NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2025 Year 12 Course Assessment Task 4 (Trial Examination)
Friday 8th August, 2025

General instructions

- Working time – 2 hours.
(plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 14)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED**: .....

Class (please ✔)

- ◯ 12MAX.1 – Miss Kim
- ◯ 12MAX.2 – Miss Chaussivert
- ◯ 12MAX.3 – Miss Lee
- ◯ 12MAX.4 – Mr Lam
- ◯ 12MXX.1 – Ms Ham
- ◯ 12MXX.2 – Mr Ho

Marker's use only.

QUESTION	1-10	11	12	13	14	Total	%
MARKS	$\overline{10}$	$\overline{17}$	$\overline{15}$	$\overline{17}$	$\overline{11}$	$\overline{70}$	

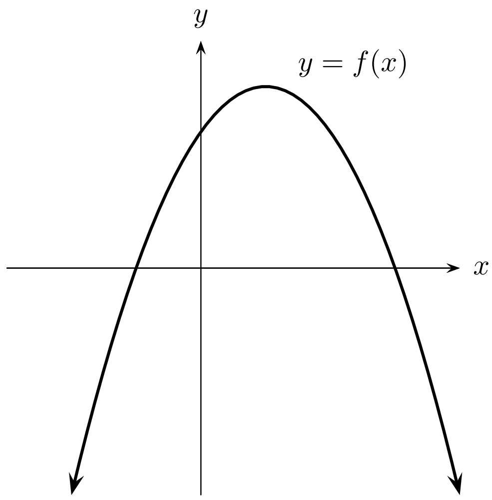
Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 14).

Questions **Marks**

1. The graph of $y = f(x)$ is shown, and the x and y axes are both drawn to the same scale. $f(x)$ then has its domain restricted such that it contains the largest possible domain for which $y = f(x)$ has an inverse function $y = f^{-1}(x)$, and the function with the restricted domain is $g(x)$. **1**



How many points of intersection are there between $y = g(x)$ and $y = g^{-1}(x)$?

(A) 1 (C) 3

(B) 2 (D) 4

2. Given $\tan\theta = \dfrac{1}{2}$, what is the exact value of $\tan\left(\theta - \dfrac{\pi}{6}\right)$? **1**

(A) $8 + 5\sqrt{3}$ (C) $\dfrac{8 + 5\sqrt{3}}{11}$

(B) $8 - 5\sqrt{3}$ (D) $\dfrac{8 - 5\sqrt{3}}{11}$

3. The equation $x^3 - 3x^2 - 18x + 40 = 0$ has roots α, β, γ. **1**

What is the value of $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\alpha\gamma}$?

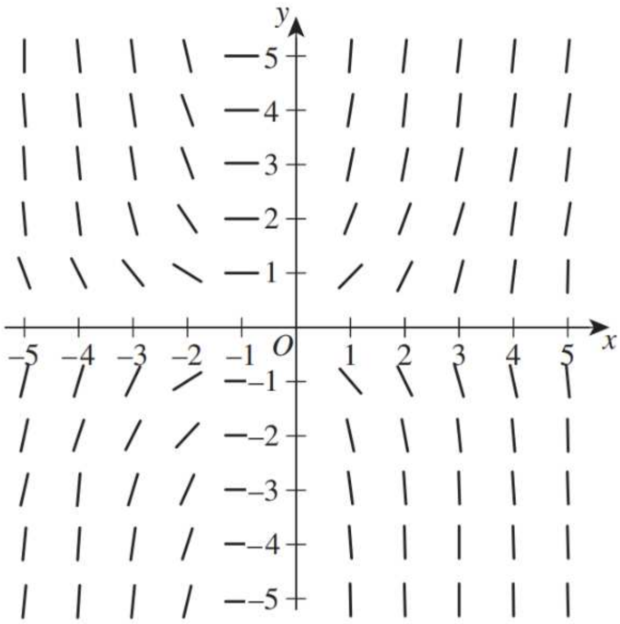
(A) $-\dfrac{1}{18}$

(B) $\dfrac{1}{18}$

(C) $-\dfrac{3}{40}$

(D) $\dfrac{3}{40}$

4. The slope field for a first order differential equation is shown below. **1**



Which of the following could be the differential equation represented?

(A) $\dfrac{dy}{dx} = (x+1)^3$

(B) $\dfrac{dy}{dx} = x(y+1)$

(C) $\dfrac{dy}{dx} = y(x+1)$

(D) $\dfrac{dy}{dx} = y(x-1)$

5. The function $f(x)$ has parametric equations: **1**

$$x = 2e^t \quad , \quad y = \cos(1 + e^{3t})$$

Which of the following is the derivative $\dfrac{dy}{dx}$?

(A) $-\dfrac{3}{2}e^{2t}\sin(1 + e^{3t})$

(B) $\dfrac{3}{2}e^{2t}\sin(1 + e^{3t})$

(C) $-6e^{2t}\sin(1 + e^{3t})$

(D) $6e^{2t}\sin(1 + e^{3t})$

6. What is the range of the function $f(x) = 2\sin^{-1} x - \tan^{-1} x$? **1**

(A) $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(B) $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(C) $y \in \left(-\dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right)$

(D) $y \in \left[-\dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right]$

Examination continues overleaf...

7. What is the value of $\displaystyle\int_0^{\frac{\pi}{4}} \sin 2\theta \cos 3\theta \, d\theta$? **1**

(A) $-\dfrac{2}{5}$

(B) $\dfrac{1}{\sqrt{2}} - \dfrac{2}{5}$

(C) $\dfrac{2}{5\sqrt{2}} - \dfrac{2}{5}$

(D) $\dfrac{3}{5\sqrt{2}} - \dfrac{2}{5}$

8. What is the primitive of $\dfrac{3f'(x)}{1 + 9\Big(f(x)\Big)^2}$? **1**

(A) $\tan^{-1}(f(3x))$

(B) $3\tan^{-1}(f(3x))$

(C) $3\ln 3(f(x))$

(D) $\tan^{-1}(3f(x))$

9. Daniel has recently been into collecting Labubus and wants to display a total of five different Labubus on his shelf. He can choose from five different Labubus from the *Have a Seat* collection and five different Labubus from the *Exciting Macaron* collection. **1**

Given that the pink *Exciting Macaron* Labubu must be displayed, how many ways can he select the remaining if he wants at least two from the *Have a Seat* collection and at least one other from the *Exciting Macaron* collection on the shelf?

(A) 100

(B) 240

(C) 252

(D) 320

10. The magnitude of the projection of the vector $\underset{\sim}{a} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ in the direction of the vector $\underset{\sim}{b}$ is $\dfrac{11}{\sqrt{10}}$. Which of the following could be equal to $\underset{\sim}{b}$? **1**

(A) $\underset{\sim}{i} + 3\underset{\sim}{j}$

(B) $3\underset{\sim}{i} + \underset{\sim}{j}$

(C) $\underset{\sim}{i} - 3\underset{\sim}{j}$

(D) $-3\underset{\sim}{i} + \underset{\sim}{j}$

Section II

60 marks
Attempt Questions 11 to 14
Allow approximately 1 hour and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (17 marks) Commence a NEW booklet. **Marks**

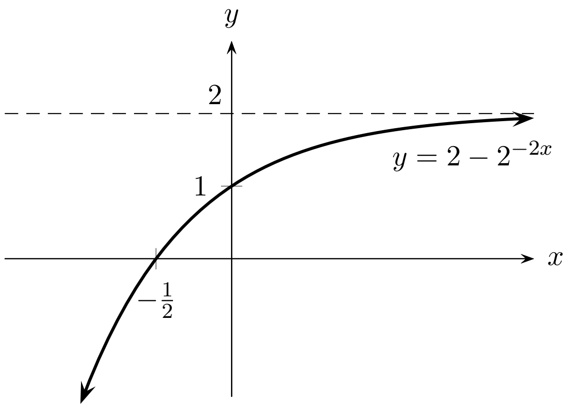
(a) Solve the inequality for x **3**

$$\frac{-x}{(x-3)(2x+1)} \leq \frac{1}{6}$$

(b) Find the term independent of x in the expansion of $\left(2x+\dfrac{3}{x^2}\right)^{15}$. **2**

You may leave your answer in the form $\dbinom{n}{k}2^a3^b$.

(c) The diagram below shows the graph of the function $f(x) = 2 - 2^{-2x}$.



On the grid supplied on page 13, sketch the following graphs by showing all important features.

i. $y^2 = f(x)$ **2**

ii. $y = \dfrac{1}{f(x)}$ **2**

(d) A group of six males and five females, including Aden and Ina are to be placed into two separate lines. Line A consists of six people and line B consists of five people. How many ways can they be arranged across the two lines if:

i. Aden and Ina are to be placed in separate lines? **2**

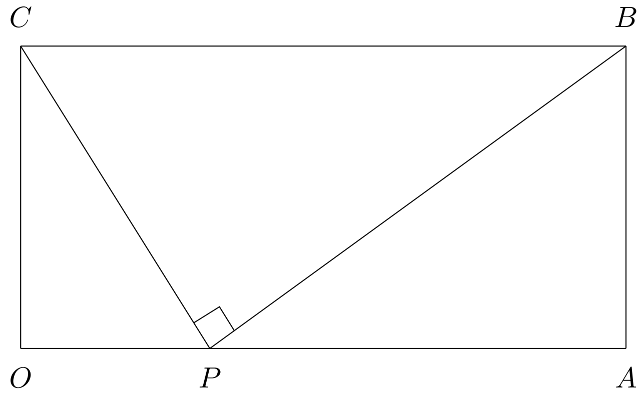
ii. The members in line A are selected first and must alternate between males and females within A? **3**

(e) Use the substitution $u = 1 + x$ to evaluate **3**

$$15\int_{-1}^{0} x\sqrt{1+x}\,dx$$

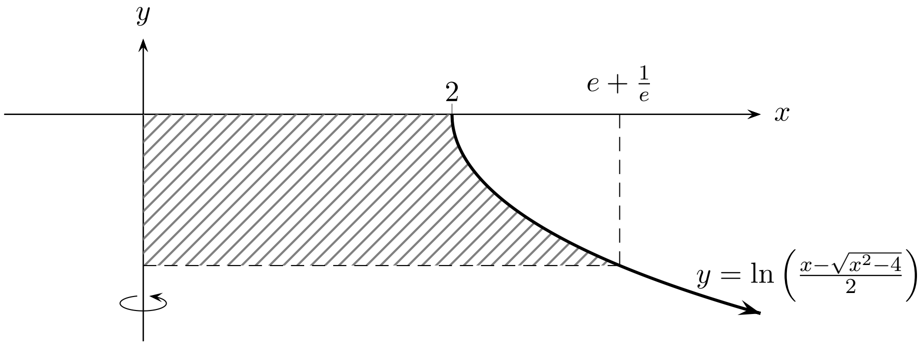
Question 12 (15 marks) Commence a NEW booklet. **Marks**

(a) In the diagram below, $OABC$ is a rectangle in which $\overrightarrow{OA} = \frac{7}{2}\underset{\sim}{i}$ and $\overrightarrow{OC} = \frac{1}{2}\underset{\sim}{j}$. P is a point on OA such that $\overrightarrow{OP} = \lambda\underset{\sim}{i}$ for some scalar parameter λ and $\angle CPB = 90°$. **4**



Use vector methods to show that $4\lambda^2 - 14\lambda + 1 = 0$ and hence find any values of λ.

(b) Below is the graph of $y = \ln\left(\dfrac{x - \sqrt{x^2 - 4}}{2}\right)$.



i. Show that the equation of the graph can also be expressed as **2**

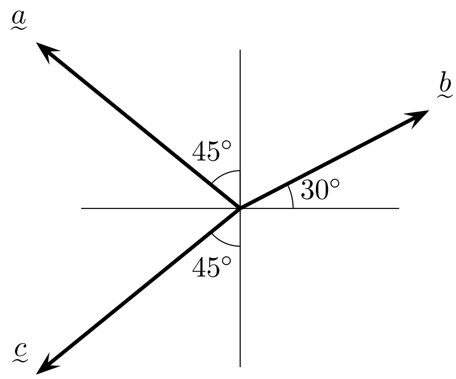
$$x = e^y + e^{-y}$$

ii. Hence find the exact volume generated when the shaded region is rotated about the y axis. **3**

Examination continues overleaf...

(c) Three students decide to test each other's strength by playing a game of tug-of-war. They each attach their rope to a weight and pull in their direction as shown in the diagram below.

Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be the pulling force of the three boys Ashton, Brendon and Conor.



It is given that $|\underset{\sim}{c}| = 250$ N. When the game starts, all three students are unable to move the weight from its starting position in any direction.

i. Show that **2**

$$\underset{\sim}{a} = \left(-\frac{1}{\sqrt{2}}|\underset{\sim}{a}|\right)\underset{\sim}{i} + \left(\frac{1}{\sqrt{2}}|\underset{\sim}{a}|\right)\underset{\sim}{j} \quad \text{and} \quad \underset{\sim}{b} = \left(\frac{\sqrt{3}}{2}|\underset{\sim}{b}|\right)\underset{\sim}{i} + \left(\frac{1}{2}|\underset{\sim}{b}|\right)\underset{\sim}{j}$$

ii. Find the exact value of $|\underset{\sim}{a}|$ and $|\underset{\sim}{b}|$. (You do not need to rationalise the denominators in your final answer). **4**

Question 13 (17 marks) Commence a NEW booklet. **Marks**

(a) i. Show that $\dfrac{1}{(n+2)!} - \dfrac{n+1}{(n+3)!} = \dfrac{2}{(n+3)!}$, where n is an integer. **1**

ii. Prove by mathematical induction that, for all positive integers n, **3**

$$\frac{1\times 2}{3!} + \frac{2\times 2^2}{4!} + \frac{3\times 2^3}{5!} + \cdots + \frac{n\times 2^n}{(n+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$

(b) i. You are given that **2**

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \quad \text{and} \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Use this to show that

$$\sin^4\theta + \cos^4\theta = \frac{1}{4}(\cos 4\theta + 3)$$

ii. By letting $x = \cos\theta$ in the quartic equation $8x^4 + 8\left(1 - x^2\right)^2 = 7$, show that $\cos 4\theta = \dfrac{1}{2}$. **2**

iii. Hence show that $\cos^2\dfrac{\pi}{12}\cos^2\dfrac{5\pi}{12} = \dfrac{1}{16}$. **3**

(c) A team of Australian wildlife conservationists are working on a project to increase the population of the mountain pygmy-possums in Kosciuszko National Park. From observations in other alpine areas which is the primary habitat of these pygmy-possums, the conservationists know that populations of more than 800 are unsustainable due to a lack of habitat. They propose the following logistic model of population growth

$$\frac{dP}{dt} = rP(800 - P)$$

where P is the number of pygmy-possums in the park, t is time in years and r is a constant. The team of conservationists initially release 100 possums into the National Park.

i. During the months of spring, when the maximum growth rate occurs, all the pygmy-possums in the monitored population find a mate and half the possum pairs successfully raise an offspring. **2**

Show that $r = \dfrac{1}{1600}$.

ii. Given that **3**

$$\frac{1}{P(800-P)} = \frac{1}{800}\left(\frac{1}{P} + \frac{1}{800-P}\right)$$

solve the differential equation to show that

$$P(t) = \frac{800}{1 + 7e^{-0.5t}}$$

iii. How long, to the nearest year, will it take for the possum population to reach 85% of the maximum sustainable population? **1**

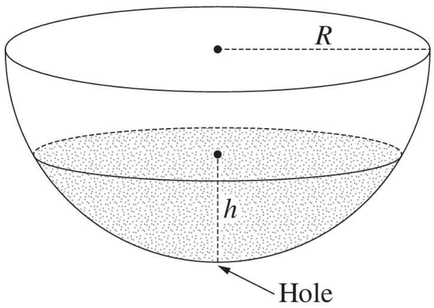
Examination continues overleaf…

Question 14 (11 marks) Commence a NEW booklet. **Marks**

(a) A hemispherical water tank has radius R cm. The tank has a hole at the bottom which allows water to drain out.

Initially the tank is empty. Water is poured into the tank at a constant rate of $2kR$ cm^3/sec, where k is a positive constant.

After t seconds, the height of the water in the tank is h cm, as shown in the diagram, and the volume of water in the tank is V cm^3.



It is known that $V = \pi\left(Rh^2 - \dfrac{h^3}{3}\right)$. (Do NOT prove this.)

While water flows into the tank and also drains out of the bottom, the rate of change of the volume of water in the tank is given by $\dfrac{dV}{dt} = k(2R - h)$.

i. Show that $\dfrac{dh}{dt} = \dfrac{k}{\pi h}$. **2**

ii. Show that the tank is full of water after $T = \dfrac{\pi R^2}{2k}$ seconds. **2**

iii. The instant the tank is full, water stops flowing into the tank, but it continues to drain out of the hole at the bottom as before. **3**

Show that the tank takes 3 times as long to empty as it did to fill.

(b) A particle is projected from the origin with velocity V metres per second at an angle θ to the horizontal. The position of the particle at time t seconds is given by the equations **4**

$$\underset{\sim}{r}(t) = \begin{pmatrix} Vt\cos\theta \\ Vt\sin\theta - \dfrac{1}{2}gt^2 \end{pmatrix}$$

where g is the acceleration due to gravity.

It is known that the time of flight is $T = \dfrac{2V\sin\theta}{g}$ if the projectile lands back at ground level (Do NOT prove this).

Let $S(t) = \left|\dot{\underset{\sim}{r}}(t)\right|^2$ be the function of the square of the speed of the particle in terms of time.

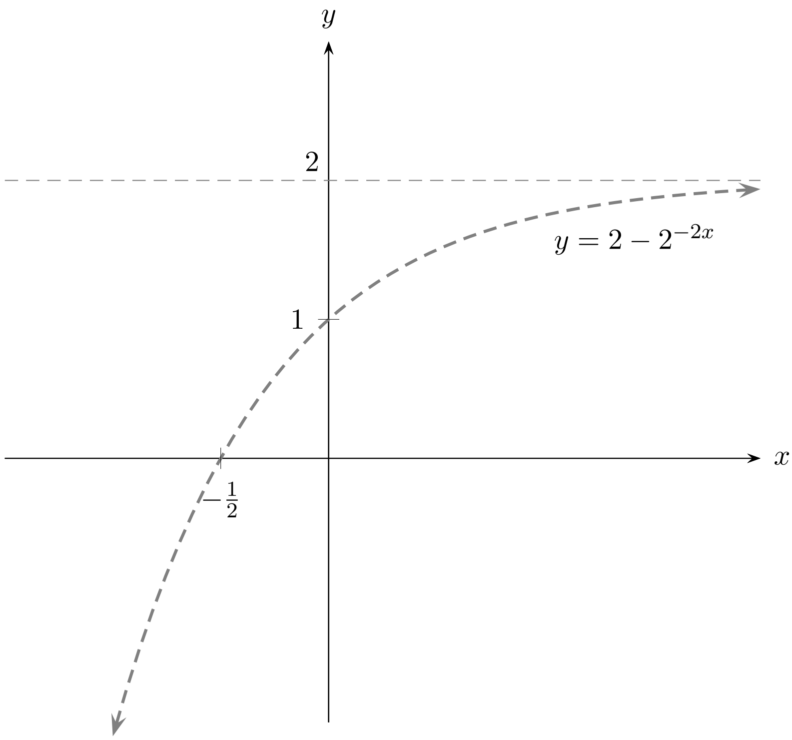
Find the initial projection angle θ, given that at the time where a quarter of the time of flight has elapsed, the speed of the particle is $\dfrac{\sqrt{7}}{4}$ of the initial projection speed.

End of paper.

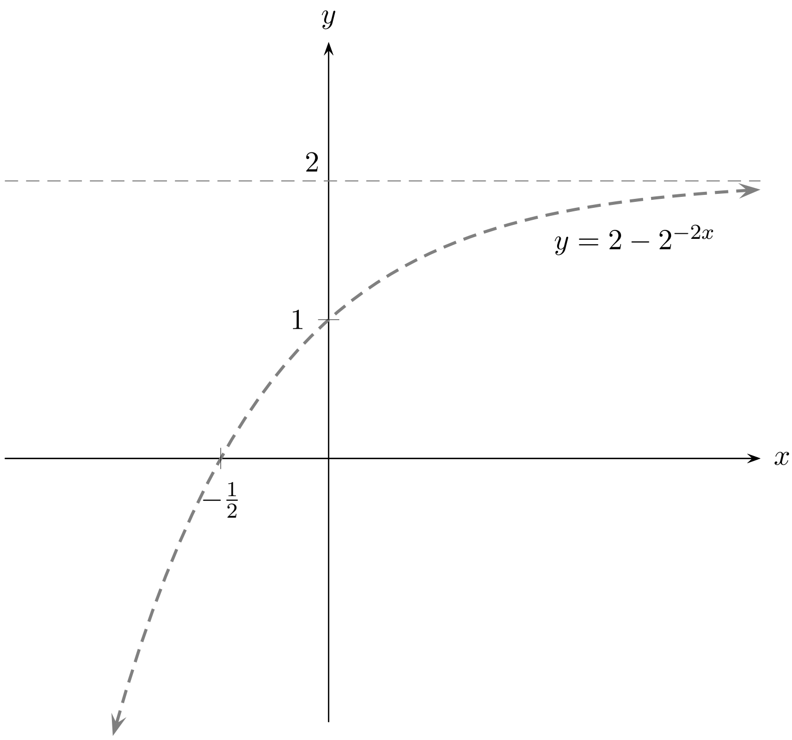
Question 11

(c) Using the graph of $f(x)$, sketch the graph of

i. $y^2 = f(x)$ **2**



ii. $y = \dfrac{1}{f(x)}$ **2**

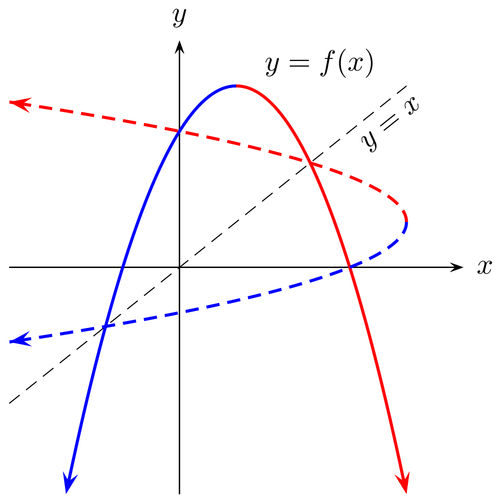


Sample Band E4 Responses

Section I

1. (A) **2.** (D) **3.** (C) **4.** (C) **5.** (A)
6. (D) **7.** (D) **8.** (D) **9.** (A) **10.** (B)

1. (D) - Take either left or eight of the parabola as the largest domain, and draw its inverse. At most, only one point of intersection will exist (red branch will intersect with red dashed branch, blue branch will intersect with blue dashed branch)



2. (D)

$$\tan\theta = \frac{1}{2}$$

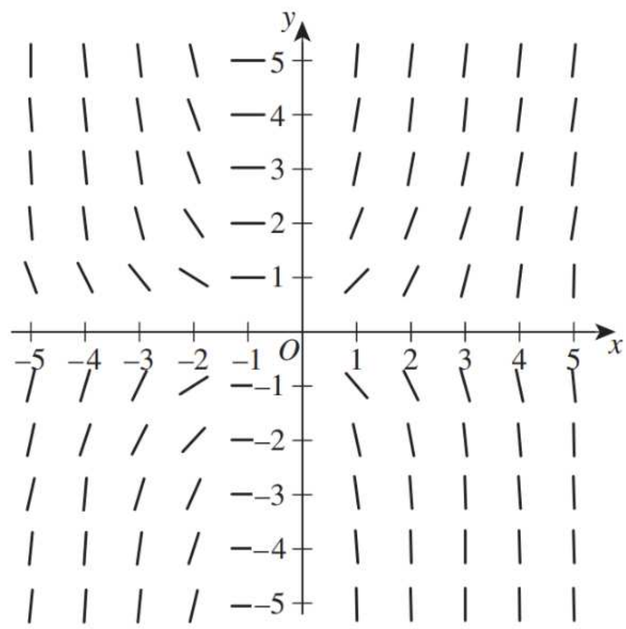
Expanding,

$$\begin{aligned}\tan\left(\theta - \frac{\pi}{6}\right) &= \frac{\tan\theta - \tan\frac{\pi}{6}}{1 + \tan\theta\tan\frac{\pi}{6}} \\ &= \frac{\frac{1}{2} - \frac{1}{\sqrt{3}}}{1 + \left(\frac{1}{2} \times \frac{1}{\sqrt{3}}\right)} \\ &= \frac{\frac{\sqrt{3}-2}{\cancel{2\sqrt{3}}}}{\frac{2\sqrt{3}+1}{\cancel{2\sqrt{3}}}} \\ &= \frac{\sqrt{3}-2}{2\sqrt{3}+1}\frac{\times(2\sqrt{3}-1)}{\times(2\sqrt{3}-1)} \\ &= \frac{6\sqrt{3} - 4\sqrt{3} + 2}{12 - 1} \\ &= \frac{8 - 5\sqrt{3}}{11}\end{aligned}$$

3. (C)

$$\begin{aligned}x^3 - 3x^2 - 18x + 40 &= 0 \\ \alpha + \beta + \gamma &= -(-3) = 3 \\ \alpha\beta\gamma &= -40 \\ \frac{1}{\alpha\beta}\frac{\times\gamma}{\times\gamma} + \frac{1}{\beta\gamma}\frac{\times\alpha}{\times\alpha} + \frac{1}{\alpha\gamma}\frac{\times\beta}{\times\beta} &= \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} \\ &= \frac{3}{-40}\end{aligned}$$

4. (C)



- At $(0,0)$, $\dfrac{dy}{dx} = 0$.

  Hence, (A) is ruled out as $\dfrac{dy}{dx} \neq 0$ when $x = 0$ and $y = 0$
- Also, when $x = -1$, $\frac{dy}{dx} = 0$.

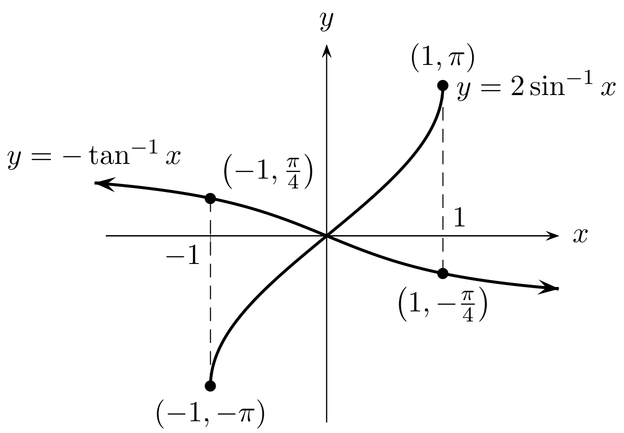
  The remaining viable option is (C) as (B) and (D) would not result in $\dfrac{dy}{dx} = 0$ when $x = -1$.

5. (A)

$$\begin{cases} x = 2e^t \\ y = \cos\left(1+e^{3t}\right) \end{cases}$$

$$\begin{aligned}
\frac{dx}{dt} &= 2e^t \\
\frac{dy}{dt} &= -3e^{3t}\sin\left(1+e^{3t}\right) \\
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\
&= \frac{-3e^{3t}\sin\left(1+e^{3t}\right)}{2e^t} \\
&= -\frac{3}{2}e^{2t}\sin\left(1+e^{3t}\right)
\end{aligned}$$

6. (D)



By addition of ordinates, the extremities of the $y = 2\sin^{-1} x$ graph will relegate the range to $\left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$.

7. (D)

$$\begin{aligned}
\sin 2\theta\cos 3\theta + \cos 3\theta\sin 2\theta &= \sin(2\theta+3\theta) \\
&= \sin 5\theta \\
\sin 2\theta\cos 3\theta - \cos 3\theta\sin 2\theta &= \sin(2\theta-3\theta) \\
&= -\sin\theta
\end{aligned}$$

Adding the two relationships,

$$\begin{aligned}
2\sin 2\theta\cos 3\theta &= \sin 5\theta - \sin\theta \\
\sin 2\theta\cos 3\theta &= \frac{1}{2}\left(\sin 5\theta - \sin\theta\right)
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \sin 2\theta\cos 3\theta\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \left(\sin 5\theta - \sin\theta\right)\, d\theta \\
&= \frac{1}{2}\left[-\frac{1}{5}\cos 5\theta + \cos\theta\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2}\left[\left(-\frac{1}{5}\cos\frac{5\pi}{4} + \cos\frac{\pi}{4}\right) - \left(-\frac{1}{5}+1\right)\right] \\
&= \frac{1}{2}\left(\frac{1}{5\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{4}{5}\right) \\
&= \frac{1}{2}\left(\frac{6}{5\sqrt{2}} - \frac{4}{5}\right) \\
&= \frac{3}{5\sqrt{2}} - \frac{2}{5}
\end{aligned}$$

8. (D) - Differentiating each option:

(A) $y = \tan^{-1}\Big(f(3x)\Big)$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{1+\Big(f(3x)\Big)^2} \times 3f'(3x) \\
&= \frac{3f'(3x)}{1+\Big(f(3x)\Big)^2}
\end{aligned}$$

Which does not match $\dfrac{3f'(x)}{1+9\Big(f(x)\Big)^2}$.

(B) $y = 3\tan^{-1}\Big(f(3x)\Big)$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3}{1+\Big(f(3x)\Big)^2} \times 3f'(3x) \\
&= \frac{9f'(3x)}{1+\Big(f(3x)\Big)^2}
\end{aligned}$$

Which does not match $\dfrac{3f'(x)}{1+9\Big(f(x)\Big)^2}$.

(C) $y = 3\ln\Big(3f(x)\Big)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{3}{3f(x)} \times 3f'(x)\\ &= \frac{3f'(x)}{f(x)}\end{aligned}$$

Which does not match $\dfrac{3f'(x)}{1+9\Big(f(x)\Big)^2}$.

(D) $y = \tan^{-1}\Big(3f(x)\Big)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+\Big(3f(x)\Big)^2} \times 3f'(x)\\ &= \frac{3f'(3x)}{1+9\Big(f(x)\Big)^2}\end{aligned}$$

9. (A)

- Of 5 spaces available, 4 spaces remain as the pink *Exciting Labubu* must be selected.
- If there are 3 *Have a seat* Labubus (and 1 more *Exciting Macaron*):

$$^5C_3 \times {}^4C_1$$

- If there are 2 *Have a seat* Labubus (and 2 more *Exciting Macaron*):

$$^5C_2 \times {}^4C_2$$

- Total number is

$$\left(^5C_3 \times {}^4C_1\right) + \left(^5C_2 \times {}^4C_2\right) = 100$$

10. (B) - let $\underset{\sim}{b} = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{aligned}\left|\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a}\right| &= \left|\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|}\right)\widehat{\underset{\sim}{b}}\right|\\ &= \frac{11}{\sqrt{10}}\widehat{\underset{\sim}{b}}\\ \therefore \frac{\begin{pmatrix} 2 \\ 5 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}}{\sqrt{x^2+y^2}} &= \frac{11}{\sqrt{10}}\\ \frac{2x+5y}{\sqrt{x^2+y^2}} &= \frac{11}{\sqrt{10}}\end{aligned}$$

Testing out the various combinations of x and y provided, (B) is the best option.

Section II

Question 11

(a) (3 marks) (Kim)

✓ [1] Understands the inequality symbol may change and hence correctly multiplies by the square of the denominator, or equivalent merit.

✓ [2] Obtains a correct polynomial inequality with zero as the subject, or equivalent merit.

✓ [3] Correct solution.

$$-\underbrace{\frac{x}{(x-3)(2x+1)}}_{\times(x-3)^2(2x+1)^2} \leq \underbrace{\frac{1}{6}}_{\times(x-3)^2(2x+1)^2}$$

$$-x(x-3)(2x+1) \leq \frac{1}{6}(x-3)^2(2x+1)^2$$

$$(x-3)^2(2x+1)^2 + 6x(x-3)(2x+1) \geq 0$$

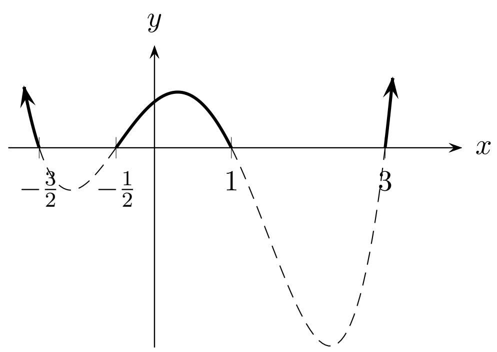
$$(x-3)(2x+1)\Big[(x-2)(2x+1)+6x\Big] \geq 0$$

$$(x-3)(2x+1)\left(2x^2+x-6x-3+6x\right) \geq 0$$

$$(x-3)(2x+1)\left(2x^2+x-3\right) \geq 0$$

$$(x-3)(2x+1)(2x+3)(x-1) \geq 0$$

Drawing polynomial helper graph,



$\therefore x \leq -\frac{3}{2}$, $-\frac{1}{2} < x \leq 1$ or $x > 3$
(Note that $x \neq 3$ and $x \neq -\frac{1}{2}$ from original inequality)

(b) (2 marks) (Kim)

✓ [1] Obtains the correct typical term, or equivalent merit.

✓ [2] Correct solution.

$$\begin{aligned}\left(2x+\frac{3}{x^2}\right)^{15} &= \sum_{k=0}^{15}\binom{15}{k}(2x)^{15-k}\left(3x^{-2}\right)^k \\ &= \sum_{k=0}^{15}\binom{15}{k}2^{15-k}3^k x^{15-k-2k} \\ &= \sum_{k=0}^{15}\binom{15}{k}2^{15-k}3^k x^{15-3k}\end{aligned}$$

Typical term:

$$t_k = \binom{15}{k}2^{15-k}3^k x^{15-3k}$$

Independent term occurs when

$$15-3k=0$$
$$k=5$$

Independent term is

$$\binom{15}{5}2^{10}3^5 \qquad \text{or} \qquad \binom{15}{10}2^{10}3^5$$

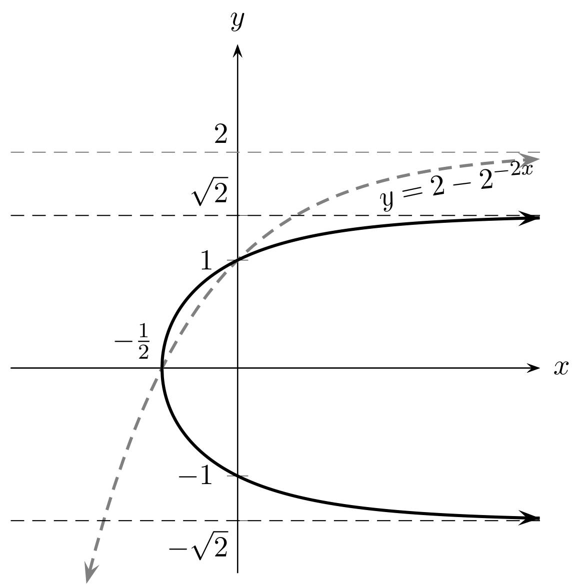
(c) (Kim)

i. (2 marks)

✓ [1] Correct shape & scale, or intercepts & asymptotes.

✓ [2] Correct solution

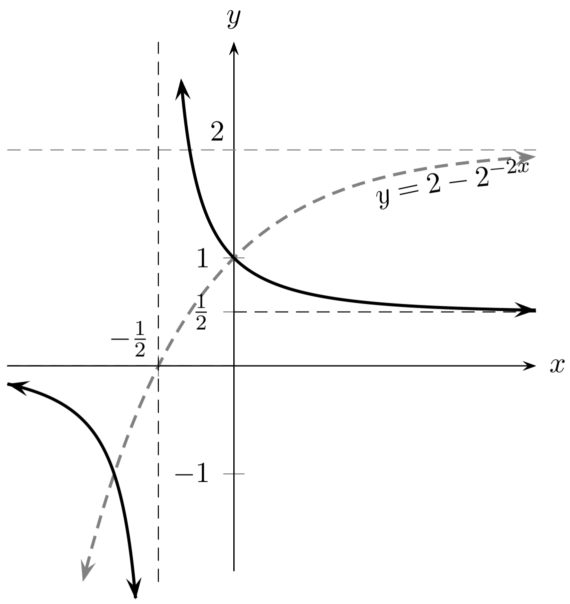
$y^2 = f(x) \Rightarrow y = \pm\sqrt{2-2^{-2x}}$



ii. (2 marks)

✓ [1] Correct shape & scale, or intercepts & asymptotes.

✓ [2] Correct solution



(d) (Lam)

i. (2 marks)

✓ [1] for correct number of groups of 5 from 9 people required for Line A, or equivalent merit.

✓ [2] for correct answer.

- Place Aden in Line A, leaving 5 spaces out of 6 to choose from - ${}^9C_5$
  - 6! ways of arranging Line A once the people have been chosen.
  - Hence, ${}^9C_5 \times 6!$ ways of arranging Line A, with Aden in Line A
- There remains 5! (= 120) others to arrange in Line B, which will include Ina.
- However, Aden and Ina can swap positions from Line A to B. Two more cases.

$${}^9C_5 \times 6! \times 5! \times 2 = 21\,772\,800 \text{ ways}$$

ii. (3 marks)

✓ [1] for correct permutation of 6 or 5 people in Line A or Line B respectively, or equivalent merit.

✓ [2] for correct identification of two cases required, or equivalent merit.

✓ [3] for correct answer.

Note: this part is unrelated to part (i).

- Place male in first, third and fifth position in Line A, and then females in second, fourth and sixth positions, which will be

$$\overbrace{6 \times 5 \times 4}^{\text{Arrangements of male}} \times \overbrace{5 \times 4 \times 3}^{\text{Arrangements of females}}$$

- Permute Line B, where there are no restrictions.

5! ways

- However, females can also commence Line A, hence double the number of cases.

$$(6\times5\times4)\times(5\times4\times3)\times5!\times2 = 1\,728\,000 \text{ ways}$$

(e) (3 marks)

✓ [1] for correct manipulation of integrand from x to u, including du, or limits of integration from x to u, or equivalent merit.

✓ [2] for correct manipulation of integrand from x to u, including du, and limits of integration from x to u, or equivalent merit.

✓ [3] for correct answer.

$$\int_{-1}^{0} x\sqrt{1+x}\,dx$$

$$\left|\begin{aligned} u &= 1+x \\ x &= u-1 \\ du &= dx \end{aligned}\right.$$

Modifying limits:

$$\begin{aligned} x &= 0 & u &= 1 \\ x &= -1 & u &= 0 \end{aligned}$$

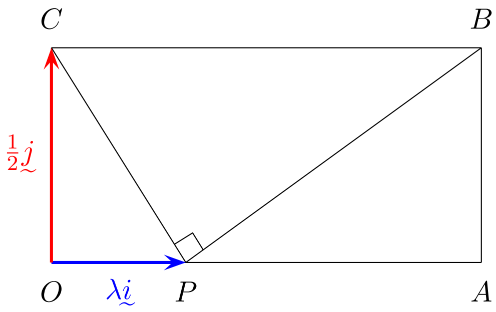
Modifying integrand:

$$\begin{aligned}
15\int_{-1}^{0} x\sqrt{1+x}\,dx &= 15\int_{u=0}^{u=1}(u-1)\sqrt{u}\,du\\
&= 15\int_0^1\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)\,du\\
&= 15\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right]_0^1\\
&= 15\left[\frac{2}{5}-\frac{2}{3}-0\right]\\
&= -4
\end{aligned}$$

Question 12(Chaussivert)

(a) (4 marks)

- ✓ [0] (If vector methods are not used)
- ✓ [1] Correctly identifies the necessary vectors $\overrightarrow{PC}$, $\overrightarrow{PA}$ and $\overrightarrow{PB}$, or equivalent merit.
- ✓ [2] Uses the dot product and equates to zero for the perpendicular vectors, or equivalent merit.
- ✓ [3] Obtains the correct quadratic equation in λ, or equivalent merit.
- ✓ [4] Correct solution.



From the diagram,

- $\overrightarrow{PC}=-\lambda\underset{\sim}{i}+\frac{1}{2}\underset{\sim}{j}$
- $\overrightarrow{PA}=\frac{7}{2}\underset{\sim}{i}-\lambda\underset{\sim}{i}=\left(\frac{7}{2}-\lambda\right)\underset{\sim}{i}$
- In $\triangle PAB$,

$$\begin{aligned}
\overrightarrow{PA}+\overrightarrow{AB}&=\overrightarrow{PB}\\
\therefore\overrightarrow{PB}&=\left(\frac{7}{2}-\lambda\right)\underset{\sim}{i}+\frac{1}{2}\underset{\sim}{j}
\end{aligned}$$

As $PC\perp PB$, then

$$\begin{aligned}
\overrightarrow{PC}\cdot\overrightarrow{PB}&=0\\
\left(-\lambda\underset{\sim}{i}+\frac{1}{2}\underset{\sim}{j}\right)\cdot\left(\left(\frac{7}{2}-\lambda\right)\underset{\sim}{i}+\frac{1}{2}\underset{\sim}{j}\right)&=0\\
-\lambda\left(\frac{7}{2}-\lambda\right)+\frac{1}{2}\times\frac{1}{2}&=0\\
-\frac{7}{2}\lambda+\lambda^2+\frac{1}{4}&=0\\
4\lambda^2-14\lambda+1&=0
\end{aligned}$$

Applying the quadratic formula,

$$\begin{aligned}
\lambda&=\frac{-(14)\pm\sqrt{(-14)^2-4(4)(1)}}{2\times 4}\\
&=\frac{14\pm\sqrt{180}}{8}\\
&=\frac{14\pm 6\sqrt{5}}{8}\\
&=\frac{7\pm 3\sqrt{5}}{4}
\end{aligned}$$

(Both values of λ still fit within the restriction of $0<\lambda<\frac{7}{2}$ as $\overrightarrow{OA}=\frac{7}{2}\underset{\sim}{i}$)

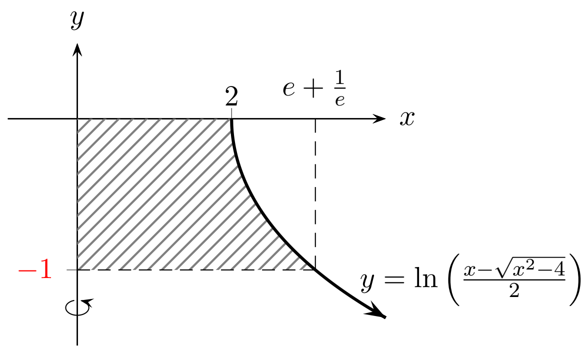
(b) i. (2 marks)

- ✓ [1] For significant progress to find x in terms of y in removing logarithmic and fractional parts, or equivalent merit.
- ✓ [2] For correct solution.

$$\begin{aligned}
y&=\ln\left(\frac{x-\sqrt{x^2-4}}{2}\right)\\
e^y&=\frac{x-\sqrt{x^2-4}}{2}\\
x-2e^y&=\sqrt{x^2-4}\\
(x-2e^y)^2&=x^2-4\\
\cancel{x^2}\underbrace{-4xe^y+4e^{2y}}_{\div 4e^y}&=\cancel{x^2}-\underbrace{4}_{\div 4e^y}\\
-x+e^y&=-e^{-y}\\
x&=e^y+e^{-y}
\end{aligned}$$

ii. (3 marks)

- ✓ [1] Correct identification of volume integral and limits, or equivalent merit.
- ✓ [2] Significant progress to finding the primitive, or equivalent merit.
- ✓ [3] Correct solution.

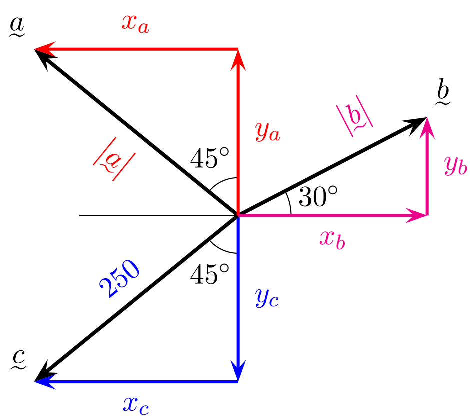


- If $x = e + \frac{1}{e}$, then in terms of the $x = e^y + e^{-y}$ expression, $y = \pm 1$.
- However the volume to be found has a negative y value as the lower limit. Hence $y = -1$ only.

$$\begin{aligned}
V &= \pi \int_a^b x^2\, dy \\
&= \pi \int_{-1}^{0} \left(e^y + e^{-y}\right)^2\, dy \\
&= \pi \int_{-1}^{0} \left(e^{2y} + 2 + e^{-2y}\right)\, dy \\
&= \pi \left[\frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y} + 2y\right]_{-1}^{0} \\
&= \pi \left[\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + 2(0)\right)\right. \\
&\qquad \left. - \left(\frac{e^{-2}}{2} - \frac{e^2}{2} + 2(-1)\right)\right] \\
&= \frac{\pi}{2}\left(e^2 - e^{-2} + 4\right)\ \text{units}^3
\end{aligned}$$

(c) i. (2 marks)

- ✓ [1] Shows one of the results for $\underset{\sim}{a}$ or $\underset{\sim}{b}$, or equivalent merit.
- ✓ [2] Correct solution.



Resolving $\underset{\sim}{a}$ into Cartesian components,

$$\begin{aligned}
x_a &= -\left|\underset{\sim}{a}\right| \sin 45^\circ \\
&= -\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \\
y_a &= \left|\underset{\sim}{a}\right| \cos 45^\circ \\
&= \frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| \\
\therefore \underset{\sim}{a} &= \left(-\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right|\right)\underset{\sim}{i} + \left(\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right|\right)\underset{\sim}{j}
\end{aligned}$$

Resolving $\underset{\sim}{b}$ into Cartesian components,

$$\begin{aligned}
x_b &= \left|\underset{\sim}{b}\right| \cos 30^\circ \\
&= \frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right| \\
y_b &= \left|\underset{\sim}{a}\right| \sin 30^\circ \\
&= \frac{1}{2}\left|\underset{\sim}{b}\right| \\
\therefore \underset{\sim}{b} &= \left(\frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{i} + \left(\frac{1}{2}\left|\underset{\sim}{b}\right|\right)\underset{\sim}{j}
\end{aligned}$$

ii. (4 marks)

- ✓ [1] Resolves $\underset{\sim}{c}$ into Cartesian components, then expresses the sum of forces to be zero due to the system being in equilibrium, or equivalent merit.
- ✓ [2] Correctly finds one of the x or y components' sum of forces, or equivalent.
- ✓ [3] Correctly finds both x or y components' sum of forces, or equivalent.
- ✓ [4] Correct solution.

Similarly from part (i),

$$\begin{aligned} x_c &= -250\cos 45^\circ \\ &= -\frac{250}{\sqrt{2}} \\ y_c &= 250\sin 45^\circ \\ &= \frac{250}{\sqrt{2}} \\ \therefore \underset{\sim}{c} &= -\frac{250}{\sqrt{2}}\underset{\sim}{i} - \frac{250}{\sqrt{2}}\underset{\sim}{j} \end{aligned}$$

As the system is in equilibrium,

$$\begin{cases} \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 0 \\ x_a + x_b + x_c = 0 \\ y_a + y_b + y_c = 0 \end{cases}$$

Sum of x components,

$$\begin{aligned} -\frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| + \frac{\sqrt{3}}{2}\left|\underset{\sim}{b}\right| - \frac{250}{\sqrt{2}} &= 0 \\ \frac{-2\left|\underset{\sim}{a}\right| + \sqrt{6}\left|\underset{\sim}{b}\right| - 500}{2\sqrt{2}} &= 0 \\ \therefore -2\left|\underset{\sim}{a}\right| + \sqrt{6}\left|\underset{\sim}{b}\right| - 500 &= 0 \quad (1) \end{aligned}$$

Sum of y components,

$$\begin{aligned} \frac{1}{\sqrt{2}}\left|\underset{\sim}{a}\right| + \frac{1}{2}\left|\underset{\sim}{b}\right| - \frac{250}{\sqrt{2}} &= 0 \\ \frac{2\left|\underset{\sim}{a}\right| + \sqrt{2}\left|\underset{\sim}{b}\right| - 500}{2\sqrt{2}} &= 0 \\ 2\left|\underset{\sim}{a}\right| + \sqrt{2}\left|\underset{\sim}{b}\right| - 500 &= 0 \qquad (2) \end{aligned}$$

(1) + (2):

$$\begin{aligned} \left(\sqrt{6}+\sqrt{2}\right)\left|\underset{\sim}{b}\right| - 1000 &= 0 \\ \left|\underset{\sim}{b}\right| &= \frac{1000}{\sqrt{6}+\sqrt{2}} \end{aligned}$$

Substitute into (2):

$$\begin{aligned} 2\left|\underset{\sim}{a}\right| + \sqrt{2}\left(\frac{1000}{\sqrt{6}+\sqrt{2}}\right) - 500 &= 0 \\ 2\left|\underset{\sim}{a}\right| + \frac{1000}{\sqrt{3}+1} &= 500 \\ \left|\underset{\sim}{a}\right| &= 250 - \frac{500}{\sqrt{3}+1} \end{aligned}$$

Question 13

(a) (Ham)

i. (1 mark)

$$\begin{aligned} &\frac{1}{(n+2)!}\frac{\times(n+3)}{\times(n+3)} - \frac{n+1}{(n+3)!} \\ &= \frac{(\cancel{n}+3)-(\cancel{n}+1)}{(n+3)!} \\ &= \frac{2}{(n+3)!} \end{aligned}$$

ii. (3 marks)

- ✓ [1] Correct proof of the base case.
- ✓ [2] Correct construction of the inductive hypothesis, and attempt to create the $(k+1)$-th proposition.
- ✓ [3] Correct solution.

Let $P(n)$ be the proposition

$$\frac{1\times 2}{3!} + \frac{2\times 2^2}{4!} + \frac{3\times 2^3}{5!} + \cdots + \frac{n\times 2^n}{(n+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$

1. Base case: $P(1)$

$$\begin{aligned} LHS: &\quad \frac{1\times 2}{3!} = \frac{1}{3} \\ RHS: &\quad 1 - \frac{2^2}{3!} = 1 - \frac{4}{6} \\ &\quad = \frac{1}{3} \end{aligned}$$

Hence $P(1)$ is true.

2. Inductive step: assume $P(k)$ is true for $1 \le k \le n$:

$$\frac{1\times 2}{3!} + \frac{2\times 2^2}{4!} + \frac{3\times 2^3}{5!} + \cdots + \frac{k\times 2^k}{(k+2)!} = 1 - \frac{2^{k+1}}{(k+2)!}$$

Examine $P(k+1)$:

$$\begin{aligned}
&\frac{1\times 2}{3!}+\frac{2\times 2^2}{4!}+\cdots+\frac{k\times 2^k}{(k+2)!}\\
&\quad+\frac{(k+1)\times 2^{k+1}}{\big((k+1)+2\big)!}\\
&=1-\frac{2^{k+1}}{(k+2)!}+\frac{(k+1)\times 2^{k+1}}{(k+3)!}\\
&=1-\frac{2^{k+1}(k+3)-(k+1)2^{k+1}}{(k+3)!}\\
&=1-\frac{2^{k+1}(2)}{(k+3)!}\\
&=1-\frac{2^{(k+1)+1}}{\big((k+1)+1\big)!}
\end{aligned}$$

Hence $P(k+1)$ is also true, and $P(n)$ is true by induction.

(b) (Ham)

i. (2 marks)

- ✓ [1] Obtains an unsimplified expression for $4\sin^4\theta+4\cos\theta$, or equivalent merit.
- ✓ [2] Correct solution.

$$\begin{cases}\cos 3\theta = 4\cos^3\theta-3\cos\theta & (1)\\ \sin 3\theta = 3\sin\theta-4\sin^3\theta & (2)\end{cases}$$

Change the subject of (1) to the cubic term,

$$\begin{aligned}
\underset{\times\cos\theta}{4\cos^3\theta} &= \underbrace{\cos 3\theta+3\cos\theta}_{\times\cos\theta}\\
4\cos^4\theta &= \cos 3\theta\cos\theta+3\cos^2\theta \qquad (\dagger)
\end{aligned}$$

Change the subject of (2) to the cubic term,

$$\begin{aligned}
\underset{\times\sin\theta}{4\sin^3\theta} &= \underbrace{3\sin\theta-\sin 3\theta}_{\times\sin\theta}\\
4\sin^4\theta &= 3\sin^2\theta-\sin 3\theta\sin\theta \qquad (\ddagger)
\end{aligned}$$

Adding ($\dagger$) with ($\ddagger$):

$$\begin{aligned}
&4\cos^4\theta+4\sin^4\theta\\
&=\cos 3\theta\cos\theta+3\cos^2\theta\\
&\qquad+3\sin^2\theta-\sin 3\theta\sin\theta\\
&=\cos 3\theta\cos\theta-\sin 3\theta\sin\theta\\
&\qquad+3\left(\cos^2\theta+\sin^2\theta\right)\\
&=\cos(3\theta+\theta)+3\\
&\therefore \cos^4\theta+\sin^4\theta=\frac{1}{4}(\cos 4\theta+3)
\end{aligned}$$

ii. (2 marks)

- ✓ [1] Correctly uses the substitution to transform the quartic into a solvable trigonometric equation, or equivalent merit.
- ✓ [2] Correct solution.

$$8x^4+8\left(1-x^2\right)^2=7$$

Let $x=\cos\theta$,

$$\begin{aligned}
8\cos^4\theta+8\left(1-\cos^2\theta\right)^2&=7\\
8\cos^4\theta+8\sin^4\theta&=7\\
\therefore \cos^4\theta+\sin^4\theta&=\frac{7}{8}
\end{aligned}$$

From (i), $\cos^4\theta+\sin^4\theta=\frac{1}{4}(\cos 4\theta+3)$:

$$\begin{aligned}
\frac{1}{4}(\cos 4\theta+3)&=\frac{7}{8}\\
\cos 4\theta+3&=\frac{7}{2}\\
\cos 4\theta&=\frac{1}{2}
\end{aligned}$$

iii. (3 marks)

✓ [1] Correctly finds values of θ for which $\cos 4\theta = \frac{1}{2}$.

✓ [2] Recognises some solutions to θ in $\cos 4\theta = \frac{1}{2}$ are the roots to the $8x^4 + 8\left(1-x^2\right)^2 = 7$

✓ [3] Correct solution.

$$\cos 4\theta = \frac{1}{2}$$

$$\begin{aligned}4\theta &= \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}\\ &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}\\ \theta &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}\end{aligned}$$

Solutions to $8x^4 + 8\left(1-x^2\right)^2 = 7$ are hence

$$\begin{aligned}x &= \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}\\ &= \pm\cos\frac{\pi}{12}, \pm\cos\frac{5\pi}{12}\end{aligned}$$

as $\cos\frac{11\pi}{12} = -\cos\frac{\pi}{12}$ and $\cos\frac{7\pi}{12} = -\cos\frac{5\pi}{12}$. Rearranging the quartic equation into general form:

$$\begin{aligned}8x^4 + 8\left(1-x^2\right)^2 &= 7\\ 8x^4 + 8\left(1 - 2x^2 + x^4\right) &= 7\\ 16x^4 - 16x^2 + 1 &= 0\end{aligned}$$

The product of roots to this quartic are:

$$\left(\cos\frac{\pi}{12}\right)\left(-\cos\frac{\pi}{12}\right)\left(\cos\frac{5\pi}{12}\right)\left(-\cos\frac{5\pi}{12}\right)$$

$$= \frac{e}{a} = \frac{1}{16}$$

$$\therefore \cos^2\frac{\pi}{12}\cos^2\frac{5\pi}{12} = \frac{1}{16}$$

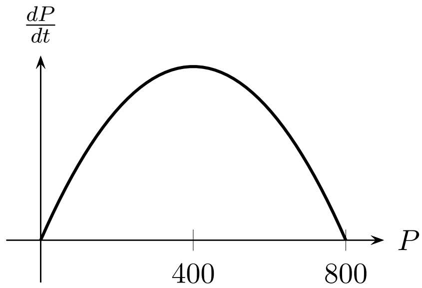
(c) (Ho)

i. (2 marks)

✓ [1] Locates the value of P and r for where the maximum occurs, or equivalent merit.

✓ [2] Correct solution.

$\frac{dP}{dt} = rP(800-P)$ is a quadratic in terms of P:



$\frac{dP}{dt}$ is at a maximum when $P = 400$. Hence

$$\frac{dP}{dt} = r \times 400(800-400) = 16\,000r$$

When ideal conditions occur, the population increase by 100 possums (200 parents $\times\frac{1}{2} = 100$)

$$\begin{aligned}100 &= 16\,000r\\ r &= \frac{1}{1600}\end{aligned}$$

ii. (3 marks)

✓ [1] Correct separates variables and uses the given expression to transform one of the integrands, or equivalent merit.

✓ [2] Finds the population equation in terms of time, with or without the correct constant of integration.

✓ [3] Correct solution.

$$\begin{aligned}\frac{dP}{dt} &= \frac{P}{1\,600}(800-P)\\ \frac{dP}{P(800-P)} &= \frac{dt}{1\,600}\end{aligned}$$

Integrating, and using the identity given

$$\begin{aligned}\frac{1}{800}\int\left(\frac{1}{P} + \frac{1}{800-P}\right)\,dP &= \int\frac{1}{1\,600}\,dt\\ \ln|P| - \ln|800-P| &= \frac{1}{2}t + C\\ \ln\left|\frac{P}{800-P}\right| &= \frac{1}{2}t + C\\ \left|\frac{P}{800-P}\right| &= e^{\frac{1}{2}t+C} = Ae^{\frac{1}{2}t}\end{aligned}$$

As $0 < P < 800$ from (i), the absolute value is not needed. Hence

$$\begin{aligned}
\frac{P}{800-P} &= Ae^{\frac{1}{2}t}\\
P &= 800Ae^{\frac{1}{2}t} - PAe^{\frac{1}{2}t}\\
P\left(1+Ae^{\frac{1}{2}t}\right) &= 800Ae^{\frac{1}{2}t}\\
P &= \frac{800Ae^{\frac{1}{2}t}}{1+Ae^{\frac{1}{2}t}} \frac{\div Ae^{\frac{1}{2}t}}{\div Ae^{\frac{1}{2}t}}\\
&= \frac{800}{\frac{1}{A}e^{-\frac{1}{2}t}+1}\\
&= \frac{800}{1+Be^{-\frac{1}{2}t}}
\end{aligned}$$

When $t = 0$, $P = 100$:

$$\begin{aligned}
100 &= \frac{800}{Be^0+1}\\
B+1 &= 8\\
B &= 7\\
\therefore P &= \frac{800}{1+7e^{-\frac{1}{2}t}}
\end{aligned}$$

iii. (1 mark)
Max sustainable population is 800 possums.

$$\begin{aligned}
P &= 0.85\times 800 = 680\\
680 &= \frac{800}{1+7e^{-\frac{1}{2}t}}\\
1+7e^{-\frac{1}{2}t} &= \frac{20}{17}\\
7e^{-\frac{1}{2}t} &= \frac{3}{17}\\
e^{-\frac{1}{2}t} &= \frac{3}{119}\\
-\frac{1}{2}t &= \ln\frac{3}{119}\\
t &= 2\ln\frac{119}{3} \approx 7.36\cdots \text{ years}
\end{aligned}$$

Question 14(Lee)

(a) i. (2 marks)

- ✓ [1] Correctly differentiates V with respect to h, or equivalent merit.
- ✓ [2] Correct solution.

$$\begin{aligned}
V &= \pi\left(Rh^2 - \frac{h^3}{3}\right)\\
\frac{dV}{dh} &= \pi\left(2Rh - \frac{1}{\cancel{3}}\times\cancel{3}h^2\right)\\
&= \pi\left(2Rh-h^2\right)
\end{aligned}$$

As $\dfrac{dV}{dt} = \dfrac{dV}{dh}\times\dfrac{dh}{dt}$,

$$\begin{aligned}
k(2R-h) &= \pi\left(2Rh-h^2\right)\times\frac{dh}{dt}\\
\frac{dh}{dt} &= \frac{k\cancel{(2R-h)}}{\pi h\cancel{(2R-h)}} = \frac{k}{\pi h}
\end{aligned}$$

ii. (2 marks)

- ✓ [1] Correct separation of variable to find an equation for t in terms of h, or equivalent merit.
- ✓ [2] Correct solution.

$$\begin{aligned}
\frac{dh}{dt} &= \frac{k}{\pi h}\\
h\,dh &= \frac{k}{\pi}\,dt
\end{aligned}$$

Integrating,

$$\begin{aligned}
\int h\,dh &= \frac{k}{\pi}\int dt\\
\frac{k}{\pi}t &= \frac{1}{2}h^2 + C\\
t &= \frac{\pi h^2}{2k} + C_2
\end{aligned}$$

When $t = 0$, $h = 0$ as the tank is initially empty:

$$\begin{aligned}
0 &= \frac{\pi}{2k}0^2 + C_2\\
\therefore C_2 &= 0
\end{aligned}$$

When the tank is full, $t = T$ and $h = R$:

$$T = \frac{\pi}{2k}R^2$$

iii. (3 marks)

- ✓ [1] Correctly calculates the related rate of change, and the correct $\frac{dV}{dt}$ to calculate a new $t-h$ equation, without the constant of integration evaluated.
- ✓ [2] Further progress to evaluate the correct constant of integration.
- ✓ [3] Correct solution.

When the tank is full, the rate of change of volume is altered to $\frac{dV}{dt} = -kh$ only as the $2kR$ term is the rate of inflow, which is stopped.

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh} \times \frac{dh}{dt} \\ -kh &= \pi h(2R-h) \times \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{k}{\pi(2R-h)}\end{aligned}$$

Separating variables

$$\begin{aligned}\int (2R-h)\,dh &= -\frac{k}{\pi}\int\,dt \\ -\frac{k}{\pi}t &= 2Rh - \frac{1}{2}h^2 + C_3 \\ t &= -\frac{\pi}{k}\left(2Rh - \frac{1}{2}h^2\right) + C_4\end{aligned}$$

When $t = 0$, $h = R$:

$$\begin{aligned}0 &= -\frac{\pi}{k}\left(2R^2 - \frac{1}{2}R^2\right) + C_4 \\ 0 &= -\frac{\pi}{k}\left(\frac{3}{2}R^2\right) + C_4 \\ C_4 &= \frac{3\pi}{2k}R^2 \\ \therefore t &= -\frac{\pi}{k}\left(2Rh - \frac{h^2}{2}\right) + \frac{3\pi}{2k}R^2\end{aligned}$$

Let the time when the water is emptied out be T_2. When $t = T_2$, $h = 0$:

$$\begin{aligned}T_2 &= \cancel{-\frac{\pi}{k}\left(2R(0) - \frac{0}{2}\right)} + \frac{3\pi}{2k}R^2 \\ &= 3\left(\frac{\pi}{2k}R^2\right) \\ &= 3T\end{aligned}$$

(b) (4 marks)

- ✓ [1] Correctly obtains the equation for $|\dot{r}(t)|^2$, or equivalent merit.
- ✓ [2] Finds the quarter of the time of flight and substitutes into $|\dot{r}(t)|^2$.
- ✓ [3] Significant progress to simplifying the expression obtained.
- ✓ [4] Correct solution.

$$\begin{aligned}\underset{\sim}{r}(t) &= \begin{pmatrix} Vt\cos\theta \\ Vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix} \\ \dot{\underset{\sim}{r}}(t) &= \begin{pmatrix} V\cos\theta \\ V\sin\theta - gt \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\left|\dot{\underset{\sim}{r}}(t)\right|^2 &= (V\cos\theta)^2 + (V\sin\theta - gt)^2 \\ &= V^2\cos^2\theta + V^2\sin^2\theta - 2Vgt\sin\theta + g^2t^2 \\ &= V^2\left(\cos^2\theta + \sin^2\theta\right) - 2Vgt\sin\theta + g^2t^2 \\ &= g^2t^2 - 2Vgt\sin\theta + V^2\end{aligned}$$

When $t = \frac{2V\sin\theta}{g} \times \frac{1}{4}$, $\left|\dot{\underset{\sim}{r}}(t)\right| = \frac{\sqrt{7}}{4}V$, i.e.

$$\begin{aligned}\left|\dot{\underset{\sim}{r}}\left(\frac{V\sin\theta}{2g}\right)\right|^2 &= \frac{7}{16}V^2 \\ \frac{7}{16}V^2 &= g^2\left(\frac{V\sin\theta}{2g}\right)^2 - 2Vg\sin\theta\left(\frac{V\sin\theta}{2g}\right) + V^2 \\ \frac{7}{16}\cancel{V^2} &= \cancel{g^2}\left(\frac{\cancel{V^2}\sin^2\theta}{4\cancel{g^2}}\right) - \cancel{V^2}\sin^2\theta + \cancelto{1}{V^2} \\ \frac{7}{16} &= -\frac{3}{4}\sin^2\theta + 1 \\ \frac{3}{4}\sin^2\theta &= \frac{9}{16} \\ \sin^2\theta &= \frac{3}{4} \\ \sin\theta &= \pm\frac{\sqrt{3}}{2}\end{aligned}$$

As $0^\circ < \theta < 90^\circ$,

$$\theta = 60^\circ \text{ only}$$