NGS NEWCASTLE GRAMMAR SCHOOL **Newcastle Grammar School**	**Student Number** ☐☐☐☐☐☐☐☐

2023 HSC TRIAL EXAMINATION

Year 12 Extension 1 Mathematics

Examiner: AF

READING TIME – 10 MINUTES

WORKING TIME – 2 HOURS

General Instructions

- Write using black pen.
- For Section I, use the multiple choice answer sheet provided.
- For Section II, start each section in a separate writing booklet.
- Write your student number clearly on each booklet.
- A NESA Reference Sheet will be provided.
- NESA-approved calculators may be used,
- All diagrams should be drawn in pencil.
- Do not remove this question paper from the examination room.

Section	Guidance	Marks Available	Your Score
SECTION I	• Type of Questions – Multiple Choice • Attempt Questions 1-10 • Allow about 15 minutes for this section	10	
SECTION II	• Type of Questions – Short Answer and Extended Response • Attempt Questions 11-14 • Allow about 1 hour 45 minutes for this section	60	
	TOTALS	70	

FINAL MARK	/70	%

Your examination paper begins overleaf.

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<u>SECTION I</u>

<u>Questions 1-10</u> (10 marks)

On the **multiple choice answer sheet provided**, shade in the best answer for each question.

1 What is the remainder when $-2x^3+8x-10$ is divided by $x+1$?

(A) -16

(B) -20

(C) -4

(D) 0

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2 Which of the following is the correct expression for $\int \frac{4}{\sqrt{9-x^2}}\,dx$?

(A) $\frac{4}{3}\sin^{-1}\left(\frac{x}{3}\right)+C$

(B) $\frac{4}{3}\sin^{-1}(3x)+C$

(C) $4\sin^{-1}\left(\frac{x}{3}\right)+C$

(D) $4\sin^{-1}(3x)+C$

3 Which of the following integrals is obtained when the substitution $u=(\ln x)^2$ is applied to $\displaystyle\int_e^{e^2} \frac{dx}{x(\ln x)^3}$?

(A) $\dfrac{1}{2}\displaystyle\int_1^4 u\,du$

(B) $2\displaystyle\int_1^4 u\,du$

(C) $\dfrac{1}{2}\displaystyle\int_1^4 u^{-2}\,du$

(D) $2\displaystyle\int_1^4 u^2\,du$

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4 In how many ways can six cards be selected from a standard deck of 52 cards without replacement so that exactly two are spades and four are clubs?

(A) 55770

(B) 230230

(C) 2676960

(D) 20358520

5 A, B and C are collinear points with position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. B lies between A and C. Given that $\left|\overrightarrow{BC}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|$, which of the following expressions is equivalent to $\underset{\sim}{c}$?

(A) $\frac{1}{2}\underset{\sim}{a} - \frac{3}{2}\underset{\sim}{b}$

(B) $\frac{3}{2}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$

(C) $\frac{3}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

(D) $\frac{3}{2}\underset{\sim}{b} - \frac{3}{2}\underset{\sim}{a}$

---

6 Let X be a discrete random variable with a binomial distribution. The mean of X is 2.4 and the variance of X is 1.92. The values of the number of independent trials, n, and probability of success, p, are:

(A) $n=3;\ p=0.2$

(B) $n=12;\ p=0.2$

(C) $n=3;\ p=0.8$

(D) $n=12;\ p=0.8$

---

7 Consider the function $f(x)=2x^3-3x^2-36x+10$. Over which of the following domains could an inverse function exist for f?

(A) $x\in[-2,\infty)$

(B) $x\in(-\infty,3]$

(C) $x\in[-3,2]$

(D) $x\in[-2,3]$

8 A particle is moving in a straight line such that its acceleration (in ms^{-2}) after t seconds is given by $\ddot{x} = \sin^2 t$. Initially, the particle is at rest at the origin. Movement to the right is considered positive.

Which of the following best describes the particle when $t = \dfrac{2\pi}{3}$ seconds?

(A) The particle is at the right of the origin, moving to the right, with increasing speed.

(B) The particle is at the right of the origin, moving to the left, with decreasing speed.

(C) The particle is at the left of the origin, moving to the right, with increasing speed.

(D) The particle is at the left of the origin, moving to the left, with decreasing speed.

---

9 A group of n people, where n is even, is chosen at random from people at a shopping centre, for a short survey. Each participant is asked to record how many other people (from the group) have the same colour hair as they do. Which of the following statements must be true?

(A) At least $\dfrac{n}{2}$ people in the group record the same number.

(B) At least $n-2$ people in the group record the same number.

(C) At least 2 people in the group record the same number.

(D) There may be no people in the group record the same number.

---

10 Consider two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ which are neither parallel nor perpendicular.
Let the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ be $\underset{\sim}{m}$, and the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ be $\underset{\sim}{n}$.
Which of the following is equivalent to $|\underset{\sim}{m}|:|\underset{\sim}{n}|$?

(A) $|\underset{\sim}{a} \cdot \underset{\sim}{b}| : |\underset{\sim}{a}||\underset{\sim}{b}|$

(B) $|\underset{\sim}{a}| : |\underset{\sim}{b}|$

(C) $|\underset{\sim}{b}| : |\underset{\sim}{a}|$

(D) $|\underset{\sim}{a}|^2 : |\underset{\sim}{b}|^2$

END OF SECTION 1

SECTION II

Question 11 (15 marks) **START A NEW BOOKLET**

a) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{12}{3+4x^2}\,dx$ **3**

b) Solve $\dfrac{x}{2x-1} \le \dfrac{1}{4}$ **3**

c) To join the "April Club", you must be born in the month of April. (There are 30 days in April.) How many members must the club have, in order to guarantee that five members share the same birthday? **2**

d) i) Write $2\sin x + 2\sqrt{3}\cos x$ in the form $A\sin(x+\alpha)$ where α is acute. **2**

ii) Hence, or otherwise, solve $2\sin x + 2\sqrt{3}\cos x = 2\sqrt{3}$ for all $x \in [0, 2\pi]$. **2**

e) A container with a height of 40cm is filled with water. The container starts leaking water at a rate of $\dfrac{5\sqrt{h}}{2h+45}$ $cm^3 \min^{-1}$ where h cm is the depth of water. The volume, V cm^3, can be given by $V = \pi(5h^2 + 225h)$. By considering t to be time in minutes, write an expression for the rate of change of the depth of the water. **3**

END OF QUESTION 11

<u>Question 12</u> (15 marks) **START A NEW BOOKLET**

a) The functions f and g are defined by $f(x)=\sqrt{2x-1}$ and $g(x)=x+2$.

i) Find $h(x)$, the inverse of the composite function $f(g(x))$. **2**

ii) Sketch the graph of $y=h(x)$. **2**

b) Show that $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$. **2**

c) Prove, by Mathematical Induction, that $5^{2n+1}+2^{2n+1}$ is divisible by 7 for any integer $n\geq 0$. **3**

d) Let $P(x)=2x^4-15x^3+2x^2+ax+b$ where a and b are real numbers. Find the values of a and b given that $(x-5)^2$ is a factor of $P(x)$. **3**

e) Consider the function $f(x)=\sin^{-1}(\cos x)$ for $x\in(0,\pi)$.

i) Prove that $f(x)$ is linear. **2**

i) Express $f(x)$ in the form $ax+b$ where $a,b\in\mathbb{R}$. **1**

END OF QUESTION 12

Question 13 (15 marks) **START A NEW BOOKLET**

a) The coefficient of x^4 in the expansion of $\left(1+2x^2\right)^n$ is 220. Find the value of n. **2**

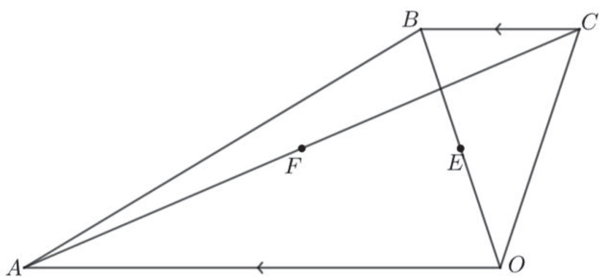
b) A box contains 3 blue marbles, 5 green marbles and k yellow marbles. A marble is chosen at random and then placed back into the box. This is repeated four times.

i) What is the probability, P, that the first marble chosen is blue? **1**

ii) Let X be the number of blue marbles chosen. What is the smallest value of k for which $Var(X)<0.8$? **3**

c) Consider $y=\sin^{-1}(x+1)+p\sin\left(\dfrac{\pi}{12}\right)$, where $p\in\mathbb{R}$ and $\sin\left(\dfrac{\pi}{12}\right)=\dfrac{\sqrt{6}-\sqrt{2}}{4}$.
Find the minimum value of p for which $y\geq 0$, for all x in the domain. **2**

d) The points O, A, B, C form a trapezium with $\overrightarrow{OA}=3\overrightarrow{CB}$. Let $\underset{\sim}{a}=\overrightarrow{OA}$ and $\underset{\sim}{c}=\overrightarrow{OC}$. Points E and F are the midpoints of OB and CA respectively.



i) Show that $\overrightarrow{EF}=\dfrac{1}{3}\underset{\sim}{a}$. **3**

ii) Justify the statement, "Hence, $CEFB$ is a parallelogram." **1**

e) Use the substitution $u^2=x+1$ to find the volume of solid formed by rotating the area bounded by the curve $y=\dfrac{x-1}{\sqrt{x+1}}$, the x axis, and the lines $x=3$ and $x=8$ about the x axis. Give your answer in exact form. **3**

END OF QUESTION 13

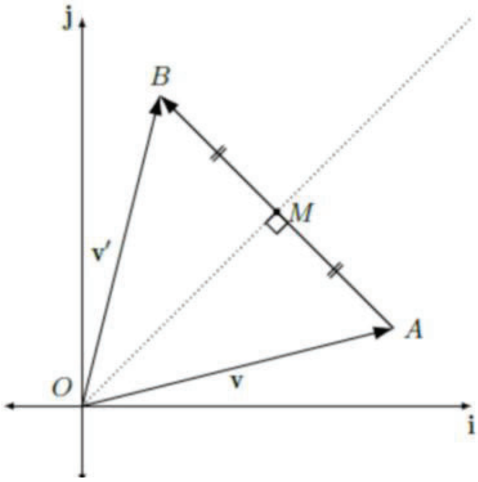
<u>Question 14</u> (15 marks) **START A NEW BOOKLET**

a) The roots of the quadratic equation $8x^2 - 5x + k = 0$ are $\sin\theta$ and $\cos 2\theta$ (for some angle θ). Find the possible values of k. **3**

b) The diagram below shows the vector $\underset{\sim}{v} = \begin{pmatrix} a \\ b \end{pmatrix}$.

The vector $\underset{\sim}{v}'$ is the reflection of $\underset{\sim}{v}$ about the dotted line $y = x$. The point M bisects interval AB and lies on the line $y = x$.

Using vector projection, show that $\underset{\sim}{v}' = \begin{pmatrix} b \\ a \end{pmatrix}$. **3**



c) A survey shows that 36% of Sydney-siders go to the beach at some point during summer. Samples of 100 people are taken to determine the proportion of people who go to the beach at some point during summer. Find the mean and the standard deviation of the distribution of these sample proportions. Estimate the probability that a sample of 100 people will contain at least 36 and at most 42 people who visited the beach this summer. **4**

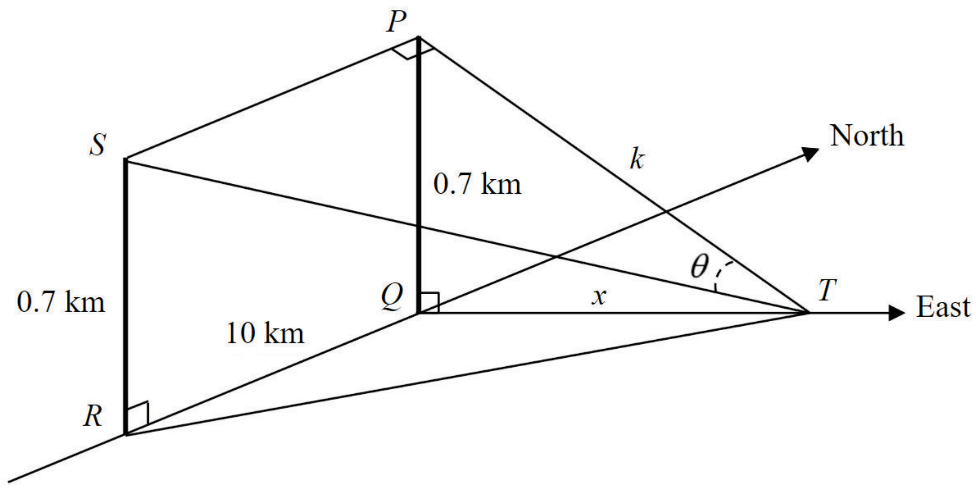
(Use the table shown below of $P(Z < z)$ where Z has a standard normal distribution.)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

QUESTION 14 CONTINUES OVERLEAF

Question 14 continued

d) PQ and SR are two towers of height $0.7\,\text{km}$. Q is $10\,\text{km}$ due North of R. A solar powered vehicle at T is travelling due East away from Q at a constant speed of $10\,\text{km/h}$. Let the distance QT be x and the distance PT be k. Let $\angle PTS = \theta$.



i) Show that $\dfrac{dk}{dt} = \dfrac{10x}{\sqrt{x^2 + 0.49}}$. **2**

ii) By first finding an expression for k in terms of θ, show that $\dfrac{dk}{d\theta} = -10\,\text{cosec}^2\theta$. **1**

iii) Find the exact rate at which θ is changing when the vehicle is $2.4\,\text{km}$ from Q. **2**

END OF PAPER

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SECTION I

Questions 1-10 (10 marks)

On the **multiple choice answer sheet provided**, shade in the best answer for each question.

1 What is the remainder when $-2x^3+8x-10$ is divided by $x+1$?

(A) -16

(B) -20

(C) -4

(D) 0

$-2(-1)^3+8(-1)-10$

$=-16$

2 Which of the following is the correct expression for $\int \frac{4}{\sqrt{9-x^2}}\,dx$?

(A) $\frac{4}{3}\sin^{-1}\left(\frac{x}{3}\right)+C$

(B) $\frac{4}{3}\sin^{-1}(3x)+C$

(C) $4\sin^{-1}\left(\frac{x}{3}\right)+C$

(D) $4\sin^{-1}(3x)+C$

↳ see reference sheet

3 Which of the following integrals is obtained when the substitution $u = (\ln x)^2$ is applied to

$$\int_e^{e^2} \frac{dx}{x(\ln x)^3}?$$

(A) $\frac{1}{2}\int_1^4 u \, du$

(B) $2\int_1^4 u \, du$

(C) $\frac{1}{2}\int_1^4 u^{-2} \, du$

(D) $2\int_1^4 u^2 \, du$

$$\int_e^{e^2} \frac{dx}{x(\ln x)^3}$$

$$\frac{1}{2}\int_e^{e^2} \frac{1}{(\ln x)^4} \times \frac{2\ln x}{x} dx$$

$$= \frac{1}{2}\int_1^4 \frac{du}{u^2}$$

$u = (\ln x)^2$

$du = \frac{2(\ln x)}{x} dx$

$x = e \rightarrow u = 1$

$x = e^2 \rightarrow u = 4$

4 In how many ways can six cards be selected from a standard deck of 52 cards without replacement so that exactly two are spades and four are clubs?

(A) 55770

(B) 230230

(C) 2676960

(D) 20358520

$^{13}C_2 \times {}^{13}C_4$

$= 78 \times 715$

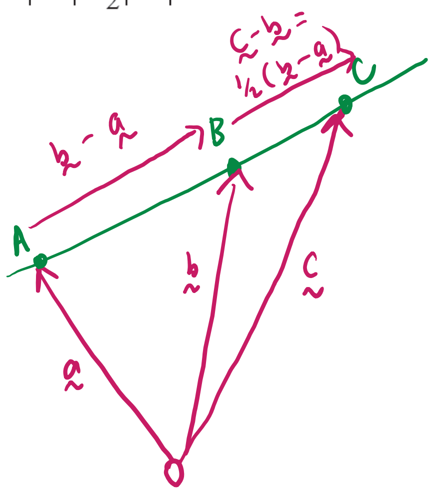
5 A, B and C are collinear points with position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. B lies between A and C. Given that $\left|\overrightarrow{BC}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|$, which of the following expressions is equivalent to $\underset{\sim}{c}$?

(A) $\frac{1}{2}\underset{\sim}{a} - \frac{3}{2}\underset{\sim}{b}$

(B) $\frac{3}{2}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$

(C) $\frac{3}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

(D) $\frac{3}{2}\underset{\sim}{b} - \frac{3}{2}\underset{\sim}{a}$



$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

$\underset{\sim}{c} = \underset{\sim}{a} + (\underset{\sim}{b} - \underset{\sim}{a}) + \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$

$\underset{\sim}{c} = -\frac{1}{2}\underset{\sim}{a} + \frac{3}{2}\underset{\sim}{b}$

6 Let X be a discrete random variable with a binomial distribution. The mean of X is 2.4 and the variance of X is 1.92. The values of the number of independent trials, n, and probability of success, p, are:

(A) $n=3;\ p=0.2$

(B) $n=12;\ p=0.2$

(C) $n=3;\ p=0.8$

(D) $n=12;\ p=0.8$

$np = 2.4$ — (1)

$npq = 1.92$ — (2)

$\frac{(2)}{(1)}$ $q = 0.8$

$\therefore p = 0.2$

$\therefore n = \frac{2.4}{0.2} = 12$

7 Consider the function $f(x)=2x^3-3x^2-36x+10$. Over which of the following domains could an inverse function exist for f?

(A) $x\in[-2,\infty)$

(B) $x\in(-\infty,3]$

(C) $x\in[-3,2]$

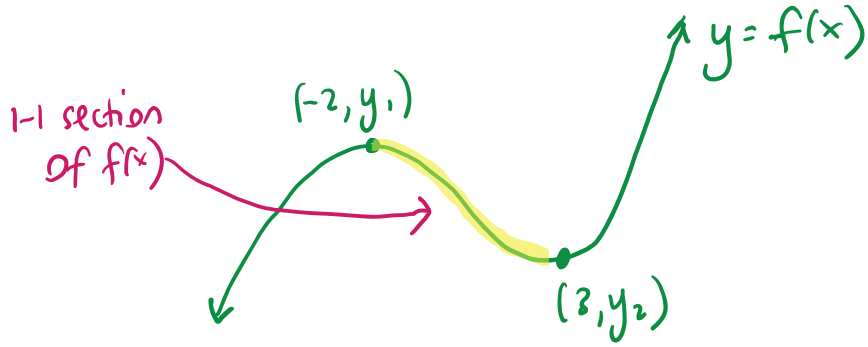
(D) $x\in[-2,3]$

$f'(x)=6x^2-6x-36$

$=6(x^2-x-6)$

$f'(x)=0:\quad (x-3)(x+2)=0$

$x=-2,3$



8 A particle is moving in a straight line such that its acceleration (in ms^{-2}) after t seconds is given by $\ddot{x}=\sin^2 t$. Initially, the particle is at rest at the origin. Movement to the right is considered positive.

Which of the following best describes the particle when $t=\dfrac{2\pi}{3}$ seconds?

(A) The particle is at the right of the origin, moving to the right, with increasing speed.

(B) The particle is at the right of the origin, moving to the left, with decreasing speed.

(C) The particle is at the left of the origin, moving to the right, with increasing speed.

(D) The particle is at the left of the origin, moving to the left, with decreasing speed.

$\boxed{\ddot{x}=\sin^2 t}$ $\qquad$ $\boxed{x(0)=0 \quad \dot{x}(0)=0}$

$\dot{x}=\int \sin^2 t\,dt$

$=\frac{1}{2}\int 1-\cos 2t\,dt$

$=\frac{1}{2}\left[t-\frac{\sin 2t}{2}\right]+C_1$

$\underline{\dot{x}(0)=0}:\ 0=\frac{1}{2}\left(0-\frac{\sin 0}{2}\right)+C_1$

$\Rightarrow C_1=0$

$\dot{x}=\frac{1}{2}\left(t-\frac{\sin 2t}{2}\right)$

$x=\frac{1}{2}\int t-\frac{\sin 2t}{2}\,dt$

$=\frac{1}{2}\left[\frac{t^2}{2}+\frac{\cos 2t}{4}\right]+C_2$

$\underline{x(0)=0};\ 0=\frac{1}{2}\left[0+\frac{\cos 0}{4}\right]+C_2 \Rightarrow C_2=-\frac{1}{8}$

$x=\frac{t^2}{4}+\frac{\cos 2t}{8}-\frac{1}{8}$

@ $t=2\pi/3$

$x=0.909\ldots$

$\dot{x}=1.263\ldots$

$\ddot{x}=0.75$

9 A group of n people, where n is even, is chosen at random from people at a shopping centre, for a short survey. Each participant is asked to record how many other people (from the group) have the same colour hair as they do. Which of the following statements must be true?

(A) At least $\frac{n}{2}$ people in the group record the same number.

(B) At least $n-2$ people in the group record the same number.

(C) At least 2 people in the group record the same number.

(D) There may be no people in the group record the same number.

N°. people = $\{1, 2, 3, \ldots n\}$ $\rightarrow$ n pigeons

Case 1: Everyone has at least one other person with same hair colour

N°s to pick: $\{1, 2, \ldots n-1\}$

$\left\lceil \frac{n}{n-1} \right\rceil = 2$

Case 2: One person has unique hair colour

That person $\{0\}$

Everyone else $(n-1)$ $\{1, 2, \ldots n-2\}$

$\Downarrow$

$\left\lceil \frac{n-1}{n-2} \right\rceil = 2$

Case 3: Two people have unique hair colour

These 2 people $\{0\}$

Everyone else $(n-2)$ $\{1, 2, 3, \ldots n-3\}$

$\left\lceil \frac{n-2}{n-3} \right\rceil = 2$

etc.

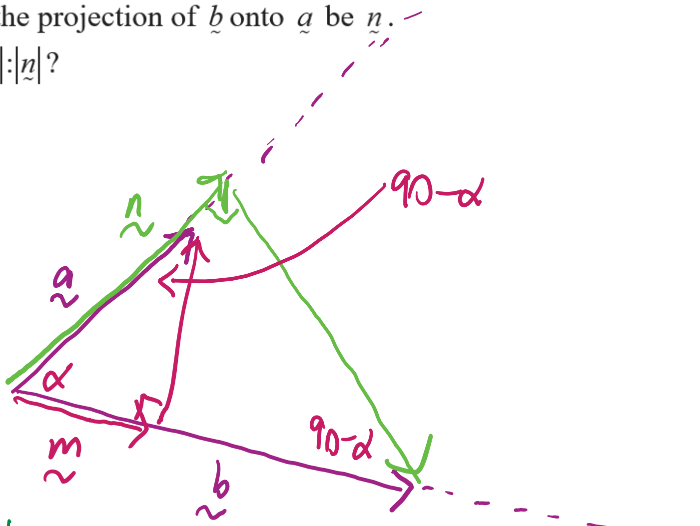
10 Consider two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ which are neither parallel nor perpendicular. Let the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ be $\underset{\sim}{m}$, and the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ be $\underset{\sim}{n}$. Which of the following is equivalent to $|\underset{\sim}{m}|:|\underset{\sim}{n}|$?

(A) $|\underset{\sim}{a}\cdot\underset{\sim}{b}|:|\underset{\sim}{a}||\underset{\sim}{b}|$

(B) $|\underset{\sim}{a}|:|\underset{\sim}{b}|$

(C) $|\underset{\sim}{b}|:|\underset{\sim}{a}|$

(D) $|\underset{\sim}{a}|^2:|\underset{\sim}{b}|^2$



$\underset{\sim}{m} = \text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = (\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$

$\therefore |\underset{\sim}{m}| = \underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}$

$\underset{\sim}{n} = \text{proj}_{\underset{\sim}{a}}\underset{\sim}{b} = (\underset{\sim}{b}\cdot\hat{\underset{\sim}{a}})\hat{\underset{\sim}{a}}$

$\therefore |\underset{\sim}{n}| = \underset{\sim}{b}\cdot\hat{\underset{\sim}{a}}$

Consider $\dfrac{|\underset{\sim}{m}|}{|\underset{\sim}{n}|} = \dfrac{\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}}{\underset{\sim}{b}\cdot\hat{\underset{\sim}{a}}}$

$= \dfrac{\underset{\sim}{a}\cdot\dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|}}{\underset{\sim}{b}\cdot\dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}}$

$= \dfrac{\frac{1}{|\underset{\sim}{b}|}\cancel{(\underset{\sim}{a}\cdot\underset{\sim}{b})}}{\frac{1}{|\underset{\sim}{a}|}\cancel{(\underset{\sim}{b}\cdot\underset{\sim}{a})}}$

$= \dfrac{\frac{1}{|\underset{\sim}{b}|}}{\frac{1}{|\underset{\sim}{a}|}} = \dfrac{|\underset{\sim}{a}|}{|\underset{\sim}{b}|}$

<u>Note</u>: These are similar Δ's (equiangular)

$\therefore \dfrac{|\underset{\sim}{m}|}{|\underset{\sim}{a}|} = \dfrac{|\underset{\sim}{n}|}{|\underset{\sim}{b}|}$ (corresponding sides are in proportion.)

$\therefore \dfrac{|\underset{\sim}{m}|}{|\underset{\sim}{n}|} = \dfrac{|\underset{\sim}{a}|}{|\underset{\sim}{b}|}$

SECTION II

Question 11 (15 marks) **START A NEW BOOKLET**

a) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{12}{3+4x^2}\,dx$ **3**

$$= 6\int_0^{\frac{1}{2}} \frac{2}{\sqrt{3}^2 + (2x)^2}\,dx$$

$$= 6 \times \frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{2x}{\sqrt{3}}\right]_0^{\frac{1}{2}}$$

$$= 2\sqrt{3}\left[\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right]$$

$$= 2\sqrt{3}\left(\frac{\pi}{6} - 0\right) = \boxed{\frac{\sqrt{3}\pi}{3}}$$

1 ~ correct integration

1 ~ correct substitution

1 ~ correct simplification

b) Solve $\dfrac{x}{2x-1} \le \dfrac{1}{4}$ **3**

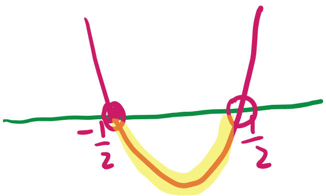
$\boxed{x \neq \frac{1}{2}}$

$$(2x-1)^{\cancel{2}} \times \frac{x}{\cancel{2x-1}} \le \frac{1}{4} \times (2x-1)^2$$

$$4 \times x(2x-1) \le \frac{1}{\cancel{4}}(2x-1)^2 \times \cancel{4}$$

$$(2x-1)\left[4x - (2x-1)\right] \le 0$$

$$(2x-1)(2x+1) \le 0$$



$$\boxed{x \in \left[-\frac{1}{2}, \frac{1}{2}\right)}$$

1 ~ multiply by square of denom. or finds critical points

1 ~ clearly shows application of inequality (graph or subst values)

1 ~ correctly applies restriction on domain

c) To join the "April Club", you must be born in the month of April. (There are 30 days in April.) How many members must the club have, in order to guarantee that five members share the same birthday? **2**

minimum number $= (30 \times 4) + 1$

$= 121$ members

(by the pigeonhole principle)

1 ~ utilises PHP (30 days of 4 members each)

1 ~ correct answer

d) i) Write $2\sin x + 2\sqrt{3}\cos x$ in the form $A\sin(x+\alpha)$ where α is acute. **2**

ii) Hence, or otherwise, solve $2\sin x + 2\sqrt{3}\cos x = 2\sqrt{3}$ for all $x \in [0, 2\pi]$. **2**

i) $2\sin x + 2\sqrt{3}\cos x \equiv A\sin(x+\alpha)$

$= A(\sin x\cos\alpha + \cos x\sin\alpha)$

$= A\cos\alpha\sin x + A\sin\alpha\cos x$

Equating coefficients: $A\cos\alpha = 2$ — ① $A\sin\alpha = 2\sqrt{3}$ — ②

② ÷ ①: $\dfrac{A\sin\alpha}{A\cos\alpha} = \dfrac{2\sqrt{3}}{2}$

1 ~ correct α

$\tan\alpha = \sqrt{3} \Rightarrow \alpha = \pi/3$

1 - correct A

①^2 + ②^2: $A^2(\underbrace{\cos^2\alpha + \sin^2\alpha}_{=1}) = 2^2 + (2\sqrt{3})^2$

$A = \sqrt{4+12} = 4$

$\therefore A\sin(x+\alpha) = \boxed{4\sin(x + \pi/3)}$

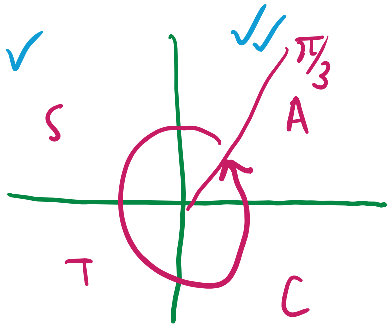
(see next page)

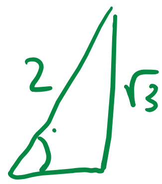
ii) $4\sin\left(x+\frac{\pi}{3}\right) = 2\sqrt{3}$

$x \in [0, 2\pi]$

$x+\frac{\pi}{3} \in \left[\frac{\pi}{3}, 2\pi+\frac{\pi}{3}\right]$

$$\sin\left(x+\frac{\pi}{3}\right) = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$





RA $= \frac{\pi}{3}$

$$x+\frac{\pi}{3} = 0+\frac{\pi}{3},\ \pi-\frac{\pi}{3},\ 2\pi+\frac{\pi}{3}$$

$$x+\frac{\pi}{3} = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{7\pi}{3}$$

$$\boxed{x = 0,\ \frac{\pi}{3},\ 2\pi}$$

1 ~ correct use of quadrants and domain

1 ~ correct use of RA. to identify & simplify solutions.

e) A container with a height of 40cm is filled with water. The container starts leaking water at a rate of $\frac{5\sqrt{h}}{2h+45}$ $cm^3 \min^{-1}$ where h cm is the depth of water. The volume, V cm^3, can be given by $V = \pi(5h^2 + 225h)$. By considering t to be time in minutes, write an expression for the rate of change of the depth of the water. **3**

$$\frac{dV}{dt} = \frac{-5\sqrt{h}}{2h+45}$$

$$V = \pi(5h^2 + 225h)$$

$$\frac{dV}{dh} = \pi(10h + 225)$$

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$= \frac{-5\sqrt{h}}{2h+45} \times \frac{1}{\pi(10h+225)}$$

$$= \frac{-\cancel{5}\sqrt{h}}{2h+45} \times \frac{1}{\cancel{5}\pi(2h+45)}$$

$$= \boxed{\frac{-\sqrt{h}}{\pi(2h+45)^2}}$$

1 - correct use of chain rule

1 ✓ use of correct $\frac{dV}{dt}$, $\frac{dh}{dV}$ (incl. neg)

1 ✓ full simplification of expression

END OF QUESTION 11

Question 12 (15 marks) **START A NEW BOOKLET**

a) The functions f and g are defined by $f(x)=\sqrt{2x-1}$ and $g(x)=x+2$.

i) Find $h(x)$, the inverse of the composite function $f(g(x))$. **2**

ii) Sketch the graph of $y=h(x)$. **2**

i) $f(g(x)) = \sqrt{2(x+2)-1}$

$= \sqrt{2x+3}$ $\Rightarrow$ Note: $x \geq -\frac{3}{2}$, $y \geq 0$

[By inspection, $h(x) = \dfrac{x^2-3}{2}$; $x \geq 0$]

OR For $f(g(x))$, let $y = \sqrt{2x+3}$

For $h(x)$, let $x = \sqrt{2y+3}$

$x^2 = 2y+3$

$x^2 - 3 = 2y$

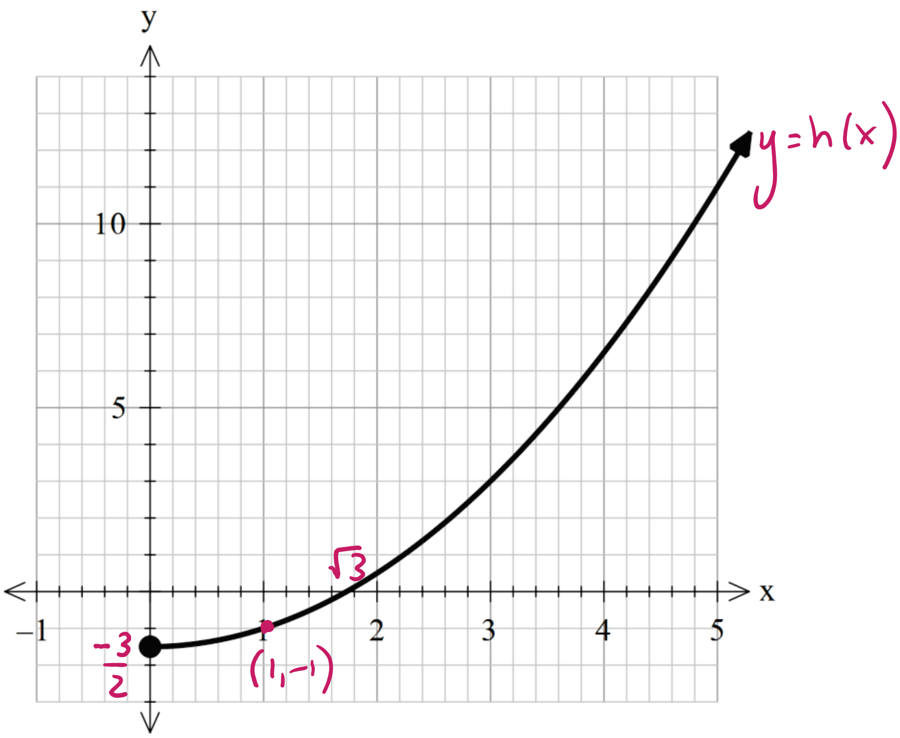
$\dfrac{x^2-3}{2} = y$

$\therefore \boxed{h(x) = \dfrac{x^2-3}{2} \; ; x \geq 0}$

1 ~ finds $f(g(x))$

1 ~ finds inverse function, $h(x)$

[$-\frac{1}{2}$ is domain not restricted] ii)



xint: $\frac{x^2-3}{2}=0$

$x^2=3$

$x=\sqrt{3}$

$(x\geq 0)$

yint: $y=\frac{0-3}{2}$

$y=\frac{-3}{2}$

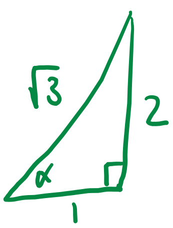
b) Show that $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$. **2**

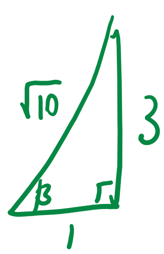
RTP: $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$

LHS $=\frac{\pi}{4}+\tan^{-1}2+\tan^{-1}3$

$\therefore$ RTP: $\tan^{-1}2+\tan^{-1}3=\frac{3\pi}{4}$

Let $\alpha=\tan^{-1}2$ $\quad\beta=\tan^{-1}3$





To find $\alpha+\beta$, consider $\tan(\alpha+\beta)$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{2+3}{1-2\times 3} = -1$$

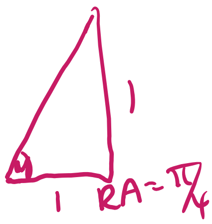
Now $0 < \alpha < \pi/2$, $0 < \beta < \pi/2$ (range of $y = \tan^{-1}x$)

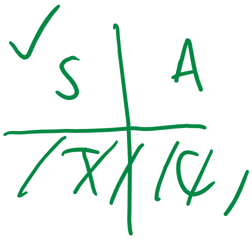
$\therefore\ 0 < \alpha+\beta < \pi$

For $\tan(\alpha+\beta) = -1$

$\alpha+\beta = \pi - \pi/4$

$= 3\pi/4$





$\therefore\ \tan^{-1}2 + \tan^{-1}3 = 3\pi/4$

$\therefore\ \boxed{\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi}$ #

$\frac{1}{2}$ ~ use $\tan\pi/4 = 1$

$\frac{1}{2}$ ~ correct use of $\tan(\alpha+\beta)$ to simplify LHS

1 ~ working & reasoning to support $\alpha+\beta = 3\pi/4$ (must include consideration of range)

c) Prove, by Mathematical Induction, that $5^{2n+1}+2^{2n+1}$ is divisible by 7 for any integer $n \geq 0$. **3**

$P(k)$: $5^{2n+1} + 2^{2n+1} = 7M \quad (m \in \mathbb{Z}) \qquad n \in \mathbb{Z} : n \geq 0$

Test $P(0)$: $5^{2(0)+1} + 2^{2(0)+1}$

$= 5 + 2 = 7 \times 1 \qquad \therefore P(0)$ is true

Assume $P(k)$ true:

$5^{2k+1} + 2^{2k+1} = 7M \quad (m \in \mathbb{Z})$

Prove $P(k+1)$ true:

RTP $5^{2(k+1)+1} + 2^{2(k+1)+1} = 7N \quad (N \in \mathbb{Z})$

$LHS = 5^{2k+3} + 2^{2k+3}$

$= 5^{2k+1} \times 5^2 + 2^{2k+1} \times 2^2$

$= 25 \times 5^{2k+1} + 4 \times 2^{2k+1}$

$= 21 \times 5^{2k+1} + 4 \times (5^{2k+1} + 2^{2k+1})$

$= 21 \times 5^{2k+1} + 4 \times 7m$ from assumption

$= 7[3 \times 5^{2k+1} + 4m]$

$= 7N$ (since $3 \times 5^{2k+1} + 4m \in \mathbb{Z}$)

$= RHS$

1 ~ base case ($\frac{1}{2}$ if n=1 used correctly)

1 ~ correct RTP and correct use of assumption

1 ~ correct simplification

Conclusion Therefore, $P(n)$ is true $\forall n \in \mathbb{Z} : n \geq 0$ by mathematical induction.

d) Let $P(x)=2x^4-15x^3+2x^2+ax+b$ where a and b are real numbers. Find the values of a and b given that $(x-5)^2$ is a factor of $P(x)$. **3**

$(x-5)^2$ is a factor $\Rightarrow$ $x=5$ is a double root (root of multiplicity 2)

i.e. $P(5)=0$ & $P'(5)=0$

$P(5)=0$: $2(5)^4-15(5)^3+2(5)^2+a(5)+b=0$

$-575+5a+b=0$

$b=575-5a$ — (1)

1 ~ correct subst. & simp. of $P(5)=0$

$P'(x)=8x^3-45x^2+4x+a$

$P'(5)=0$: $8(5)^3-45(5)^2+4(5)+a=0$

$-105+a=0$

$a=105$ — (2)

1 ~ correct subst. & simp. of $P'(5)=0$ (or equivalent merit)

1 ~ simultaneous solution

Sub (2) in (1): $b=575-5(105)$

$b=50$

$\therefore$ $\boxed{a=105,\ b=50}$

e) Consider the function $f(x)=\sin^{-1}(\cos x)$ for $x\in(0,\pi)$.

i) Prove that $f(x)$ is linear. **2**

i) Express $f(x)$ in the form $ax+b$ where $a,b\in\mathbb{R}$. **1**

i) $f(x) = \sin^{-1}(\cos x)$

$f'(x) = \dfrac{1}{\sqrt{1-\cos^2 x}} \times -\sin x$

$= \dfrac{-\sin x}{\sqrt{\sin^2 x}}$

$= \dfrac{-\sin x}{\sin x} = -1$ which is constant $\therefore f(x)$ is linear as the gradient is a constant for all x in the domain

1 ~ derivative of $f(x)$

1 ~ explanation re: constant gradient $\Rightarrow$ linear.

ii) $f(x) = ax+b$

$a=-1$ $(f'(x)=-1 \Rightarrow$ part i))

$f(x) = -x+b$

b is the y-intercept $\Rightarrow f(0) = \sin^{-1}(\cos 0)$

$= \sin^{-1}(1)$

$= \pi/2$

$\therefore \boxed{f(x) = -x+\pi/2 \quad \text{for } x\in(0,\pi)}$

1 ~ correct a & b

END OF QUESTION 12

Question 13 (15 marks) **START A NEW BOOKLET**

a) The coefficient of x^4 in the expansion of $(1+2x^2)^n$ is 220. Find the value of n. **2**

$$T_3 = {}^nC_2(1)^{n-2}(2x^2)^2$$

Let coeff = 220: $220 = {}^nC_2 \times 1^{n-2} \times 2^2$

$$220 = {}^nC_2 \times 1 \times 4$$

$${}^nC_2 = 55$$

$$\frac{n!}{(n-2)!\,2!} = 55$$

$$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 2! \times 55$$

$$n(n-1) = 110$$

$$n^2 - n - 110 = 0$$

$$(n-11)(n+10) = 0$$

$$n = -10, 11$$

but $n > 0$ (to yield x^4)

$$\therefore \boxed{n = 11}$$

1 ~ equates coeff. of T_3 with 220

1 ~ solves for n
($-\frac{1}{2}$ if negative case not disregarded)
or shows guess/check/refine process

b) A box contains 3 blue marbles, 5 green marbles and k yellow marbles. A marble is chosen at random and then placed back into the box. This is repeated four times.

i) What is the probability, P, that the first marble chosen is blue? **1**

ii) Let X be the number of blue marbles chosen. What is the smallest value of k for which $Var(X) < 0.8$? **3**

i) $P(\text{blue first}) = \dfrac{3}{8+k}$ 1 ~ correct expression

ii) $X \sim Bin\left(4, \dfrac{3}{8+k}\right)$ $Var(X) < 0.8$

$$4 \times \left(\frac{3}{8+k}\right) \times \left(1 - \left(\frac{3}{8+k}\right)\right) < 0.8$$

1 ~ correct inequality statement

$$\frac{3}{8+k} \times \frac{5+k}{8+k} < 0.2$$

1 ~ solves for values of k

$$3(5+k) < 0.2(8+k)^2$$

$$15(5+k) < (8+k)^2$$

$$75 + 15k < 64 + 16k + k^2$$

$$k^2 + k - 11 > 0$$

1 ~ applies context to answer the question.

Consider $k^2 + k - 11 = 0$

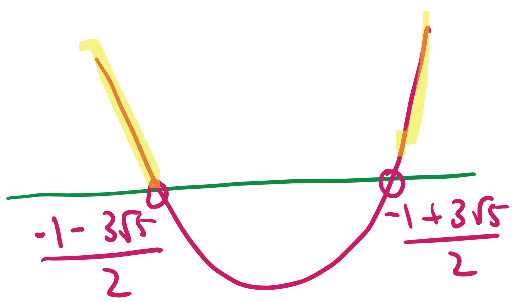
$$k = \frac{-1 \pm \sqrt{1^2 - 4(1)(-11)}}{2(1)}$$

$$k = \frac{-1 \pm \sqrt{45}}{2}$$

$$= \frac{-1 \pm 3\sqrt{5}}{2}$$

Note: $k \in \mathbb{Z}^+$

(# marbles)

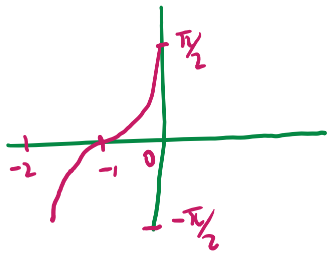


$k > 2.854...$

For smallest possible value, $\boxed{k=3}$

c) Consider $y = \sin^{-1}(x+1) + p\sin\left(\frac{\pi}{12}\right)$, where $p \in \mathbb{R}$ and $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$. **2**

Find the minimum value of p for which $y \geq 0$, for all x in the domain.



smallest y-value of $\sin^{-1}(x+1)$ is $-\frac{\pi}{2}$

$\therefore\ p\sin\left(\frac{\pi}{12}\right) \geq \frac{\pi}{2}$ for $y \geq 0$

$p \times \frac{\sqrt{6}-\sqrt{2}}{4} \geq \frac{\pi}{2}$ (given)

$p(\sqrt{6}-\sqrt{2}) \geq \frac{\cancel{4}^{2}\pi}{\cancel{2}}$

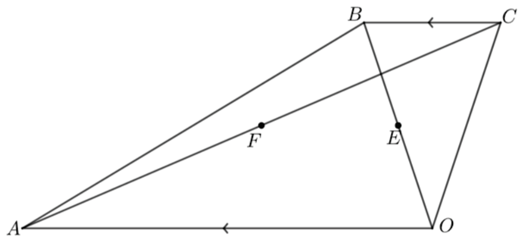
$p \geq \frac{2\pi}{\sqrt{6}-\sqrt{2}}$

$\therefore$ minimum value for p is $\boxed{\frac{2\pi}{\sqrt{6}-\sqrt{2}}}$

1 ~ utilise range of $y = \sin^{-1}(x+1)$ to conclude $p\sin\left(\frac{\pi}{12}\right) \geq \frac{\pi}{2}$

1 ~ correct answer.

d) The points O, A, B, C form a trapezium with $\overrightarrow{OA} = 3\overrightarrow{CB}$. Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{c} = \overrightarrow{OC}$.
Points E and F are the midpoints of OB and CA respectively.



i) Show that $\overrightarrow{EF} = \frac{1}{3}\underset{\sim}{a}$. **3**

ii) Justify the statement, "Hence, $CEFB$ is a parallelogram." **1**

$\overrightarrow{OA} = 3\overrightarrow{CB}$ (given)

$\underset{\sim}{a} = 3\overrightarrow{CB}$

$\boxed{\overrightarrow{CB} = \frac{1}{3}\underset{\sim}{a}}$

$\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$

$\therefore \boxed{\overrightarrow{CF} = \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{c})}$

(F is midpoint of CA)

$\overrightarrow{BO} = \overrightarrow{BC} + \overrightarrow{CO}$

$= -\overrightarrow{CB} + -\overrightarrow{OC}$

$= -\frac{1}{3}\underset{\sim}{a} - \underset{\sim}{c}$ (from previous)

$\Rightarrow \overrightarrow{EO} = \frac{1}{2}\left(-\frac{1}{3}\underset{\sim}{a} - \underset{\sim}{c}\right)$

(E is midpoint of BO)

i) $\overrightarrow{EF} = \overrightarrow{EO} + \overrightarrow{OC} + \overrightarrow{CF}$

$= \left[\frac{1}{2}\left(-\frac{1}{3}\underset{\sim}{a} - \underset{\sim}{c}\right) + (\underset{\sim}{c}) + \left(\frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{c})\right)\right]$

$= -\frac{1}{6}\underset{\sim}{a} - \cancel{\frac{1}{2}\underset{\sim}{c}} + \cancel{\underset{\sim}{c}} + \frac{1}{2}\underset{\sim}{a} - \cancel{\frac{1}{2}\underset{\sim}{c}}$

$\boxed{\overrightarrow{EF} = \frac{1}{3}\underset{\sim}{a}}$ #

1 ~ find $\overrightarrow{EO}$

1 ~ find $\overrightarrow{CF}$

1 - correct solution

ii) $\vec{CB} = \frac{1}{3}\underset{\sim}{a}$ and $\vec{EF} = \frac{1}{3}\underset{\sim}{a}$

which means that $\vec{CB}$ & $\vec{EF}$ are equal vectors

i.e. They are parallel (same direction) and have the same magnitude (size).

Considering these vectors as opposite sides of the quadrilateral CEFB, this means that they are both parallel & equal in length, which are minimum conditions for CEFB to be a parallelogram

1 ~ use of equal vectors to show that there is a pair of sides that are both equal (size) and parallel.

e) Use the substitution $u^2 = x + 1$ to find the volume of solid formed by rotating the area bounded by the curve $y = \dfrac{x-1}{\sqrt{x+1}}$, the x axis, and the lines $x = 3$ and $x = 8$ about the x axis. Give your answer in exact form. **3**

$V = \pi\int_a^b y^2\,dx$

Let $u^2 = x+1 \Rightarrow u = \sqrt{x+1}$

$\Downarrow$

$u^2 - 2 = x - 1$

$(u^2-2)^2 = (x-1)^2$

$du = \dfrac{1}{2\sqrt{x+1}}\,dx$

@ $x = 3 \rightarrow u = 2$

$x = 8 \rightarrow u = 3$

$$V = \pi\int_3^8 \frac{(x-1)^2}{x+1}\,dx$$

$$= 2\pi\int_3^8 \frac{(x-1)^2}{\sqrt{x+1}}\cdot\frac{1}{2\sqrt{x+1}}\,dx$$

$$= 2\pi\int_2^3 \frac{(u^2-2)^2}{u}\,du$$

$$= 2\pi\int_2^3 \frac{u^4-4u^2+4}{u}\,du$$

$$= 2\pi\int_2^3 u^3 - 4u + \frac{4}{u}\,du$$

$$= 2\pi\left[\frac{u^4}{4} - 2u^2 + 4\ln|u|\right]_2^3$$

$$= 2\pi\left[\left(\frac{3^4}{4} - 2(3)^2 + 4\ln 3\right) - \left(\frac{2^4}{4} - 2(2)^2 - 4\ln 2\right)\right] = \boxed{2\pi\left(\frac{25}{4} + 4\ln\frac{3}{2}\right) \text{ units}^3}$$

1 ~ correct subst. incl. change of bounds

1 ~ correct integration

1 ~ simplification

END OF QUESTION 13

Question 14 (15 marks) **START A NEW BOOKLET**

a) The roots of the quadratic equation $8x^2 - 5x + k = 0$ are $\sin\theta$ and $\cos 2\theta$ (for some angle θ). Find the possible values of k. **3**

let $\alpha = \sin\theta$ $\quad \beta = \cos 2\theta$

$\alpha + \beta = -b/a$

$\sin\theta + \cos 2\theta = \frac{5}{8}$

$\sin\theta + (1 - 2\sin^2\theta) = \frac{5}{8}$

$16\sin^2\theta - 8\sin\theta - 3 = 0$

$(4\sin\theta + 1)(4\sin\theta - 3) = 0$

$\sin\theta = -\frac{1}{4}, \frac{3}{4}$ — (1)

$\alpha\beta = \frac{c}{a}$:

$\sin\theta\cos 2\theta = k/8$

$\sin\theta(1 - 2\sin^2\theta) = k/8$

$k = 8\sin\theta(1 - 2\sin^2\theta)$ — (2)

Sub results from (1) into (2):

$\sin\theta = -\frac{1}{4}$: $\quad k = 8 \times -\frac{1}{4} \times \left(1 - 2\left(-\frac{1}{4}\right)^2\right)$

$= -\frac{7}{4}$

$\sin\theta = \frac{3}{4}$: $\quad k = 8 \times \frac{3}{4} \times \left(1 - 2\left(\frac{3}{4}\right)^2\right)$

$= -\frac{3}{4}$

$\therefore \boxed{k = -\frac{3}{4}, -\frac{7}{4}}$

1 ~ states correct eqⁿs for $\alpha + \beta$ and $\alpha\beta$

OR

Substitutes both $\sin\theta$, $\cos 2\theta$ and writes a simultaneous eqⁿ to combine

OR

creates correct factored exp & equates coefficients

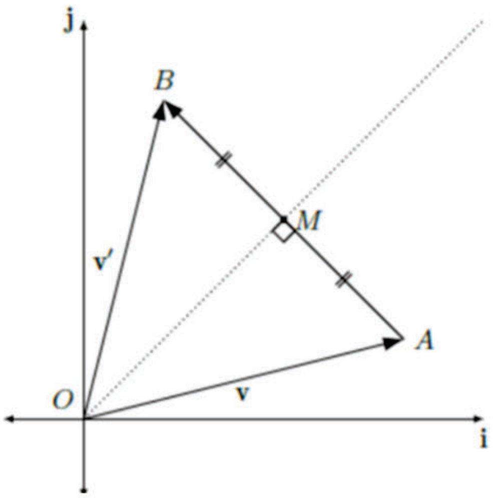
1 ~ solves for $\sin\theta$

1 ~ uses values of $\sin\theta$ correctly in solving for k

b) The diagram below shows the vector $\underset{\sim}{v} = \begin{pmatrix} a \\ b \end{pmatrix}$.

The vector $\underset{\sim}{v}'$ is the reflection of $\underset{\sim}{v}$ about the dotted line $y = x$. The point M bisects interval AB and lies on the line $y = x$.

Using vector projection, show that $\underset{\sim}{v}' = \begin{pmatrix} b \\ a \end{pmatrix}$. **3**



let $\underset{\sim}{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be a vector along $y = x$

$$\overrightarrow{OM} = \text{proj}_{\underset{\sim}{w}} \underset{\sim}{v} = (\underset{\sim}{v} \cdot \hat{\underset{\sim}{w}}) \hat{\underset{\sim}{w}}$$

$|\underset{\sim}{w}| = \sqrt{1^2 + 1^2}$
$= \sqrt{2}$

$\hat{\underset{\sim}{w}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \left[\begin{pmatrix} a \\ b \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \left[a \times 1 + b \times 1 \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{a+b}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{pmatrix} \text{ — } 1$$

$$\text{Now } \overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM}$$

$$= -\overrightarrow{OA} + \overrightarrow{OM}$$

$$= -\underset{\sim}{v} + \overrightarrow{OM}$$

$$= -\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{pmatrix} \quad \text{[from (1)]}$$

$$= \begin{pmatrix} -a + \frac{a+b}{2} \\ -b + \frac{a+b}{2} \end{pmatrix} = \begin{pmatrix} \frac{b-a}{2} \\ \frac{a-b}{2} \end{pmatrix} \quad -(2)$$

Now $\overrightarrow{AM} = \overrightarrow{MB}$ (from given information; M is the midpoint of AB)

Also, $\underset{\sim}{v}' = \overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB}$

$$= \begin{pmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{pmatrix} + \begin{pmatrix} \frac{b-a}{2} \\ \frac{a-b}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a+b+b-a}{2} \\ \frac{a+b+a-b}{2} \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \quad \#$$

1 ~ finds vector projection of $\underset{\sim}{v}$ onto a vector on $y = x$

1 ~ utilises $\overrightarrow{AM} = \overrightarrow{MB}$ and addition of vectors in ΔOMB

1 ~ successfully shows $\underset{\sim}{v}' = \begin{pmatrix} b \\ a \end{pmatrix}$

c) A survey shows that 36% of Sydney-siders go to the beach at some point during summer. Samples of 100 people are taken to determine the proportion of people who go to the beach at some point during summer. Find the mean and the standard deviation of the distribution of these sample proportions. Estimate the probability that a sample of 100 people will contain at least 36 and at most 42 people who visited the beach this summer. **4**

(Use the table shown below of $P(Z < z)$ where Z has a standard normal distribution.)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

$p = 0.36$, $n = 100$

$\mu_{\hat{p}} = \hat{p} = \underline{0.36}$

$Var(\hat{p}) = \dfrac{p(1-p)}{n}$

1 ~ $\mu_{\hat{p}}$

1 ~ $\sigma_{\hat{p}}$

1 ~ finds z-scores using $\mu_{\hat{p}}$, $\sigma_{\hat{p}}$

1 ~ uses table correctly to find probability

$\sigma_{\hat{p}} = \sqrt{Var(\hat{p})}$

$= \sqrt{\dfrac{0.36 \times 0.64}{100}}$

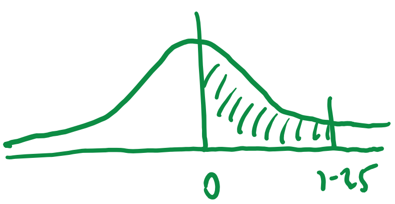
$= \underline{0.048}$

For 36 people

$\hat{p} = \dfrac{36}{100} = 0.36$; $z = \dfrac{0.36 - 0.36}{0.048} = 0$

For 42 people

$\hat{p} = \dfrac{42}{100} = 0.42$; $z = \dfrac{0.42 - 0.36}{0.048} = 1.25$

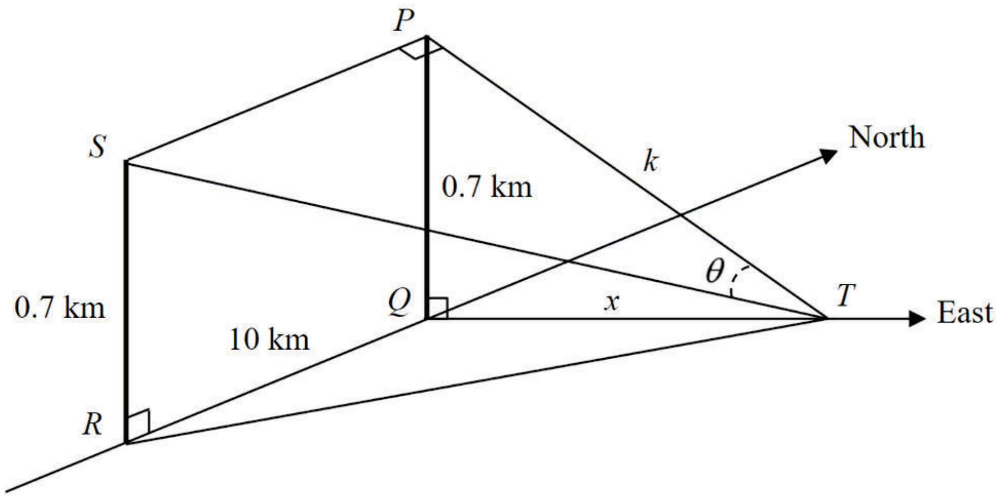


$P(0 < z < 1.25)$

$= 0.8944 - 0.5$ (from table)

$= \boxed{0.3944}$

d) PQ and SR are two towers of height 0.7km. Q is 10km due North of R. A solar powered vehicle at T is travelling due East away from Q at a constant speed of 10km/h. Let the distance QT be x and the distance PT be k. Let $\angle PTS = \theta$.



i) Show that $\frac{dk}{dt} = \frac{10x}{\sqrt{x^2+0.49}}$. **2**

$k^2 = x^2 + 0.7^2$ (Pythagoras)

$k = \sqrt{x^2+0.49}$

$\frac{dk}{dx} = \frac{1}{\cancel{2}\sqrt{x^2+0.49}} \times \cancel{2}x$

$= \frac{x}{\sqrt{x^2+0.49}}$

$\frac{dk}{dt} = \frac{dk}{dx} \times \frac{dx}{dt}$

$= \frac{x}{\sqrt{x^2+0.49}} \times 10 = \boxed{\frac{10x}{\sqrt{x^2+0.49}}}$ #

1 ~ shows differentiation of $k = \sqrt{x^2+0.49}$ to find $\frac{dk}{dx}$ (with sufficient detail)

1 ~ shows use of chain rule & given rate.

ii) By first finding an expression for k in terms of θ, show that $\frac{dk}{d\theta} = -10\operatorname{cosec}^2\theta$. **1**

In ΔPST, $\tan\theta = \frac{10}{k} \Rightarrow k = \frac{10}{\tan\theta} = 10\cot\theta$

Method One

$$k = \frac{10\cos\theta}{\sin\theta}$$

u	v
$\cos\theta$	$\sin\theta$
u'	v'
$-\sin\theta$	$\cos\theta$

$$\frac{dk}{d\theta} = 10 \times \frac{vu' - uv'}{v^2}$$

$$= 10 \times \left(\frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}\right)$$

$$= -10\left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\right)$$

$$= \frac{-10}{\sin^2\theta} = -10\operatorname{cosec}^2\theta$$

Method Two

$$k = 10(\tan\theta)^{-1}$$

$$\frac{dk}{d\theta} = 10 \times -1\,(\tan\theta)^{-2} \times \sec^2\theta$$

$$= \frac{-10}{\tan^2\theta} \times \sec^2\theta$$

$$= -10 \times \frac{\cancel{\cos^2\theta}}{\sin^2\theta} \times \frac{1}{\cancel{\cos^2\theta}}$$

$$= -10\operatorname{cosec}^2\theta$$

1 ~ shows either quotient rule or chain rule with sufficient detail to show result.

iii) Find the exact rate at which θ is changing when the vehicle is 2.4km from Q. **2**

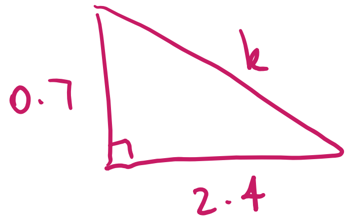
$$\frac{d\theta}{dt} = \frac{d\theta}{dk} \times \frac{dk}{dt}$$

$$= \frac{-1}{10\operatorname{cosec}^2\theta} \times \frac{10x}{\sqrt{x^2+0.49}}$$

$$= \frac{-1}{10 \times \left(\frac{\sqrt{17}}{4}\right)^2} \times \frac{10(2.4)}{\sqrt{2.4^2+0.49}}$$

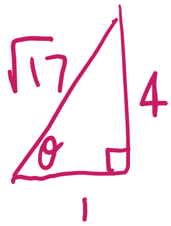
$$= \frac{-384}{425}$$

When $x = 2.4$



$$k = \sqrt{0.7^2+2.4^2} = 2.5$$

$$\tan\theta = \frac{10}{k} = \frac{10}{2.5} = \frac{4}{1}$$



$\therefore$ At $x = 2.4$ km, the angle θ is $\boxed{\text{decreasing}}$ at a rate of $\boxed{\frac{384}{425} \text{ radians/hr}}$

1 – correct subst. of $\frac{d\theta}{dk}$, $\frac{dk}{dt}$ into chain rule

1 – correct evaluation (including value of $\operatorname{cosec}\theta$)

END OF PAPER