Student No:	

Mathematics Extension 1

2025 HSC – Year 12, Assessment Task 3, 2025

General Instruction:

- Reading time – 10 minutes
- Working time – 120 minutes
- Write using black/blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks:

Section I – 10 marks (page 1-3)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 4–9)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Class Teacher (please tick one name)

- ◯ Mr Berry
- ◯ Ms Cai
- ◯ Mr Ireland
- ◯ Ms Lee
- ◯ Ms Moss
- ◯ Mr Umakanthan
- ◯ Dr Vranešević

Question No.	1-10	11	12	13	14	Total	%
Marks	/10	/15	/15	/15	/15	/70	/100

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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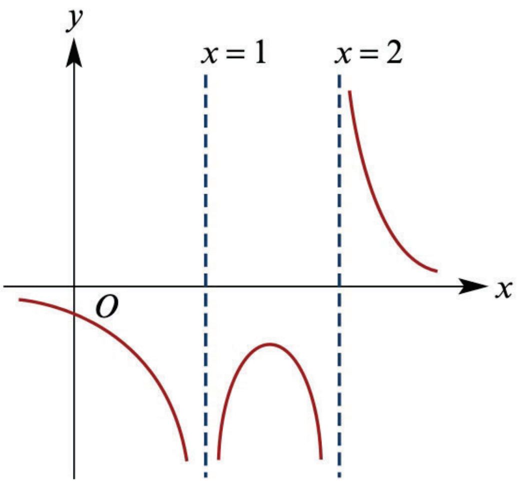
1. Consider the graph on the right. Which function does the graph represent?

A) $y = \dfrac{1}{(x-1)(x-2)}$

B) $y = \dfrac{x}{(x-1)(x-2)^2}$

C) $y = \dfrac{(x-1)(x-2)}{x}$

D) $y = \dfrac{1}{(x-1)^2(x-2)}$



2. Which of the following is equivalent to the expression $\sqrt{\dfrac{4+4\cos 2x}{1-\cos 2x}}$?

A) $2\cot^2 x$

B) $2\cot x$

C) $2\tan x$

D) $2\tan^2 x$

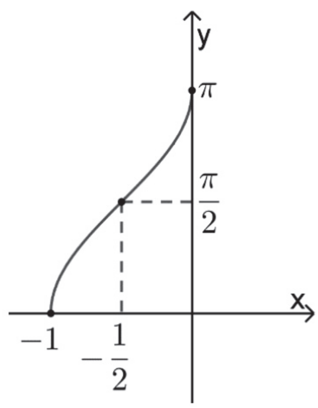
3. The diagram shows the graph of a function. Which function does the graph represent?

A) $y = \dfrac{\pi}{2} - \sin^{-1}(2x-1)$

B) $y = \dfrac{\pi}{2} + \sin^{-1}(2x+1)$

C) $y = \cos^{-1}(2x+1)$

D) $y = \cos^{-1}(2x-1)$



NOT TO SCALE

4. In a class of 15 students, there are 9 boys and 6 girls. The class elects four representatives for the student council. How many different groups of representatives are possible?

   A) 1365

   B) 3240

   C) 5005

   D) 32760

5. Let α , β and γ be the roots of $3x^3 + 8x - 1 = 0$. What is the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$?

   A) $-\dfrac{8}{3}$

   B) $\dfrac{1}{3}$

   C) $\dfrac{1}{8}$

   D) 8

6. What is the coefficient of x^8 in the expansion of $\left(2x^3 - \dfrac{1}{x}\right)^{12}$?

   A) $-{}^{12}C_8 \times 2^4$

   B) ${}^{12}C_8 \times 2^4$

   C) ${}^{12}C_8 \times 2^5$

   D) $-{}^{12}C_7 \times 2^5$

7. What is the exact value of $\displaystyle\int_{-1}^{1} \sqrt{4 - x^2}\,dx$? Use the substitution $x = 2\sin\theta$.

   A) $\dfrac{\pi}{6} + \dfrac{\sqrt{3}}{4}$

   B) $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$

   C) $\dfrac{2\pi}{3} + \sqrt{3}$

   D) $\dfrac{4\pi}{3} + 2\sqrt{3}$

8. The projection of vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ onto vector $\mathbf{i} + \mathbf{j}$ is

   A) $\dfrac{1}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

   B) $\mathbf{i} - \dfrac{1}{2}\mathbf{j}$

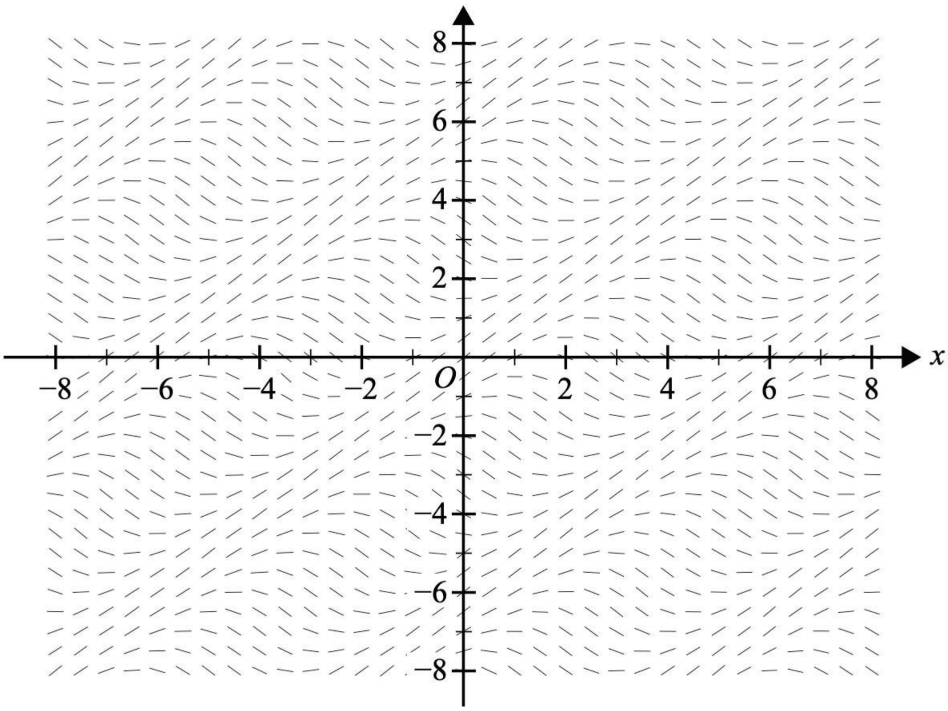
   C) $\dfrac{1}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j}$

   D) $-\dfrac{1}{2}\mathbf{i} + \mathbf{j}$

9. A particle is projected from a window 9 metres above the horizontal ground, at an angle of θ above the horizontal, where $\tan\theta = \dfrac{3}{4}$ and the initial velocity is 20 metres/second. What is the maximum height above the ground reached the particle? Equation of motions are $\ddot{y} = -10$ and $\dot{y} = -10t + V\sin\theta$.

   A) 6.5 metres
   B) 7.2 metres
   C) 15.5 metres
   D) 16.2 metres

10. The differential equation that has the diagram below as its direction field is



   A) $\dfrac{dy}{dx} = \sin(y - x)$
   B) $\dfrac{dy}{dx} = \cos(y - x)$
   C) $\dfrac{dy}{dx} = \dfrac{1}{\cos(y - x)}$
   D) $\dfrac{dy}{dx} = \dfrac{1}{\sin(y - x)}$

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

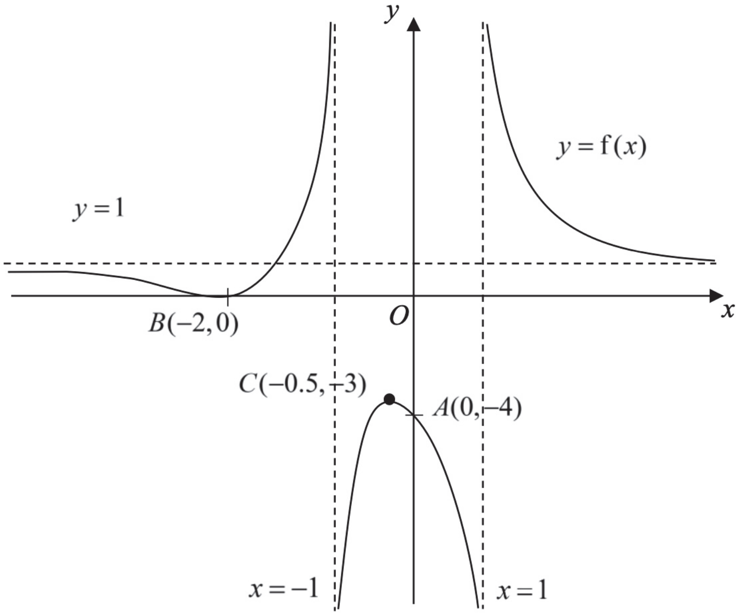
Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11. (15 marks) Use the Question 11 Writing Booklet

(a) What is the solution to the inequality $\dfrac{3}{x(2x-1)} \geq 1$? **3**

(b) The diagram below shows the graph of $f(x)$. It cuts the y–axis at $A(0,-4)$, has a local minimum at $B(-2,0)$ and a local maximum at $C(-0.5,-3)$. The equation of the horizontal asymptote is $y = 1$, and the vertical asymptotes are $x = -1$ and $x = 1$.



Sketch, on a separate diagram, the graph of

$$y^2 = f(x),$$ **3**

stating clearly the equations of all asymptotes, coordinates of any points of intersection with both axes and the coordinates of the points corresponding to A, B and C (if any and when possible).

(c) Consider the vectors given by $\mathbf{a} = m\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{b} = \hat{\mathbf{i}} + m\hat{\mathbf{j}}$, where $m \in \mathbb{R}$. Find the value(s) of m if the acute angle between $\mathbf{a}$ and $\mathbf{b}$ is 60° **2**

Question 11 continues on next page

(d) The letters from the word VOLUME are placed at random on the circumference of a circle. Each letter is used only once.

(i) Find the number of different arrangements that can be formed. **1**

If one of the arrangements is chosen at random, find the probability that

(ii) all the vowels will be together, **2**

(iii) the vowels and consonants will alternate. **1**

(e) Express y as a function of x, by solving the differential equation **3**

$$\sqrt{2-x^2}\,\frac{dy}{dx} = \frac{1}{2-y}, \quad \text{given } y(1) = 0.$$

Question 12. (15 marks) Use the Question 12 Writing Booklet

(a) Evaluate

$$\int_0^{\frac{\pi}{12}} \sin^2 3x\,dx$$ **2**

(b) (i) Use t-formulae to show $\dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$ **1**

(ii) Hence, find the exact value of $\tan 15^\circ$ **1**

(c) Show that

(i)

$$5 - 4e^x - e^{2x} = 3^2 - (e^x + 2)^2,$$ **1**

(ii) hence, find

$$\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$$ **1**

(d) What is the solution to the equation $5\left(\cos\left(x+\tan^{-1}\dfrac{4}{3}\right)\right) = 0.5$ for $0 \le x \le 2\pi$, and $0 < \alpha < \dfrac{\pi}{2}$, where $\alpha = \tan^{-1}\dfrac{4}{3}$? Leave your answer correct to 2 d.p. **2**

(e) Two tugboats are applying simultaneous forces to a container ship as it attempts to come in to port. One tugboat is applying a force of 3500 N in a westerly direction and the second tugboat is applying a 2400 N force directly north. What is the magnitude and direction of the resultant force being applied to the container ship? **2**

(f) An ingot of aluminium is placed in a room with a surrounding air temperature of 25°C and allowed to cool. It loses heat according to Newton's law of cooling, $\dfrac{dT}{dt} = -k(T-A)$ where T is the temperature of the metal in degrees Celsius at time t minutes, A is the surrounding air temperature and k is a positive constant. The initial temperature of the aluminium is 1350°C. After 10 minutes the temperature of the aluminium is 720°C.

(i) Show that $T = A + Be^{-kt}$, where B is a constant, is a solution of $\dfrac{dT}{dt} = -k(T-A)$. **1**

(ii) Find the value of k. **2**

(iii) What total time elapses for the aluminium to cool to 50°C? **2**

Question 13. (15 marks) Use the Question 13 Writing Booklet

(a) (i) Graph, on the same axes: $y = \sec x$ and $y = \tan x$ for $0 \leq x < \frac{\pi}{2}$ **2**

(ii) The area bounded by the positive y-axis, the curves $y = \sec x$ and $y = \tan x$, and the lines $x = 0$, $x = N$, where $0 < N < \dfrac{\pi}{2}$, is rotated about the x-axis. Find the volume of revolution in terms of N. **2**

(iii) Find the volume as N approaches $\dfrac{\pi}{2}$. **1**

(b) In a certain country, the population of deer was estimated in 1990 to be 150 000. The population growth is given by the logistic equation

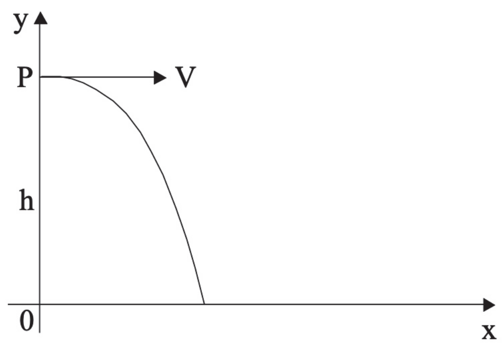
$$\frac{dP}{dt} = 0.1P\left(\frac{C-P}{C}\right)$$

where t is the number of years after1990 and C is the carrying capacity. In the year 2020, the population of deer was estimated to be 900 000. Prove that

(i) $\dfrac{C}{P(C-P)} = \dfrac{1}{P} + \dfrac{1}{C-P}$ **1**

(ii) to show that the carrying capacity is approximately 1 220 000. **4**

(c) A particle is projected horizontally from a point P, h metres above O, with a velocity of V metres per second. The equations of motion of the particle are $\ddot{x} = 0$ and $\ddot{y} = -g$ with the position of the particle at time t given by $x = Vt$ and $y = h - \dfrac{1}{2}gt^2$ (**do not prove these**).



A canistor containing a life raft is dropped from a plane to a stranded sailor. The plane is travelling at a constant velocity of 216 km/h, at a height of 120 metres above sea level, along a path that passes above the sailor.

(i) How long will the canister take to hit the water? (Take g = 10 m/s^2.) **2**

(ii) A current is causing the sailor to drift at a speed of 3.6 km/h in the same direction as the plane is travelling. The canister is dropped from the plane when the horizontal distance from the plane to the sailor is D metres. What values can D take if the canister lands at most 50 metres from the stranded sailor? **3**

Question 14. (15 marks) Use the Question 14 Writing Booklet

(a) (i) By considering the sum of the terms of an arithmetic series, show that

$$(1+2+3+\ldots+n)^2=\frac{1}{4}n^2(n+1)^2.$$ **1**

(ii) By using the Principle of Mathematical Induction prove that

$$1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2,\ \text{for all } n\geq 1.$$ **4**

(b) N monic quadratic trinomials are given,

$$y=x^2+a_1x+b_1,$$
$$y=x^2+a_2x+b_2,$$
$$y=x^2+a_3x+b_3,$$
$$\ldots$$
$$y=x^2+a_nx+b_n,$$

with their zeros x_0 and x_1, x_0 and x_2, x_0 and x_3, ... , x_0 and x_n, respectively, where

$x_0, x_1, x_2, \ldots, x_n \in \mathbb{R}$.

For the quadratic trinomial

$$y=x^2+\frac{a_1+a_2+a_3+\ldots+a_n}{n}x+\frac{b_1+b_2+b_3+\ldots+b_n}{n}$$

(i) prove that x_0 is one of the zeros, **2**

(ii) hence, find the other zero. **2**

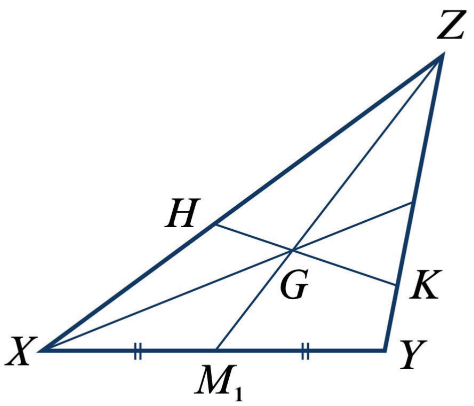
Question 14 continues on next page

(c) Let points 0, A, B and C be points on the same plane and let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. Assume that $\mathbf{a}$ and $\mathbf{b}$ are not parallel. If points A, B and C are collinear and

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \text{ where } \alpha, \beta \in \mathbb{R}.$$

(i) show that $\alpha + \beta = 1$. **2**

In the figure, the point G is the centroid of a triangle (i.e. the point where the lines joining each vertex to the midpoint of the opposite side meet).
A line passing through G meets ZX and ZY at points H and K respectively, with $ZH = hZX$ and $ZK = kZY$.



(ii) Prove that **3**

$$\overrightarrow{ZG} = \frac{2}{3}\overrightarrow{ZM_1},$$

by firstly proving that centroid G divides each of segments $\overrightarrow{ZM_1}$, $\overrightarrow{XM_2}$ and $\overrightarrow{YM_3}$ in the ratio 2 : 1, that is:

$$\overrightarrow{ZG} : \overrightarrow{GM_1} = 2:1,$$
$$\overrightarrow{XG} : \overrightarrow{GM_2} = 2:1,$$
$$\overrightarrow{YG} : \overrightarrow{GM_3} = 2:1,$$

where M_1, M_2, M_3 are the midpoint of the opposite sides to vertices Z, X, Y, respectively.

(iii) Using the results of (i) and (ii), find the value of **1**

$$\frac{1}{h} + \frac{1}{k}$$

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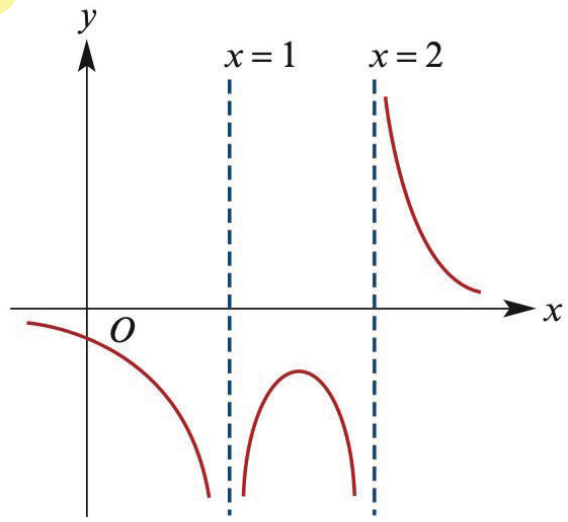
End of Examination

Q1



look at:

* $(x-1)^n(x-2)^m$, $n, m = 1, 2$
* its reciprocal
* sign

as sign change around $x=2$, this must be single zero

for guide curve $(x-1)^n(x-2)$, $\therefore m=1$

* sign around $x=1$ does not change $\therefore x=1$ is double zero $\therefore (x-1)^2(x-2)$ is guided curve

$\therefore$ D

Q2

$$\sqrt{\frac{4+4\cos 2x}{1-\cos 2x}} = 2\sqrt{\frac{1+\cos 2x}{1-\cos 2x}}$$

$$= 2\sqrt{\frac{2\cos^2 x}{2\sin^2 x}} \qquad \left(\begin{array}{l}\cos 2x = 2\cos^2 x - 1 \\ \cos 2x = 1 - 2\sin^2 x\end{array}\right)$$

$$= 2\sqrt{\cot^2 x} = 2\cot x \qquad \therefore \text{B}$$

Q3 $y = \frac{\pi}{2} + \sin^{-1}(2x+1)$ $\therefore$ B

Q4 $^{15}C_4 = 1365$ $\therefore$ A

Q5 $3x^3 + 8x - 1 = 0$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = ?$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$$

$$\alpha + \beta + \gamma = -\frac{0}{3} = 0$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{8}{3}$$

$$\alpha\beta\gamma = -\frac{-1}{3} = \frac{1}{3}$$

$$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{8/3}{1/3} = 8$$

$\therefore$ D

Q6 $\left(2x^3 - \frac{1}{x}\right)^{12} = \sum_{r=0}^{12} \left(2x^3\right)^{12-r}\left(-\frac{1}{x}\right)^r$

$T_{r+1} = {}^{12}C_r \times 2^{12-r} \times x^{36-3r} \times (-1)^r (x)^{-r}$

$= {}^{12}C_r \times 2^{12-r} \times x^{36-4r} \times (-1)^r$

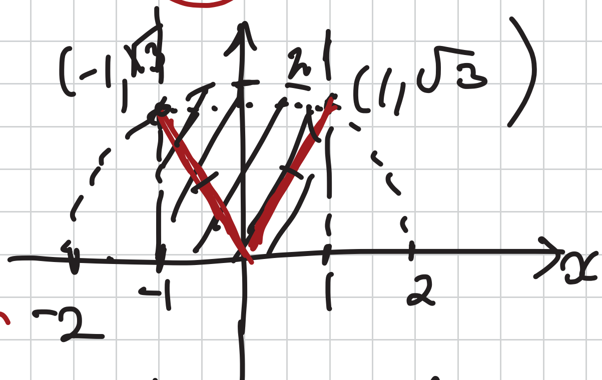
term with a coefficient of x^8

$36-4r=8 \Rightarrow 4r=28 \Rightarrow r=7$

$T_8 = {}^{12}C_7 \times (-1)^7 \times 2^{12-7} \times x^8 = -{}^{12}C_7 \times 2^5 x^8$

$\therefore$ coeff is $-{}^{12}C_7\, 2^5$ $\therefore$ D

Q7 $\int_{-1}^{1} \sqrt{4-x^2}\, dx$



using substitution

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$x=-1$, $-1 = 2\sin\theta$

$\therefore \sin\theta = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6}$

$x = 1$, $1 = 2\sin\theta$

$\therefore \sin x = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

or using graph:

$A_1 = 2 \times \frac{1}{2} \times 1 \times \sqrt{3}$

$= \sqrt{3}$

$A_2 = \frac{1}{2}\, 2^2 \times \frac{\pi}{3}$

$= \frac{2\pi}{3}$

$\therefore A = \frac{2\pi}{3} + \sqrt{3}$

$$\int_{-1}^{1}\sqrt{4-x^2}\,dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\sqrt{4-4\sin^2\theta}\times 2\cos\theta\,d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(2\cos\theta)^2\,d\theta$$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos^2\theta\,d\theta$$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\left(1+\frac{1}{2}\cos 2\theta\right)dx$$

$$= 2\left[x+\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= 2\left[\left(\frac{\pi}{6}+\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}-\frac{1}{2}\sin\left(-\frac{\pi}{3}\right)\right)\right]$$

$$= 2\left[\frac{2\pi}{6}+\frac{1}{2}\times\frac{\sqrt{3}}{2}+\frac{1}{2}\frac{\sqrt{3}}{2}\right]$$

$$= \frac{2\pi}{3}+\sqrt{3} \quad \therefore \quad c$$

Q8 $\underset{\sim}{u} = 2\hat{\underset{\sim}{i}} - \hat{\underset{\sim}{j}}$

$\hat{\underset{\sim}{i}} + \hat{\underset{\sim}{j}}$

$$\operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}$$

$$\operatorname{proj}_{\underset{\sim}{i}+\underset{\sim}{j}} (2\underset{\sim}{i} - \underset{\sim}{j}) = \frac{(2i - j) \cdot (i + j)}{(i+j) \cdot (i+j)} (i + j)$$

$$= \frac{2 - 1}{1 + 1} (i + j)$$

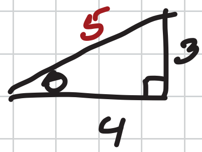
$$= \frac{1}{2} (\hat{\underset{\sim}{i}} + \hat{\underset{\sim}{j}})$$

$$= \frac{1}{2} i + \frac{1}{2} \underset{\sim}{j}$$

∴ A

Q9 $\ddot{y} = -10$

$\tan\theta = \frac{3}{4}$



$\dot{y} = -10t + v\sin\theta$

$\therefore \sin = \frac{3}{5}$

when $t=0$, $v = 20 \Rightarrow$

$\cos = \frac{4}{5}$

$\Rightarrow v\sin\theta = 20 \times \frac{3}{5} = 12$

$\therefore \dot{y} = -10t + 12$

$\therefore y = -5t^2 + 12t + c$, when $t=0$, $y=9$

$\therefore 9 = c$ $\quad \therefore y = -5t^2 + 12t + 9$

Max height occurs when $\dot{y} = 0$

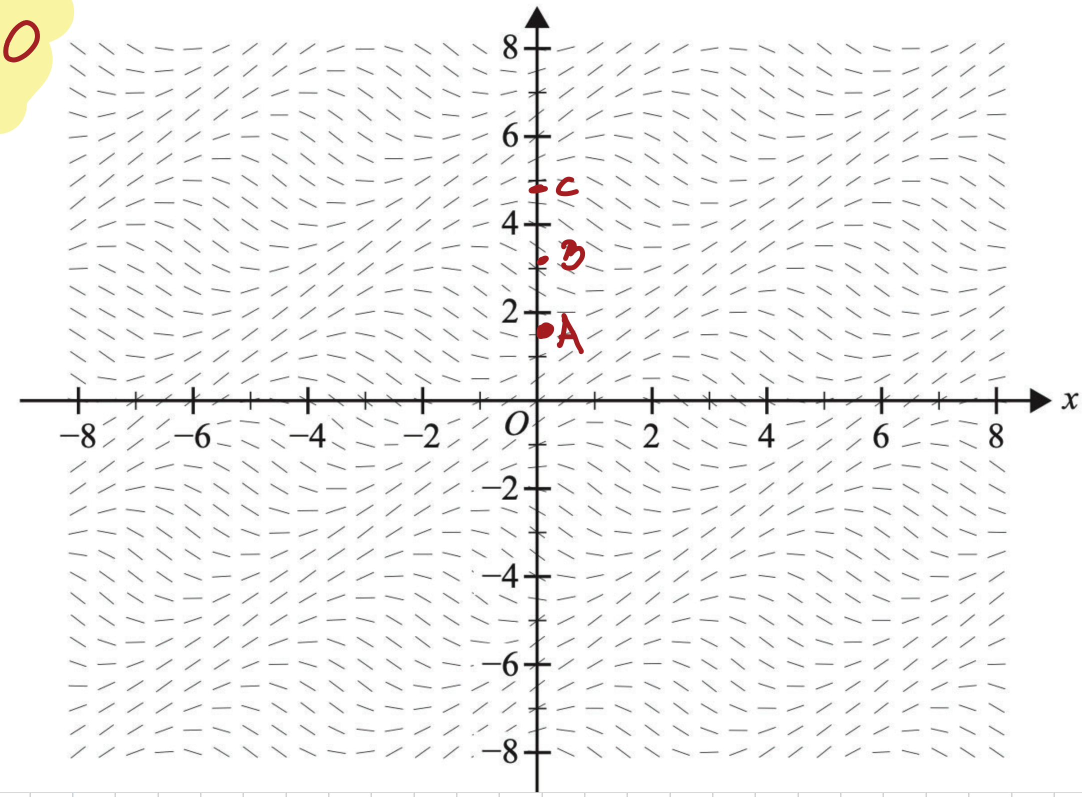
$\therefore 0 = -10t + 12 \Rightarrow t = \frac{12}{10} = \frac{6}{5}$ (time of max.)

$\therefore y = -5\left(\frac{6}{5}\right)^2 + 12 \times \frac{6}{5} + 9$

$= 16.2\text{m}$

$\therefore$ D

Q10



* slope depends only on the difference y-x, not on x and y individually ⇒ the pattern that repeats or shift is based on the value of y-x
* the repeated pattern has a period of 2π along lines where $y=x$ = const. ⇒ A & D can't be solutions
* when $y-x=0$ $(y=x)$, $\cos(0)=1$ ⇒ slope is 1

Q10 continues

when $y-x=\frac{\pi}{2}$ (point $A(0,\frac{\pi}{2})$ on graph),

$\cos\frac{\pi}{2}=0 \Rightarrow$ slope is $0 \Rightarrow$ C) can't be solution

when $y-x=\pi$ (e.g. point $B(0,\pi)$),

$\cos\pi=-1 \Rightarrow$ slope -1.

when $y-x=\frac{3\pi}{2}$ (point $C(0,\frac{3\pi}{2})$),

$\cos\frac{3\pi}{2}=0 \Rightarrow$ slope again 0

$\therefore$ ⊛ The pattern repeats every 2π, with slopes transitioning smoothly between -1 and 1

⊛ The direction field shows oscillatory behavior along lines parallel to $y=x$ (where $y-x=$ const.)

* along the line $y=x$, slope is 1
* along the line $y=x+\frac{\pi}{2}$, the slope is 0
* along the line $y=x+\pi$, the slope is -1

* this creates a wave-like pattern that shifts as you move perpendicular to the line $y=x$.

∴ MC 10 B

1 D
2 B
3 B
4 A
5 D
6 D
7 C
8 A
9 D
10 B

Q11

a)

$$\frac{3}{x(2x-1)} \geq 1 \qquad x \neq 0 \text{ or } x \neq \frac{1}{2}$$

$$x^2(2x-1)^2 \times \frac{3}{x(2x-1)} \geq 1 \times x^2(2x-1)^2$$

$$3x(2x-1) \geq x^2(2x-1)^2 \qquad x \neq 0 \text{ and } x \neq 2$$

$$3x(2x-1) - x^2(2x-1)^2 \geq 0$$

$$x(2x-1)[3 - x(2x-1)] \geq 0$$

$$x(2x-1)(2x^2 - x - 3)] \leq 0$$

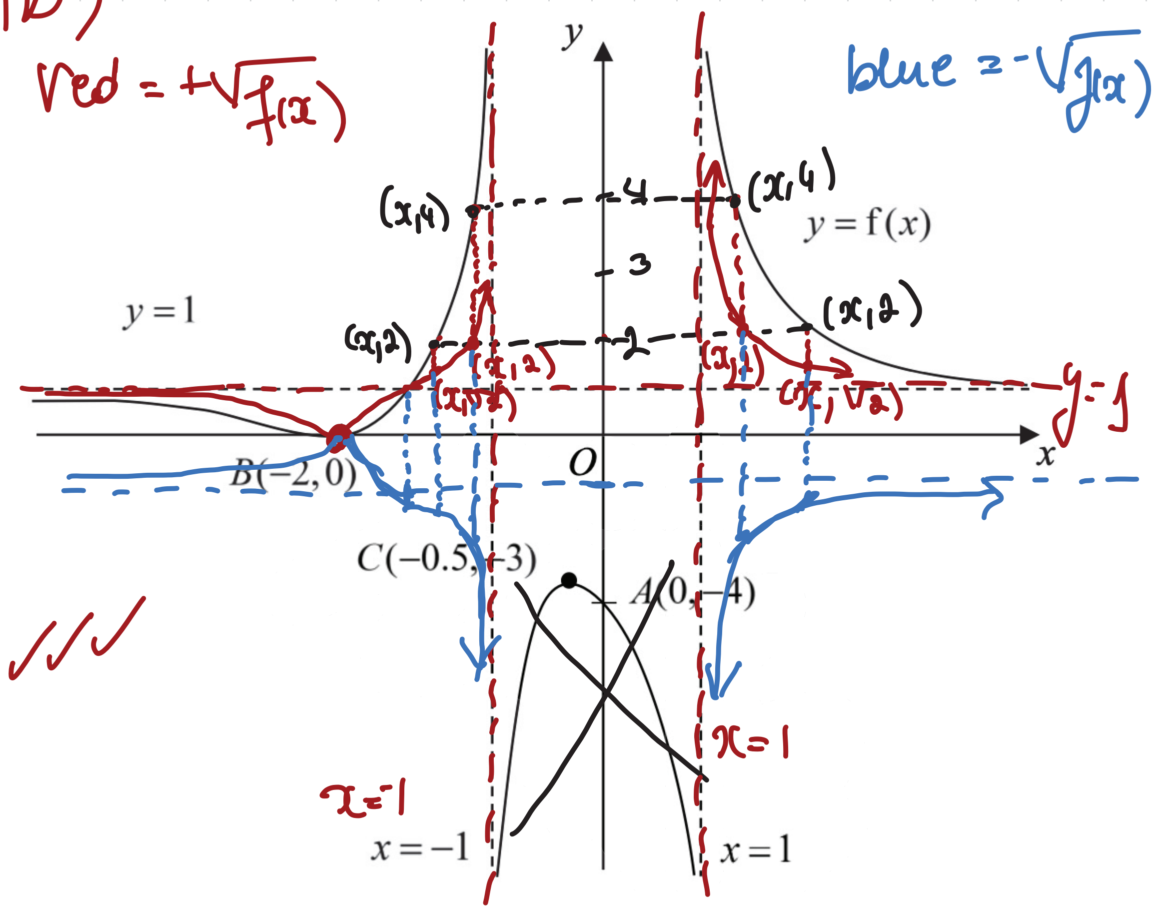
$$x(2x-1)(2x-3)(x+1) \leq 0$$

Critical points are $-1, 0, \frac{1}{2}$ and $\frac{3}{2}$

Test values in each region

$-1 \leq x < 0$ and $\frac{1}{2} < x \leq \frac{3}{2}$ wrong boundaries mark off

11b)



* delete all for $f(x) < 0$
* for $0 < f(x) < 1 \Rightarrow +\sqrt{f(x)}$ closer to $y=1$
* $y=1$ stays the asymptote
* $B(-2,0)$ stays the point on both $\pm\sqrt{f(x)}$
* points: $(x,2) \to (x,\sqrt{2})$
  for both $(x,4) \to (x,2)$
  $x>0$ & $x<0$
* $x=-1$ & 1 stays
* $-\sqrt{f(x)}$ reflect red curve around x-axis

7)(c) $\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ $\quad m \in \mathbb{R}$

$\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}$ $\quad m = ?$

$\measuredangle(\underset{\sim}{a}, b) = 60^\circ$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = m + m = 2m$$

$$|\underset{\sim}{a}| = \sqrt{m^2+1} \qquad |\underset{\sim}{b}| = \sqrt{1+m^2}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos 60^\circ$$

$$2m = \left(\sqrt{1+m^2}\right)^2 \frac{1}{2}$$

$$4m = 1+m^2$$

$$m^2 - 4m + 1 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 1}}{2}$$

$$m = \frac{4 \pm \sqrt{12}}{2}$$

$$m = 2 \pm \sqrt{3}$$ ✓✓

11 d)

(i) Number of Different Arrangements

For a circular arrangement of n distinct items, the number of distinct arrangements (accounting for rotational symmetry) is $(n-1)!$. Here, the word "VOLUME" has 6 distinct letters: V, O, L, U, M, E.

- Number of letters: $n = 6$.
- Number of distinct circular arrangements: $(6-1)! = 5! = 120$.

Thus, the number of different arrangements is **120**.

(ii) Probability that All Vowels Are Together

To find the probability that all vowels are together, we first count the number of favorable arrangements where all vowels (O, E, U) are adjacent, then divide by the total number of arrangements.

- **Identify vowels and consonants**:
  - Vowels: O, E, U (3 vowels).
  - Consonants: V, L, M (3 consonants).
- **Treat vowels as a single "block"**:
  - Group the 3 vowels (O, E, U) into one super-unit (a "vowel block").
  - This reduces the problem to arranging 4 units in a circle: the vowel block, V, L, and M.
  - Number of circular arrangements of 4 distinct units: $(4-1)! = 3! = 6$.
- **Arrange the vowels within the block**:
  - The 3 vowels (O, E, U) can be arranged among themselves in $3! = 6$ ways.
- **Total favorable arrangements**:
  - Total number of arrangements where vowels are together: $6 \times 6 = 36$.
- **Probability**:
  - Total arrangements: 120 (from part (i)).
  - Probability = Favorable arrangements / Total arrangements = $\frac{36}{120} = \frac{3}{10}$.

Thus, the probability that all vowels are together is $\frac{3}{10}$.

12d)

(iii) Probability that Vowels and Consonants Alternate

For vowels and consonants to alternate in a circular arrangement, the letters must be placed such that each vowel is followed by a consonant and vice versa. Since there are 3 vowels (O, E, U) and 3 consonants (V, L, M), we can have two possible patterns in a circle of 6 positions:

- **Pattern 1**: Vowel, Consonant, Vowel, Consonant, Vowel, Consonant (V-C-V-C-V-C).
- **Pattern 2**: Consonant, Vowel, Consonant, Vowel, Consonant, Vowel (C-V-C-V-C-V).

These two patterns are distinct in a circular arrangement because rotating one does not produce the other (e.g., V-C-V-C-V-C cannot be rotated to C-V-C-V-C-V).

- **Calculate arrangements for Pattern 1 (V-C-V-C-V-C)**:
  - Arrange the 3 vowels in the 3 vowel positions (1st, 3rd, 5th).
    - Number of ways to arrange O, E, U: $3! = 6$.
  - Arrange the 3 consonants in the 3 consonant positions (2nd, 4th, 6th).
    - Number of ways to arrange V, L, M: $3! = 6$.
  - Total arrangements for Pattern 1: $6 \times 6 = 36$.
- **Calculate arrangements for Pattern 2 (C-V-C-V-C-V)**:
  - Arrange the 3 consonants in the 3 consonant positions (1st, 3rd, 5th): $3! = 6$.
  - Arrange the 3 vowels in the 3 vowel positions (2nd, 4th, 6th): $3! = 6$.
  - Total arrangements for Pattern 2: $6 \times 6 = 36$.
- **Total favorable arrangements**:
  - Since the two patterns are mutually exclusive, add the arrangements:
    - Total = $36 + 36 = 72$.
- **Probability**:
  - Total arrangements: 120 (from part (i)).
  - Probability = Favorable arrangements / Total arrangements = $\frac{72}{120} = \frac{3}{5}$.

Thus, the probability that vowels and consonants alternate is $\frac{3}{5}$.

Final Answer

(i) Number of different arrangements: **120**.

(ii) Probability that all vowels are together: $\frac{3}{10}$.

(iii) Probability that vowels and consonants alternate: $\frac{3}{5}$.

11c)

11e)

$$\sqrt{2-x^2}\,\frac{dy}{dx}=\frac{1}{2-y}$$

$$(2-y)dy=\frac{dx}{\sqrt{2-x^2}}$$

$$\int(2-y)dy=\int\frac{dx}{\sqrt{2-x^2}}$$

$$2y-\frac{y^2}{2}=\sin^{-1}\frac{x}{\sqrt{2}}+C$$

$y(1)=0:\qquad 0=\sin^{-1}\frac{1}{\sqrt{2}}+C$

$$C=-\frac{\pi}{4}$$

$$y^2-4y+4=\frac{\pi}{2}-2\sin^{-1}\frac{x}{\sqrt{2}}+4$$

$$(y-2)^2=4+\frac{\pi}{2}-2\sin^{-1}\frac{x}{\sqrt{2}}$$

$$y-2=\pm\sqrt{4+\frac{\pi}{2}-2\sin^{-1}\frac{x}{\sqrt{2}}}$$

$$y=2\pm\sqrt{4+\frac{\pi}{2}-2\sin^{-1}\frac{x}{\sqrt{2}}}$$

But $y(1)=0$ so $y=2-\sqrt{4+\frac{\pi}{2}-2\sin^{-1}\frac{x}{\sqrt{2}}}$

12a) $\int_0^{\frac{\pi}{12}} \sin^2 3x \, dx =$

(using $\sin^2 a = \frac{1}{2}(1-\cos 2a)$)

$= \frac{1}{2}\int_0^{\pi/12} (1-\cos 6x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{12}}$

$= \frac{1}{2}\left[\frac{\pi}{12} - \frac{1}{6}\sin 6\times\frac{\pi}{12} - 0 + \frac{1}{6}\sin 0\right]$

$= \frac{\pi}{24} - \frac{1}{6}$

Q12b)

(i) $\text{LHS} = \dfrac{1-\cos\theta}{\sin\theta}$

$= \dfrac{1 - \dfrac{1-t^2}{1+t^2}}{\dfrac{2t}{1+t^2}}$

$= \dfrac{1+t^2-1+t^2}{2t}$

$= \dfrac{2t^2}{2t}$

$= t$

$= \tan\dfrac{\theta}{2}$

$= \text{RHS}$

(ii) $\tan 15^\circ = \dfrac{1-\cos 30^\circ}{\sin 30^\circ}$

$= \dfrac{1-\dfrac{\sqrt{3}}{2}}{\dfrac{1}{2}}$

$= 2-\sqrt{3}$

Q12c) i) $LHS = -(e^{2x} + 4e^{x} + 4) + 9$

$= 3^2 - (e^{x}+2)^2 = RHS$

ii) $\int \frac{e^{x}}{\sqrt{3^2-(e^{x}+2)}}\,dx = \int \frac{e^{x}}{\sqrt{1-\left(\frac{e^{x}+2}{3}\right)^2}}\,dx$

$= \sin^{-1}\left(\frac{e^{x}+2}{3}\right) + C$

12 d)

$$5(\cos(x+\tan^{-1}\frac{4}{3})) = 0.5$$

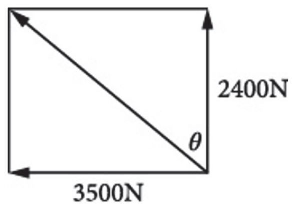
$$\cos(x+\tan^{-1}\frac{4}{3}) = 0.1$$

$$x+\tan^{-1}\frac{4}{3} = 1.470628906,\ (2\pi - 1.470628906)$$

$$x = 0.543336876,\ 3.885261184$$

$$= 0.54,\ 3.89$$

12e) **(d)**



Resultant force $= \sqrt{3500^2 + 2400^2} = 100\sqrt{1801} = 4244$ N

$\theta = \tan^{-1}\left(\frac{3500}{2400}\right) = \tan^{-1}\left(\frac{35}{24}\right) = 55°34'$

Resultant force is 4244 N in the direction N55°34′W

12 j)

i) $T = A + Be^{-kt}$ satisfies the equation $\frac{dT}{dt} = -k(T - A)$

i) $\frac{dT}{dt} = -kBe^{-kt}$

$= -k(T-A)$

ii) $A = 25$ (surrounding air temperature). Also $t = 0$ and $T = 1350$

$T = 20 + Be^{-kt}$

$1350 = 25 + Be^{-k\times 0}$

$B = 1325$

Also $t = 10$ and $T = 720$

$720 = 25 + 1325e^{-k\times 10}$

$e^{-10k} = \frac{695}{1325}$

$-10k = \log_e \frac{139}{265}$

$k = -\frac{1}{10}\log_e \frac{139}{265} = 0.064525589...$

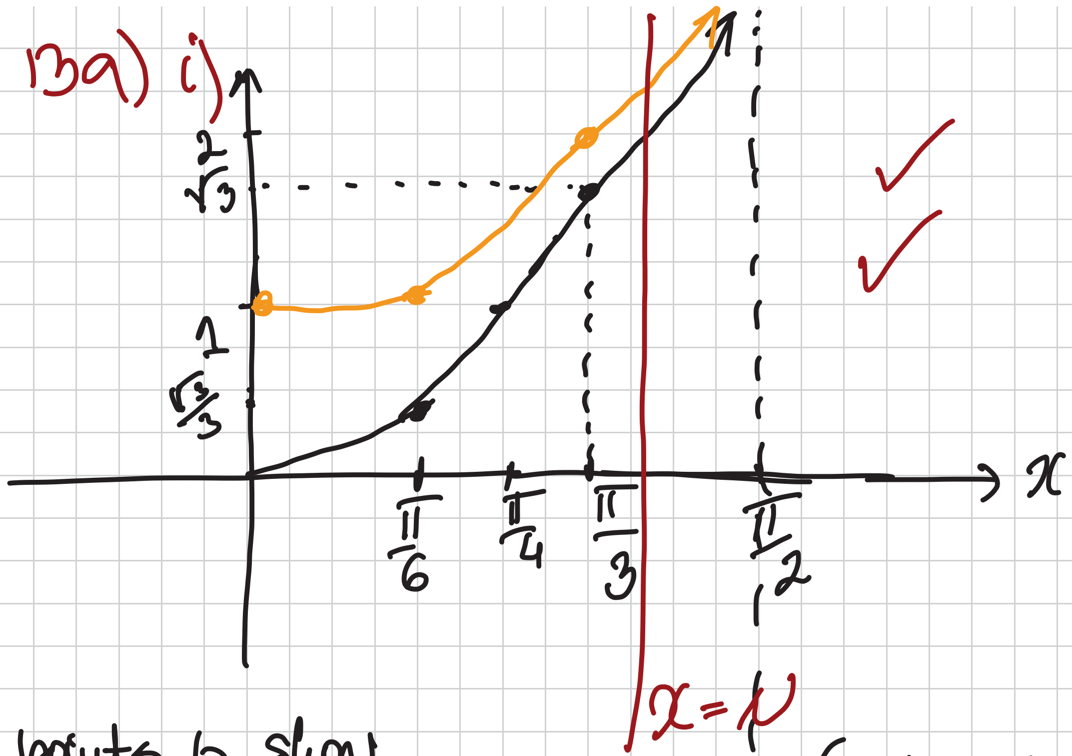
iii) $50 = 25 + 1325e^{-0.064525589...\times t}$

$e^{-0.064525589...\times t} = \frac{25}{1325}$

$-0.064525589...\times t = \log_e \frac{25}{1325}$

$t = \log_e \frac{25}{1325} \div -0.064525589... = 61.53$ minutes

13a) i)



points to show

for $\tan x$: $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{3}\right), \left(\frac{\pi}{4}, 1\right), \left(\frac{\pi}{3}, \sqrt{3}\right)$

$\tan x \to \infty$, for $x = \frac{\pi}{2}$

for $\sec x$: $\left(\frac{\pi}{6}, \frac{2}{\sqrt{3}}\right), \left(\frac{\pi}{4}, \frac{2}{\sqrt{2}}\right), \left(\frac{\pi}{3}, 2\right)$

$\sec x \to \infty$, for $x = \frac{\pi}{2}$

$\tan x$ & $\sec x$ don't intersect

ii) $$V = \pi \int_0^N (y_1^2 - y_2^2)\,dx$$

$$= \pi \int_0^N (\sec^2 x - \tan^2 x)\,dx$$

Q 13a) ii $V = \pi \int_0^N 1 \, dx$

$= \pi x \Big|_0^N$ ✓

$= \pi N$ units^3

iii) $\lim_{N \to \frac{\pi}{2}} V = \frac{\pi^2}{2}$ units^3 ✓

13.6)

i) $\dfrac{C}{P(C-P)} = \dfrac{1}{P} + \dfrac{1}{C-P}$

$RHS = \dfrac{1}{P} + \dfrac{1}{C-P}$

$= \dfrac{C \cancel{-P} \cancel{+P}}{P(C-P)}$

$= \dfrac{C}{P(C-P)} = LHS$ ✓

ii) $t=0$ in 1990, $P_0 = 150000$

$\dfrac{dP}{dt} = 0.1P\left(\dfrac{C-P}{C}\right)$

$\dfrac{dP}{dt} = \dfrac{P}{10}\left(\dfrac{C-P}{C}\right)$

$\dfrac{dt}{dP} = \dfrac{10C}{P(C-P)}$

$\int dt = \int 10 \dfrac{C}{P(C-P)} \, dP$

$t = 10 \int \left(\dfrac{1}{P} + \dfrac{1}{C-P}\right) dP$ ✓

$= 10 \left[\ln|P| - \ln|C-P| \right] + C$

$t = 10 \ln\left(\frac{P}{C-P}\right) + c \quad , \text{as } C, P > 0$

and $C > P$

when $t = 0$, $P = 150\,000$

$0 = 10 \ln\left(\frac{150\,000}{C - 150000}\right) + c$

$\therefore c = -10 \ln\left(\frac{150000}{C-150000}\right) = 10 \ln\left(\frac{C - 150\,000}{150\,000}\right)$

$\therefore t = 10 \ln\left(\frac{P}{C-P}\right) + \ln\left(\frac{C - 150\,000}{150\,000}\right)$

when $t = 30$, $P = 900\,000$

$30 = 10 \ln\left(\frac{900\,000}{C - 900\,000}\right) + 10 \ln\left(\frac{C - 150000}{150000}\right)$

$3 = \ln\left[\frac{90(C - 150000)}{15(C - 900000)}\right]$

$e^3 = \frac{6C - 900000}{C - 900\,000}$

$e^3 C - e^3 \times 900000 = 6C - 900\,000$

$C(e^3 - 6) = 900000\,(e^3 - 1)$

$C = 900000 \, \frac{e^3 - 1}{e^3 - 6} = \frac{19.08553692}{14.08553692}$

$$C = 900000 \times 1.354974044$$

$$C = 1219476.639$$

$$C \approx 1220000$$

Q13 c) i)

(ii) $V = 216 \text{ km/h} = \frac{216\,000}{3600} \text{ m/s} = 60 \text{ m/s}$.

$h = 120 \text{ m}$, $g = 10 \text{ m/s}^2$.

Canister hits the water when $y = 0$.

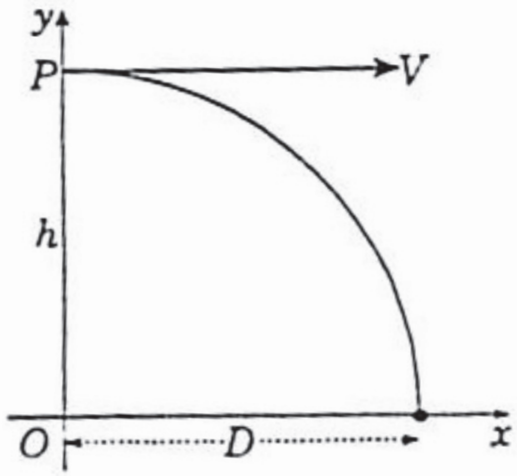
$\therefore \quad 0 = 120 - 5t^2$

$\therefore \quad t^2 = 24$

$\therefore \quad t = \sqrt{24} = 2\sqrt{6} \text{ s} \quad (t \neq -2\sqrt{6})$.

$\therefore$ It hits the water after $2\sqrt{6}$ seconds.

Q13(c)(ii)



Sailor drifts at speed of 3·6 km/h

$$= \frac{3 \cdot 6 \times 1000}{3600} \text{ m/s}$$

$$= 1 \text{ m/s.}$$

$d = 1 \times t = 2\sqrt{6}$

$\therefore$ Sailor drifts a distance of $2\sqrt{6}$ metres ($= 4 \cdot 9$ m) in the time it takes for the canister to hit the water.

Now, when $t = 2\sqrt{6}$, $x = Vt = 60 \times 2\sqrt{6}$

$\therefore$ $x = 120\sqrt{6}$ m. ($= 293 \cdot 94$ m)

Without drifting,

$$120\sqrt{6} - 50 \leq D \leq 120\sqrt{6} + 50$$

$\therefore$ *With* drifting,

$$118\sqrt{6} - 50 \leq D \leq 118\sqrt{6} + 50.$$

That is, D can take approx. values between 239 m and 339 m.

Q14 a) i) $1+2+3+4+\cdots+n$ arithmetic series

$\therefore S = \frac{n}{2}(a+l)$, where $a=1$, $l=n$

$\therefore S = \frac{n}{2}(1+n)$

$\therefore (1+2+3+\cdots+n)^2 = S^2$

$= \left[\frac{n}{2}(1+n)\right]^2$

$= \frac{n^2}{4}(1+n)^2$

$= \frac{1}{4}n^2(1+n)^2$ ✓

ii)

Step 1: Base Case

Check the statement for the smallest value, $n = 1$: ✓

- Left side: $1^3 = 1$
- Right side: $(1+2+\cdots+1)^2 = 1^2 = 1$ Since $1 = 1$, the base case holds.

Step 2: Inductive Hypothesis

Assume the statement is true for some positive integer $k \geq 1$. That is, assume:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1+2+3+\cdots+k)^2$$

We know the sum $1+2+\cdots+k = \frac{k(k+1)}{2}$, so the right side can be written as:

$$\left(\frac{k(k+1)}{2}\right)^2$$

4 a) ii)

Step 3: Inductive Step

Prove the statement for $n = k + 1$:

- Left side: $1^3 + 2^3 + \cdots + k^3 + (k+1)^3$
- Using the inductive hypothesis, this is:

$$(1 + 2 + \cdots + k)^2 + (k+1)^3$$

- Substitute the sum $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$:

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

- Right side for $n = k + 1$:

$$(1 + 2 + \cdots + (k+1))^2$$

The sum $1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}$, so:

$$\left(\frac{(k+1)(k+2)}{2}\right)^2$$

✓

Step 4: Show Equality

We need to show:

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

- Compute the left side:

$$\left(\frac{k(k+1)}{2}\right)^2 = \frac{k^2(k+1)^2}{4}$$

$$(k+1)^3 = (k+1) \cdot (k+1)^2$$

So the left side is:

$$\frac{k^2(k+1)^2}{4} + (k+1)^3$$

Factor out $(k+1)^2$:

✓

$$(k+1)^2 \left(\frac{k^2}{4} + (k+1)\right)$$

14 a) ii)

Simplify the expression inside:

$$\frac{k^2}{4} + (k+1) = \frac{k^2}{4} + \frac{4(k+1)}{4} = \frac{k^2+4k+4}{4} = \frac{(k+2)^2}{4}$$

Thus:

$$(k+1)^2 \cdot \frac{(k+2)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

- The right side is:

$$\left(\frac{(k+1)(k+2)}{2}\right)^2 = \frac{(k+1)^2(k+2)^2}{4}$$

✓

Both sides are equal:

$$\frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

✓

Step 5: Conclusion

The base case holds, and the inductive step shows that if the statement is true for $n = k$, it is true for $n = k+1$. By the Principle of Mathematical Induction, the statement $1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2$ is true for all $n \geq 1$.

14 b)(i)

$y = x^2 + a_1 x + b_1$, with zeros x_0 & $x_1 \Rightarrow 0 = x_0^2 + a_1 x_0 + b_1$ (1)

$y = x^2 + a_2 x + b_2$, with zeros x_0 & $x_2 \Rightarrow 0 = x_0^2 + a_2 x_0 + b_2$ (2)

$\vdots$

$y = x^2 + a_n x + b_n$, with zeros x_0 & $x_n \Rightarrow 0 = x_0^2 + a_n x_0 + b_n$ (3)

(1) + (2) + (3):

$0 = n x_0^2 + (a_1 + a_2 + \dots + a_n) x_0 + b_1 + b_2 + \dots + b_n$ ✓

$0 = x_0^2 + \frac{1}{n}(a_1 + a_2 + \dots + a_n) x_0 + \frac{1}{n}(b_1 + b_2 \dots + b_n)$

$\therefore$ x_0 is one zero of

$y = x^2 + \frac{a_1 + a_2 + \dots + a_n}{n} x + \frac{1}{n}(b_1 + b_2 + \dots + b_n)$ ✓

(ii) $\therefore$ the other zero must be real as well, as $\Delta > 0 \Rightarrow$ the other zero $\tilde{x}$

Using relation for the sum of roots for all polynomials

$x_0 + x_1 = -a_1$ (1)

$x_0 + x_2 = -a_2$ (2)

$\vdots$

$x_0 + x_n = -a_n$ (n) ✓

$x_0 + \tilde{x} = -\frac{1}{n}(a_1 + a_2 + \dots + a_n) = \frac{1}{n}(-a_1 - a_2 - \dots - a_n)$

$\therefore x_0 + \tilde{x} = \frac{1}{n}(x_0 + x_1 + x_0 + x_2 + \dots + x_0 + x_n)$

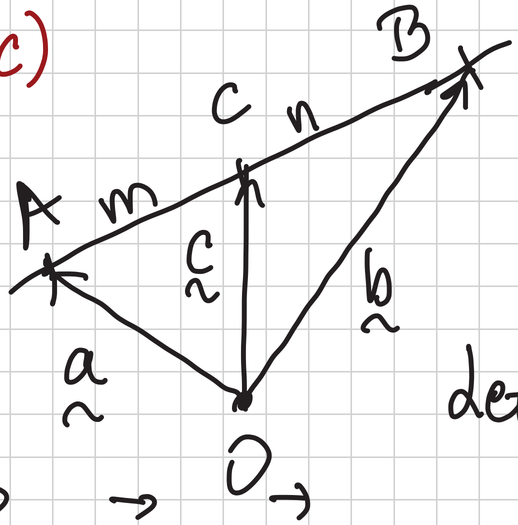
(from (1), (2) ... (n))

$x_0 + \tilde{x} = \frac{1}{n} \times n x_0 + \frac{1}{n}(x_1 + x_2 + \dots + x_n)$

$\therefore \tilde{x} = \frac{1}{n}(x_1 + x_2 + x_3 + \dots + x_n)$ ✓

14 c)

(i)



$\underset{\sim}{a} = \vec{OA}$

$\underset{\sim}{b} = \vec{OB}$

$\underset{\sim}{c} = \vec{OC}$

$\underset{\sim}{a} \nparallel \underset{\sim}{b}$

let $\vec{AC} : \vec{CB} = m : n$

$\therefore \vec{AC} = \dfrac{m}{m+n}$

$m, n \in \mathbb{R}$

$$\vec{OC} = \vec{OA} + \vec{AC}$$

$$\underset{\sim}{c} = \underset{\sim}{a} + \frac{m}{m+n}\vec{AB}$$

$$= \underset{\sim}{a} + \frac{m}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$$

$$= \underset{\sim}{a}\left(1 - \frac{m}{m+n}\right) + \frac{m}{m+n}\underset{\sim}{b} \quad ✓$$

$$= \frac{n}{m+n}\underset{\sim}{a} + \frac{m}{m+n}\underset{\sim}{b}$$

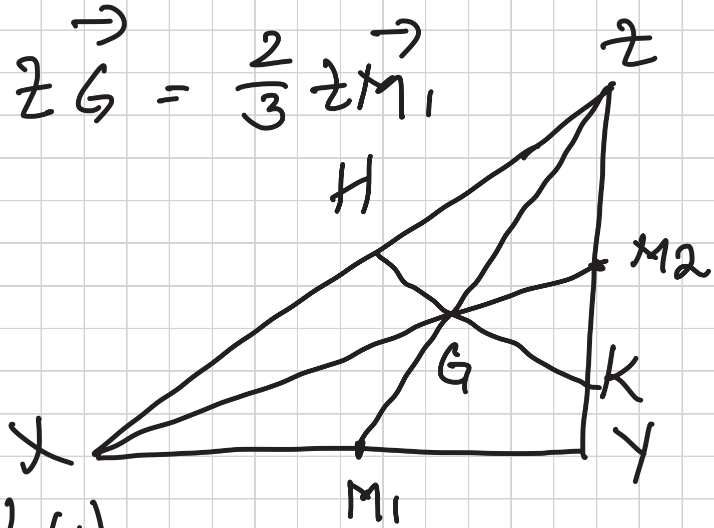
$$\therefore \underset{\sim}{c} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{b}, \quad \lambda = \frac{n}{m+n}, \quad \mu = \frac{m}{m+n}$$

$$\therefore \lambda + \mu = \frac{n}{m+n} + \frac{m}{m+n}$$

$$= 1 \quad \square \quad ✓$$

(4c)

ii) to prove $\vec{ZG} = \frac{2}{3}\vec{ZM_1}$



let;

$\vec{XY} = \underset{\sim}{z}$

$\vec{YZ} = \underset{\sim}{x}$

$\vec{ZX} = \underset{\sim}{y}$

$$\left.\begin{aligned}\vec{ZG} &= \gamma \vec{ZM_1}\\ \vec{XG} &= \delta \vec{XM_2}\end{aligned}\right\} (1)$$

need to prove that $\gamma = \delta = \frac{2}{3}$

$$\begin{aligned}\vec{ZG} &= \vec{ZX} + \vec{XG}\\ &= \underset{\sim}{y} + \delta \vec{XM_2}\\ &= \underset{\sim}{y} + \delta\left(\vec{XY} + \tfrac{1}{2}\vec{YZ}\right) = \underset{\sim}{y} + \delta\left(\underset{\sim}{z} + \tfrac{1}{2}\underset{\sim}{x}\right) \quad (2)\end{aligned}$$

$$\begin{aligned}\vec{ZM_1} &= \vec{ZX} + \vec{XM_1}\\ &= \underset{\sim}{y} + \tfrac{1}{2}\underset{\sim}{z} \quad (3)\end{aligned}$$

from (1) $\rightarrow$ $\vec{ZG} = \gamma \vec{ZM_1} \Rightarrow$

$\vec{ZG} - \gamma \vec{ZM_1} = \vec{0}$

from (2)&(3): $\therefore \underset{\sim}{y} + \delta\left(\underset{\sim}{z} + \tfrac{1}{2}\underset{\sim}{x}\right) - \gamma\left(\underset{\sim}{y} + \tfrac{1}{2}\underset{\sim}{z}\right) = \vec{0}$ (4)

then $z\vec{X} + x\vec{Y} = z\vec{Y}$

$\therefore \quad \underset{\frown}{y} + \underset{\sim}{z} = -\underset{\frown}{x} \Rightarrow \underset{\sim}{x} = -\underset{\frown}{y} - \underset{\sim}{z} \quad (5)$

$\therefore$ (4) becomes

$$\underset{\frown}{y} + \delta\left(\underset{\sim}{z} - \frac{1}{2}\underset{\frown}{y} - \frac{1}{2}\underset{\sim}{z}\right) - \gamma\left(\underset{\frown}{y} + \frac{1}{2}\underset{\sim}{z}\right) = \vec{0}$$

$$\underset{\frown}{y}\left(1 - \frac{\delta}{2} - \gamma\right) - \underset{\sim}{z}\left(\frac{1}{2}\delta - \frac{1}{2}\gamma\right) = 0$$

$$\underset{\frown}{y}\left(1 - \frac{\delta}{2} - \gamma\right) = \underset{\frown}{z}\left(\frac{1}{2}\delta - \frac{1}{2}\gamma\right)$$

$\therefore \underset{\frown}{y}$ & $\underset{\frown}{z}$ are not collinear

$\therefore \left(1 - \frac{\delta}{2} - \gamma\right) = 0$ and $\frac{1}{2}\delta - \frac{1}{2}\gamma = 0$

$\Downarrow \qquad\qquad \Downarrow$

$\delta = \gamma$

$1 - \frac{\gamma}{2} - \gamma = 0$

$-\frac{3\gamma}{2} = -1$

$\therefore \gamma = \frac{2}{3} \Rightarrow \delta = \frac{2}{3}$

14 c) (iii)

$\overrightarrow{ZG} = \alpha \overrightarrow{ZH} + \beta \overrightarrow{ZK} = \frac{2}{3} \overrightarrow{ZM}$, using (ii)

$\overrightarrow{ZG} = \frac{2}{3}\left(\frac{1}{2} \overrightarrow{ZX} + \frac{1}{2} \overrightarrow{ZY}\right)$

$= \frac{2}{3}\left(\frac{1}{2} \frac{1}{h} \overrightarrow{ZH} + \frac{1}{2} \frac{1}{k} \overrightarrow{ZK}\right)$

$= \frac{1}{3h} \overrightarrow{ZH} + \frac{1}{3k} \overrightarrow{ZK}$

$\therefore \overrightarrow{ZG} = \frac{1}{3h} \overrightarrow{ZH} + \frac{1}{3k} \overrightarrow{ZK}$

using (i)

$\therefore \frac{1}{3h} + \frac{1}{3k} = 1$ ✓

$\therefore \frac{1}{h} + \frac{1}{k} = 3$