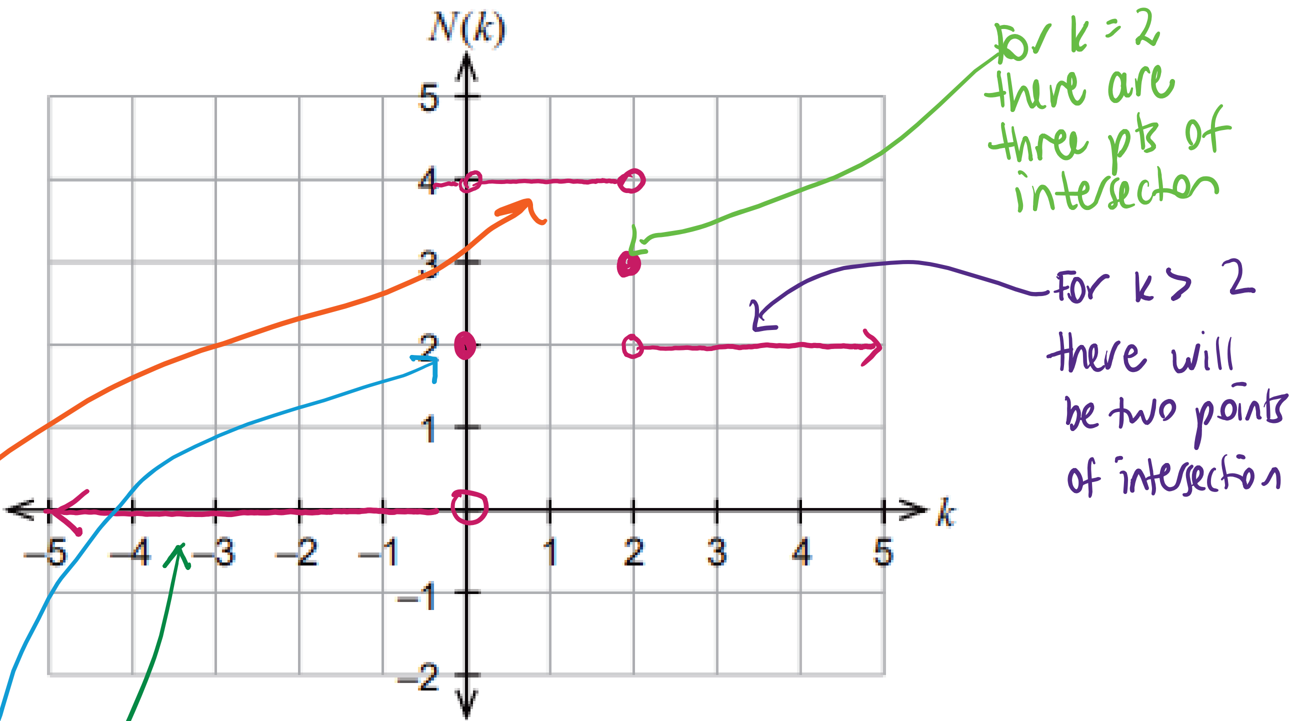
f) The sketch of the graph of $y = f(x)$ is shown below.



Consider the equation $|f(x)| = k$ where k is any real constant.

Define function $N(k) =$ the number of real solutions to the equation $|f(x)| = k$. 2

Sketch the graph of the function $N(k)$ on the axes provided (**separate answer sheet**).



For $k<0$, there are no points of intersection $\Rightarrow N(k)=0$

For $k=0$, there are two points of intersection $\begin{pmatrix} x=-2, \\ x=2 \end{pmatrix}$

For $0<k<2$ there are four points of intersection

Question 13

a) i) Show that $\dfrac{1}{(n+1)!}-\dfrac{n+1}{(n+2)!}=\dfrac{1}{(n+2)!}$ 1

ii) Use mathematical induction to show that, for all integers $n \geq 1$, 3

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\dots\dots\dots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$$

i) LHS =

$$= \frac{1}{(n+1)!} - \frac{n+1}{(n+2)!}$$

$\frac{1}{2}$ $$= \frac{(n+2)}{(n+2)(n+1)!} - \frac{n+1}{(n+2)!}$$

$$= \frac{(n+2)-(n+1)}{(n+2)!}$$

$\frac{1}{2}$ $$= \frac{\cancel{n}+2-\cancel{n}-1}{(n+2)!}$$

$$= \frac{1}{(n+2)!} = \text{RHS}$$

ii) P(n):

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\dots\frac{n}{(n+1)!} = 1-\frac{1}{(n+1)!}$$

Consider P(1)

$$\text{LHS} = \frac{1}{(1+1)!} = \frac{1}{2!} = \frac{1}{2}$$

$$\text{RHS} = 1-\frac{1}{(1+1)!} = 1-\frac{1}{2} = \frac{1}{2}$$ 1

$\therefore$ P(1) is true

Assume P(k) true:

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\dots+\frac{k}{(k+1)!} = 1-\frac{1}{(k+1)!}$$

Prove P(k+1) true:

$$\text{RTP: } \frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\dots+\frac{k}{(k+1)!}+\frac{(k+1)}{(k+2)!} = 1-\frac{1}{(k+1+1)!}$$

$$\text{LHS} = \left[1-\frac{1}{(k+1)!}\right]+\frac{k+1}{(k+2)!}$$ 1 from assumption

$$= 1-\left[\frac{1}{(k+1)!}-\frac{k+1}{(k+2)!}\right]$$ $\frac{1}{2}$

$$= 1-\frac{1}{(k+2)!}$$ $\frac{1}{2}$ from part (i)

$= \text{RHS}$ $\therefore$ P(k+1) is true

Conclusion

Hence P(n) is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

b) Consider the function $f(x)=\log_4 x+\log_5 x \quad (x>0)$

i) Explain why the function has an inverse function. 2

ii) Show that the inverse function is given by $f^{-1}(x)=e^{x\left(\frac{\ln 4\times\ln 5}{\ln 20}\right)}$ 2

iii) Hence, or otherwise, solve $\log_4 x+\log_5 x=2$. Give your answer correct to three decimal places. 1

i) $f(x) = \frac{\ln x}{\ln 4} + \frac{\ln x}{\ln 5}$

method one

$f'(x) = \frac{1}{x\ln 4} + \frac{1}{x\ln 5}$ } 1

$= \left(\frac{1}{\ln 4} + \frac{1}{\ln 5}\right)\frac{1}{x}$

$= \frac{k}{x} \quad (k>0)$

$\therefore f'(x) > 0 \quad \forall x > 0$ } 1

ie. $f(x)$ is a monotonic increasing (ie one-to-one) function $\Rightarrow f^{-1}(x)$ is a function

method two:

$f(x) = \frac{\ln x}{\ln 4} + \frac{\ln x}{\ln 5}$

$= \frac{\ln 5 \ln x + \ln 4 \ln x}{\ln 4 \ln 5}$

$= \frac{\ln 5 + \ln 4}{\ln 4 \ln 5}\ln x$ } 1

$= k \times \ln x \quad (k>0)$

1

Since $f(x) = k\ln x$ is a monotonic increasing function - - - -

ii) For $\underline{y = f^{-1}(x)}$: $x = \log_4 y + \log_5 y$

$$x = \frac{\ln y}{\ln 4} + \frac{\ln y}{\ln 5}$$ $\frac{1}{2}$

$$x = \frac{\ln 5 \ln y + \ln 4 \ln y}{\ln 4 \ln 5}$$ $\frac{1}{2}$

$$x = \frac{\ln y \left(\ln 5 + \ln 4\right)}{\ln 4 \ln 5}$$

$$x = \ln y \times \frac{\ln 20}{\ln 4 \ln 5}$$

$$\therefore \ln y = x \times \frac{\ln 4 \ln 5}{\ln 20}$$ $\frac{1}{2}$

$$\therefore \boxed{f^{-1}(x) = e^{x\left(\frac{\ln 4 \ln 5}{\ln 20}\right)}}$$ $\frac{1}{2}$ #

iii) $f(x) = 2 \iff f^{-1}(2)$

$$f^{-1}(2) = e^{2\left(\frac{\ln 4 \times \ln 5}{\ln 20}\right)}$$

$$= 4.435123557....$$

$$\boxed{\doteqdot 4.435} \text{ (3dp)}$$ 1

c) An efficiency management consultant analysis finds that 89% of trains run on time. On a particular day there are 1000 train trips.

Let X be the binomial random variable representing the number of on-time trains.

i) Find $E(X)$ 1

ii) Show that X has a standard deviation of approximately 10 1

iii) Justify the feasibility of using a normal approximation for this distribution, and use a normal approximation to find the probability that fewer than 90% of the trains run on time. 2

i) $X \sim Bin(1000, 0.89)$

$E(x) = 1000 \times 0.89$

$= \boxed{890}$ 1

ii) $\sigma = \sqrt{1000 \times 0.89 \times 0.11}$

$= 9.89444 \ldots \approx \boxed{10}$ (0dp) # 1

iii) $np = 1000 \times 0.89 = 890$

$nq = 1000 \times 0.11 = 110$

$\boxed{\text{Both} > 10}$ 1

$\therefore$ Okay to use a normal approx.

$X \sim N(890, 10^2)$

90% of trains
$= 0.9 \times 1000 = 900$

$P(x < 900)$
$= P(z < 1)$
$= 0.5 + \frac{0.68}{2} = \boxed{0.84}$ 1

d) Given that $y = \tan^{-1}(2x)$ is a solution to the differential equation:

$$(a+bx^2)\frac{d^2y}{dx^2} + c\frac{dy}{dx} = \frac{8(2-\tan(y))}{\sec^2(y)} \qquad 3$$

find the values of a, b and c where a, b and $c \in \mathbb{R}$.

$$y = \tan^{-1}(2x) \Rightarrow \underbrace{\tan y = 2x}_{1a} \Rightarrow \underbrace{\overbrace{\tan^2 y + 1}^{\sec^2 y} = 4x^2 + 1}_{1b}$$

$\frac{1}{2}$ both $\tan y$, $\sec^2 y$

$$\frac{dy}{dx} = \frac{1}{1^2 + (2x)^2} \times 2 = \frac{2}{1+4x^2} \quad -2$$

$\frac{1}{2}$

$$\frac{d^2y}{dx^2} = 2 \times -(1+4x^2)^{-2} \times 8x = \frac{-16x}{(1+4x^2)^2} \quad -3$$

$\frac{1}{2}$

Sub 1a, 1b, 2, 3 in D.E :

$$(a+bx^2) \times \frac{-16x}{(1+4x^2)^2} + c \times \frac{2}{1+4x^2} = \frac{8(2-2x)}{1+4x^2}$$

$\frac{1}{2}$ all subst.

$$-16x(a+bx^2) + 2c(1+4x^2) = 16(1-x)(1+4x^2)$$

$$-16ax - 16bx^3 + 2c + 8cx^2 = (16-16x)(1+4x^2)$$

$$-16bx^3 + 8cx^2 - 16ax + 2c = -64x^3 + 64x^2 - 16x + 16$$

Equating coefficients:

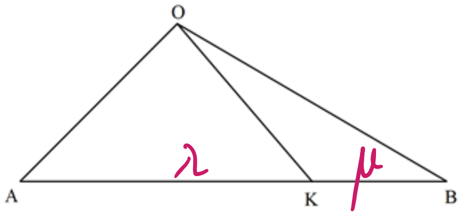
$-16b = -64$	$8c = 64$	$-16a = -16$
$b = 4$	$c = 8$	$a = 1$

equate coeff 1

$$\boxed{\begin{aligned} a &= 1 \\ b &= 4 \\ c &= 8 \end{aligned}}$$

Question 14 (15 marks) **START A NEW BOOKLET**

a) In triangle OAB, K divides AB in the ratio $\lambda:\mu$ (that is $AK:KB = \lambda:\mu$)



i) Show that $\overrightarrow{AK} = \left[\dfrac{\lambda}{\lambda+\mu}\right]\left[\overrightarrow{OB} - \overrightarrow{OA}\right]$ 1

ii) Hence, prove that $\overrightarrow{OK} = \left[\dfrac{1}{\lambda+\mu}\right]\left[\lambda\overrightarrow{OB} + \mu\overrightarrow{OA}\right]$ 2

iii) Use the result in part ii) to find the position vector of a point that divides the line connecting $A(2,5)$ and $B(12,20)$ in the ratio $2:3$ 2

i) $\dfrac{|\overrightarrow{AK}|}{|\overrightarrow{AB}|} = \dfrac{AK}{AB} = \dfrac{\lambda}{\lambda+\mu}$ (given)

$\overrightarrow{AK} = \dfrac{\lambda}{\lambda+\mu}\overrightarrow{AB}$

$\therefore \boxed{\overrightarrow{AK} = \dfrac{\lambda}{\lambda+\mu}\left[\overrightarrow{OB} - \overrightarrow{OA}\right]}$ 1

ii) $\overrightarrow{OK} = \overrightarrow{OA} + \overrightarrow{AK}$

$= \overrightarrow{OA} + \dfrac{\lambda}{\lambda+\mu}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$ $\frac{1}{2}$ from i)

$= \dfrac{(\lambda+\mu)\overrightarrow{OA} + \lambda\left(\overrightarrow{OB} - \overrightarrow{OA}\right)}{\lambda+\mu}$ $\frac{1}{2}$

$$= \frac{\vec{OA}(\cancel{\lambda} + \mu - \cancel{\lambda}) + \lambda \vec{OB}}{\lambda + \mu}$$

$$= \boxed{\frac{1}{\lambda+\mu}\left[\lambda \vec{OB} + \mu \vec{OA}\right]} \quad 1$$

\#

iii) $\vec{OA} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $\quad \vec{OB} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$ $\quad \lambda : \mu = 2 : 3$

$$\vec{OK} = \frac{1}{2+3}\left[2\begin{pmatrix} 12 \\ 20 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right] \quad 1$$

$$= \frac{1}{5}\begin{pmatrix} 24 + 6 \\ 40 + 15 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 30 \\ 55 \end{pmatrix} \quad \frac{1}{2}$$

$$\boxed{\vec{OK} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}} \quad \frac{1}{2}$$

b) Solve the differential equation $\dfrac{dy}{dx}=x\sqrt{(1-x^2)(1-y^2)}$, given that $y(1)=0$. 3

Express y as a function of x.

$$\int \frac{dy}{\sqrt{1-y^2}} = -\frac{1}{2}\int -2x\sqrt{1-x^2}\,dx$$ $\frac{1}{2}$ (separate variables)

$\frac{1}{2}$

$$\sin^{-1} y = \underbrace{-\frac{1}{2} \times \frac{2(1-x^2)^{3/2}}{3}}_{1} + C$$

$$\sin^{-1} y = -\frac{1}{3}\sqrt{(1-x^2)^3} + C$$

$y(1)=0$: $\sin^{-1} 0 = -\frac{1}{3}\sqrt{(1-1^2)^3} + C$ $\frac{1}{2}$ correct subst.

$$0 = -\frac{1}{3} \times 0 + C \quad \Rightarrow \quad C = 0$$

$$\sin^{-1} y = -\frac{1}{3}\sqrt{(1-x^2)^3}$$

$$\boxed{y = \sin\left(-\frac{1}{3}\sqrt{(1-x^2)^3}\right)}$$ $\frac{1}{2}$

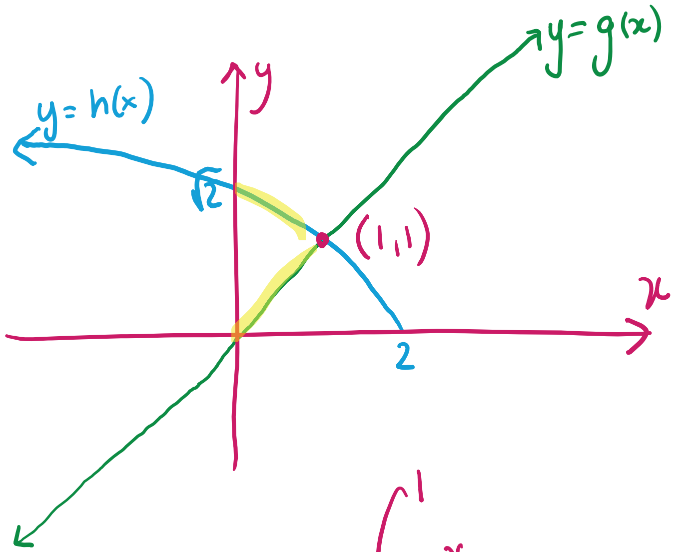
c) Consider the function with the rule $f(x)=\dfrac{x}{\sqrt{2-x}}$

i) Explain why $y=f(x)$ is on/above the x axis between $x=0$ and $x=1$. 1

ii) Find the exact area of the region enclosed by the graph $y=f(x)$, the x axis and the line $x=1$. (You will need to use the substitution $u=2-x$.) 2

iii) The region described in part i) is rotated about the x-axis to form a solid of revolution. Find the volume of this solid of revolution. (You will need to use division of polynomials to re-write the integrand.) 3

i)



$f(x)=\dfrac{x}{\sqrt{2-x}}=\dfrac{g(x)}{h(x)}$

From graph,

$g(x)\geq 0$ for $x\in[0,1]$

$h(x)>0$ for $x\in[0,1]$

Since $\dfrac{\text{pos/zero}}{\text{pos}}\geq 0$ 1

$\therefore f(x)\geq 0$ for $x\in[0,1]$

ii) Area $=\displaystyle\int_0^1 \frac{x}{\sqrt{2-x}}\,dx$

$u=2-x \Rightarrow x=2-u$

$du=-\,dx$

$\frac{1}{2}$ $\begin{cases} x=0 & u=2 \\ x=1 & u=1 \end{cases}$

$=\displaystyle\int_2^1 \underbrace{\frac{2-u}{\sqrt{u}}\times -du}_{\frac{1}{2}}$

$=\displaystyle\int_1^2 \frac{2}{\sqrt{u}}-\frac{u}{\sqrt{u}}\,du$

$=\displaystyle\int_1^2 2u^{-\frac{1}{2}}-u^{\frac{1}{2}}\,du$

$$= \left[2 \times 2u^{1/2} - \frac{2}{3}u^{3/2} \right]_1^2$$ (1/2) integration

$$= \left[4\sqrt{u} - \frac{2u\sqrt{u}}{3} \right]_1^2$$

$$= \left(4\sqrt{2} - \frac{2 \times 2\sqrt{2}}{3}\right) - \left(4\sqrt{1} - \frac{2 \times 1\sqrt{1}}{3}\right)$$

$$= 4\sqrt{2} - \frac{4}{3}\sqrt{2} - 4 + \frac{2}{3}$$

$$= \boxed{\frac{8\sqrt{2} - 10}{3}} \text{ units}^2$$ (1/2) substitution & simplification

ii) $$V = \pi \int_0^1 \left(\frac{x}{\sqrt{2-x}} \right)^2 dx$$

$$= \pi \int_0^1 \frac{x^2}{2-x}\, dx$$ (1/2)

poly. div. (1)

$$\begin{array}{r|l} & -x - 2 \\ \hline -x+2 & \cancel{x^2} + 0x + 0 \\ & \cancel{x^2} - 2x \\ \hline & 2x + 0 \\ & 2x - 4 \\ \hline & 4 \end{array}$$

$$= \pi \int_0^1 \frac{(2-x)(-x-2) + 4}{2-x} \, dx$$

$$\left. = \pi \int_0^1 -x - 2 + \frac{4}{2-x} \, dx \right\} \quad \frac{1}{2}$$

$$= -\pi \int_0^1 x + 2 - \frac{4}{2-x} \, dx$$

$$= -\pi \left[\frac{x^2}{2} + 2x + 4 \ln|2-x| \right]_0^1 \quad \frac{1}{2}$$

$$= -\pi \left[\left(\frac{1}{2} + 2 + \cancel{4 \ln 1} \right) - \left(0 + 0 + 4 \ln 2 \right) \right]$$

$$= -\pi \left(\frac{5}{2} - 4 \ln 2 \right)$$

$$= \boxed{\pi \left(4 \ln 2 - \frac{5}{2} \right)} \text{ units}^3 \quad \frac{1}{2}$$