North Sydney Girls High School

2025 **HSC TRIAL EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading Time – 10 minutes
- Working Time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 9 – 15)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

NAME:____________________ **TEACHER:**____________________

STUDENT NUMBER:

Question	1-10	11	12	13	14	Total
Mark	/10	/15	/15	/16	/14	/70

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1 What is the remainder when $P(x) = x^3 - 2x^2 + x + 5$ is divided by $(x-2)$?

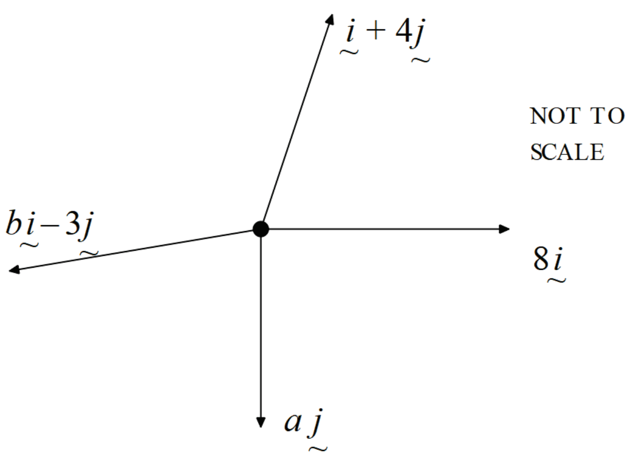
A. -13

B. 5

C. 7

D. 8

2 A particle has four ropes attached to it. The ropes are being used to pull the particle in four directions, as shown in the diagram.



If the particle is in equilibrium while the ropes are being pulled, what are the values of a and b?

A. $a = -9,\ b = -8$

B. $a = -9,\ b = -1$

C. $a = -1,\ b = -9$

D. $a = -1,\ b = 9$

3 What is the range of $y = \pi + \tan^{-1}(x)$?

A. All real y

B. $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

C. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

D. $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$

4 How many arrangements in a row of the letters of the word SECRETS begin and end with E?

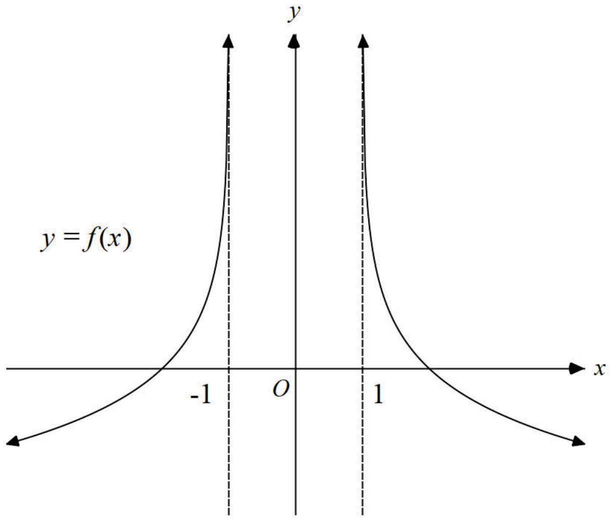
A. $\frac{7!}{2! \times 2!}$

B. $\frac{7!}{2!}$

C. $\frac{5!}{2! \times 2!}$

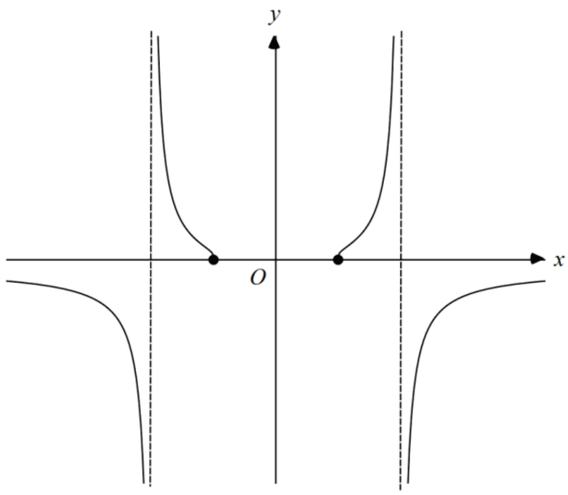
D. $\frac{5!}{2!}$

5 The graph of $y = f(x)$ is shown below.

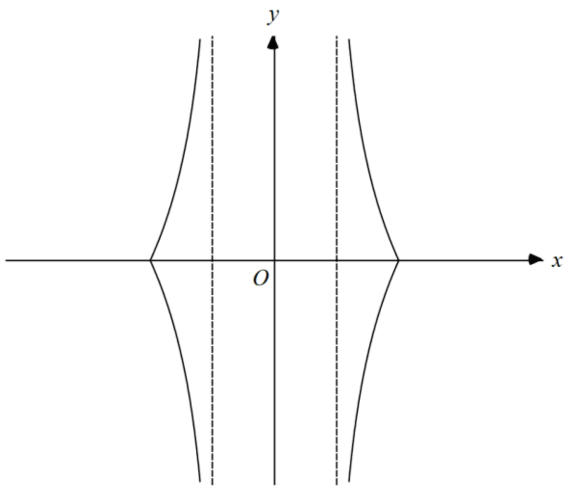


Which of the following best represents the graph of $y^2 = f(x)$?

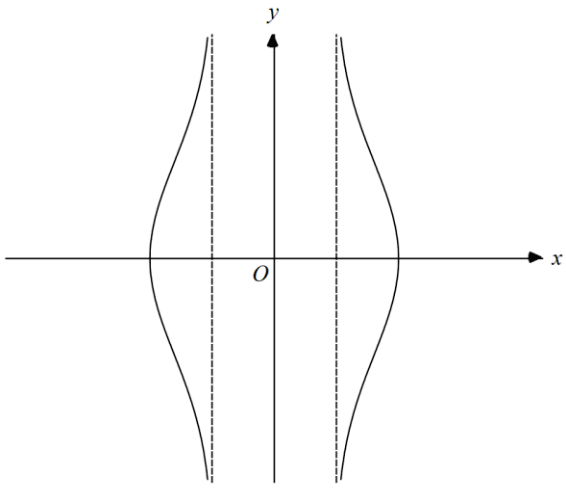
A.



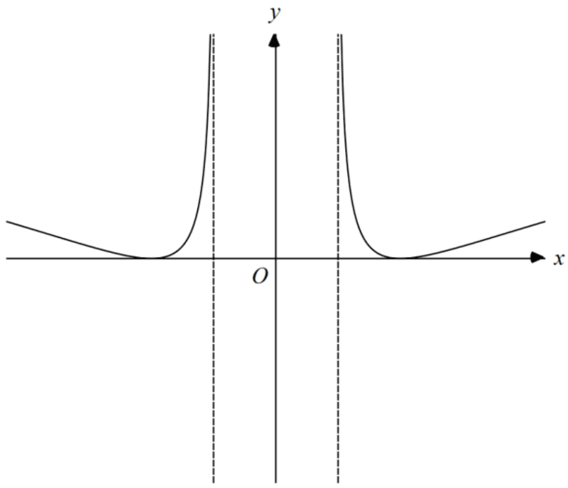
B.



C.



D.



6 The surface area of a sphere is increasing at a rate of $0.05\,\text{mm}^2/\text{s}$.

At what rate is the radius changing when it is 4 mm?

A. $\dfrac{1}{160\pi}\,\text{mm/s}$

B. $\dfrac{1}{640\pi}\,\text{mm/s}$

C. $\dfrac{1}{1280\pi}\,\text{mm/s}$

D. $\dfrac{3}{5120\pi}\,\text{mm/s}$

7 What is the magnitude of the projection of $\underset{\sim}{u} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ onto $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$?

A. $\dfrac{1}{\sqrt{85}}$

B. $\dfrac{19}{\sqrt{85}}$

C. $\dfrac{1}{85}$

D. $\dfrac{1}{\sqrt{26}}$

8 Which expression is equivalent to $\int \frac{1}{\sqrt{7-4x^2}}\,dx$?

A. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{7}}\right)+C$

B. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{7}\right)+C$

C. $\frac{1}{\sqrt{7}}\sin^{-1}\left(\frac{2x}{\sqrt{7}}\right)+C$

D. $\frac{1}{7}\sin^{-1}\left(\frac{4x}{\sqrt{7}}\right)+C$

9 When expanded, which expression contains a non-zero constant term?

A. $\left(x^2-\frac{1}{\sqrt{x}}\right)^3$

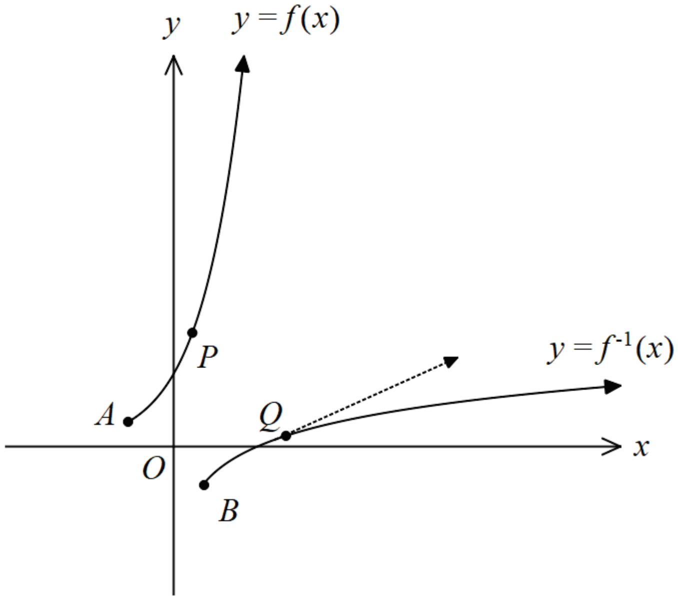
B. $\left(x^2-\frac{1}{\sqrt{x}}\right)^6$

C. $\left(x^2-\frac{1}{\sqrt{x}}\right)^8$

D. $\left(x^2-\frac{1}{\sqrt{x}}\right)^{10}$

10 Two particles move simultaneously along the curves $y = f(x)$ starting from A and $y = f^{-1}(x)$ starting from B respectively on the Cartesian plane as shown below.

When the particle on $y = f(x)$ reaches P, it is one unit vertically above the origin. At that moment, the particle on $y = f^{-1}(x)$ reaches point Q and is one unit to the right of the origin. The velocity vector of the particle at Q is also shown.



If the velocity vector of the particle at Q is $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, which of the following could represent the velocity vector at P?

A. $\begin{pmatrix} 2 \\ \sqrt{3} \end{pmatrix}$

B. $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$

C. $\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

D. $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

End of Section I

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Solve $\dfrac{4x}{2-x} \geq 3$. **3**

(b) Find the derivative of $y = x\arctan(2x)$. **2**

(c) The parametric equations of a curve are given below. **2**

$$x = e^{-4t} \quad \text{and} \quad y = e^{12t} - 1$$

Find the Cartesian equation of this curve, stating any restrictions on the domain and range.

(d) Two prefects sit around a circular table with six other students.

(i) Find the number of possible seating arrangements. **1**

(ii) Find the number of seating arrangements if the prefects are not seated together. **1**

(e) Use the substitution $u = e^x + 3$ to find $\displaystyle\int_0^{\ln 6} \frac{e^x}{\left(e^x+3\right)^{\frac{3}{2}}}\,dx$. **3**

(f) Using the substitution $t = \tan\theta$, solve $\sec 2\theta - 2\tan 2\theta = 1$ for $0 \leq \theta \leq \pi$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) (i) Evaluate $(\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{c}$, given that $\underset{\sim}{a}=\begin{pmatrix}5\\-1\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}-4\\2\end{pmatrix}$ and $\underset{\sim}{c}=\begin{pmatrix}3\\7\end{pmatrix}$. **2**

(ii) Hence, find the size of the acute angle between $\underset{\sim}{a}+\underset{\sim}{b}$ and $\underset{\sim}{c}$. **2**
Give your answer to the nearest degree.

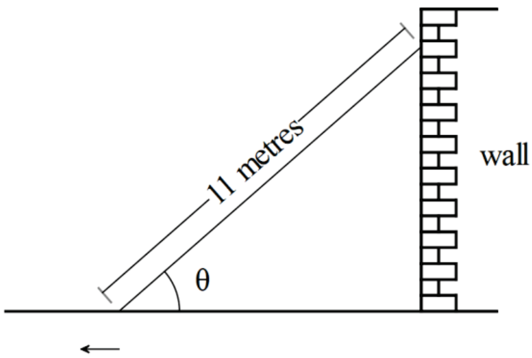
(b) Determine the coefficient of x^2 in the expansion of $(3-2x)(2+x)^4$. **3**

(c) Use mathematical induction to prove that, for integers $n\geq 1$, **3**

$$\ln\left(\frac{1}{1+2}\right)+\ln\left(\frac{2}{2+2}\right)+\ln\left(\frac{3}{3+2}\right)+\ldots+\ln\left(\frac{n}{n+2}\right)=\ln\left(\frac{2}{(n+1)(n+2)}\right)$$

Question 12 continues on page 11

(d) An 11-metre ladder rests with one end against a wall and the other end on horizontal ground, which is level with the base of the wall, and x metres from the bottom of the wall. The end that is in contact with the ground is sliding away from the wall at a constant rate of 0.1m/s. **3**



Find the rate at which the angle, θ, in radians, between the ground and the ladder is decreasing when the ladder is 6 metres away from the wall.

(e) An ice-cream parlour offers five flavours of ice-cream. Patrons can choose to have one, two or three scoops of ice-cream, and can choose any of the flavours for each scoop. **2**

On a busy day, 500 patrons bought at least one scoop each from this ice-cream parlour.

Explain, using the pigeonhole principle, why at least 10 patrons bought the exact same combination of flavours.

End of Question 12

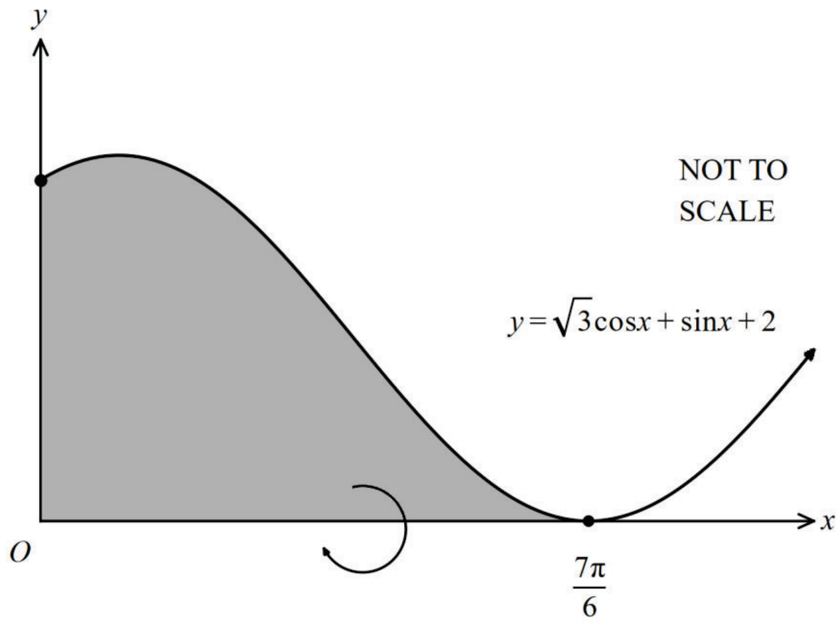
Question 13 (16 marks) Use a SEPARATE writing booklet

(a) (i) Show that $2\sin 5x \cos x = \sin 6x + \sin 4x$. **1**

(ii) Hence, find $\int \frac{\sin 4x + \sin 6x}{\cos 4x + \cos 6x}\, dx$. **2**

(b) (i) Write $\sqrt{3}\cos x + \sin x$ in the form $R\cos(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) A region bounded by the curve $y = \sqrt{3}\cos x + \sin x + 2$ and the coordinate axes is shown below. The curve touches the x-axis at $x = \frac{7\pi}{6}$. **3**



Hence, or otherwise, find the volume of the solid formed when the shaded region is rotated about the x-axis.

Question 13 continues on page 13

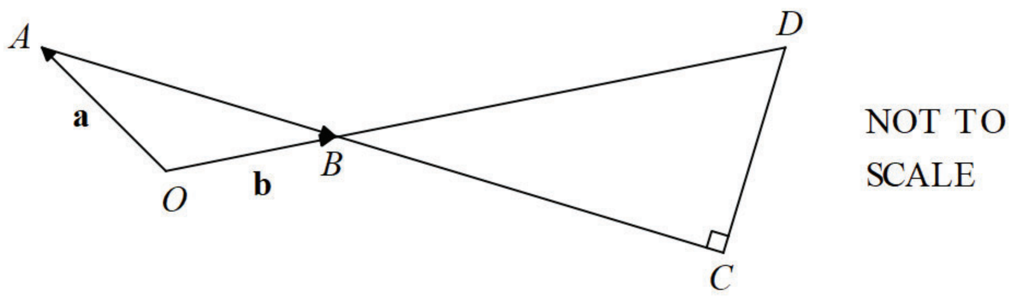
(c) (i) Show that $\sec\left(2\sin^{-1}(x)\right)$ can be expressed as $\dfrac{1}{1-2x^2}$. **2**

(ii) Hence, sketch the function $f(x)=\sec\left(2\sin^{-1}(x)\right)$. **2**

(d) In the diagram below, the points A and B have respective position vectors **a** and **b**, relative to a fixed point O. **4**

The vector $\overrightarrow{AB}$ is extended to a point C such that $\left|\overrightarrow{AB}\right| : \left|\overrightarrow{AC}\right| = 1:4$.

The vector $\overrightarrow{OB}$ is extended to point D such that $\left|\overrightarrow{OB}\right| : \left|\overrightarrow{OD}\right| = 1:k$, where k is a positive constant.



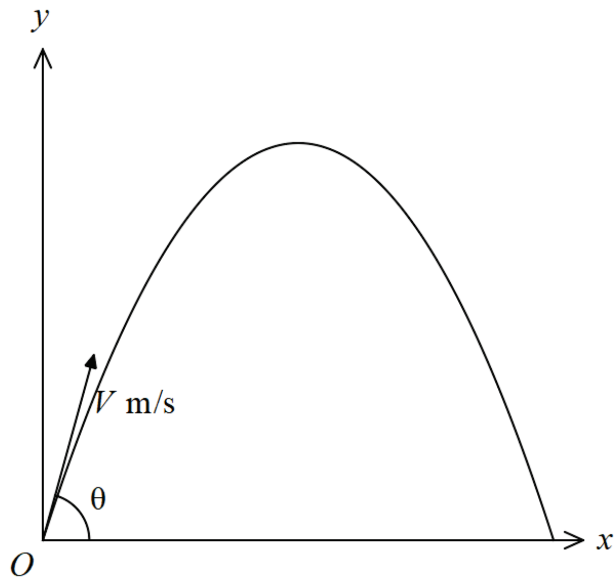
If the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular, show that

$$k = \frac{3|\mathbf{a}|^2 - 7\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2}{|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}}$$

End of Question 13

Question 14 (14 marks) Use a SEPARATE writing booklet

(a) A projectile is launched from the origin at ground level with an initial velocity V m/s at an angle θ above the horizontal.



The displacement vector of the particle t seconds after launch is given by

$$r(t) = \begin{pmatrix} Vt\cos\theta \\ -\frac{1}{2}gt^2 + Vt\sin\theta \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

(i) Show that the maximum height attained by the projectile is given by **3**

$$\frac{V^2\sin^2\theta}{2g}$$

(ii) Hence, determine the set of values of θ for which the maximum height reached by the projectile is less than its range. **3**

Give your answers to the nearest degree.

Question 14 continues on page 15

(b) The function $f\left(x\right)=\dfrac{1+kx}{x^2-1}$ is continuous in the domain $-1<x<1$. **3**

Find the values of k such that the inverse function $f^{-1}\left(x\right)$ exists.

(c) The equation $px^3+qx+r=0$ has a double root.

(i) Show that $27pr^2+4q^3=0$. **3**

(ii) Hence, find the exact value of the y coordinates of the stationary points on the curve $y=2x^3-3x-1$ without the use of calculus. **2**

End of paper

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2025	**HSC TRIAL EXAMINATION**

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NAME: Solutions TEACHER: ____________

STUDENT NUMBER: | | | | | | | | |

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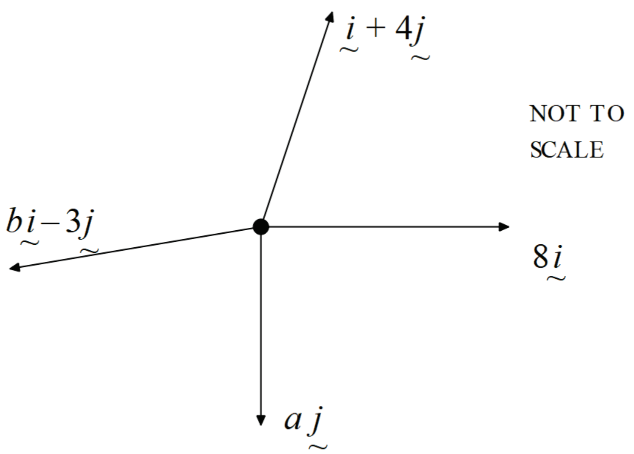

If the particle is in equilibrium while the ropes are being pulled, what are the values of a and b?

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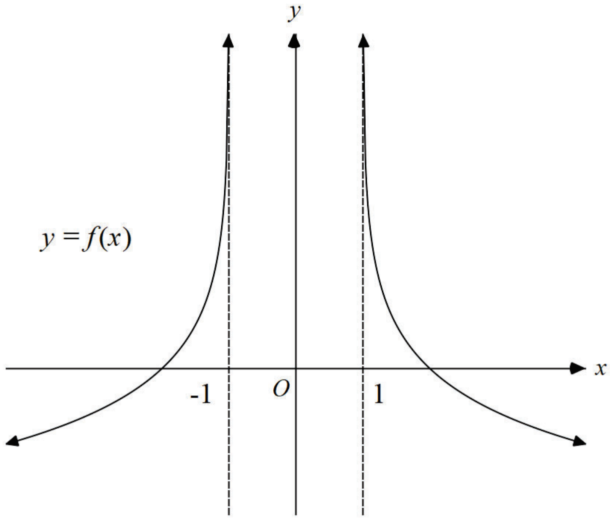

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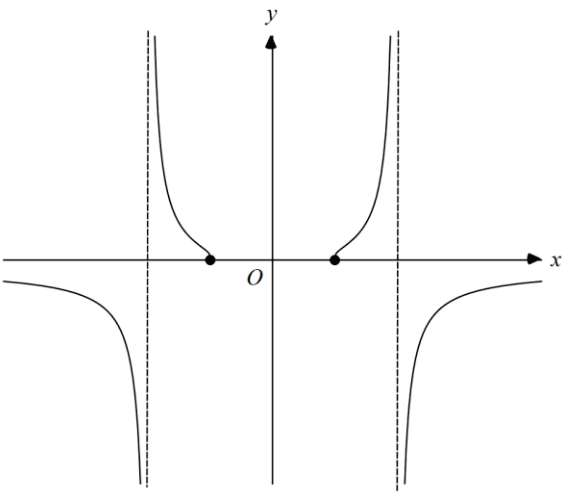

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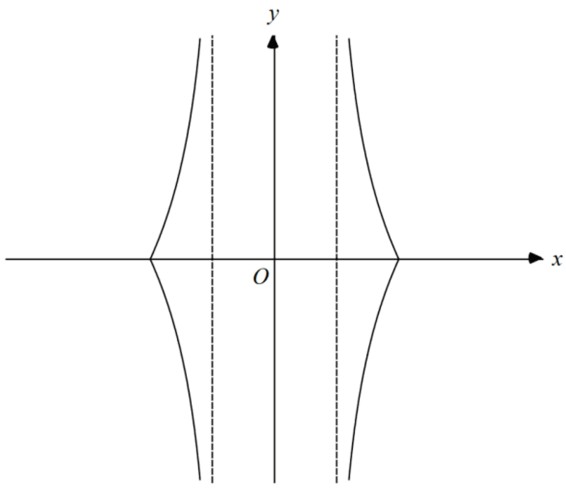
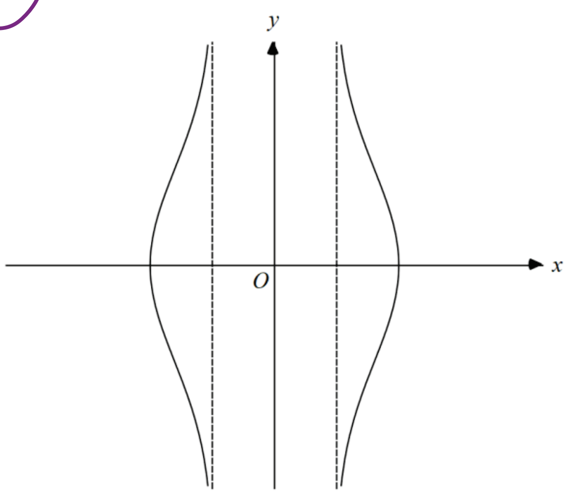
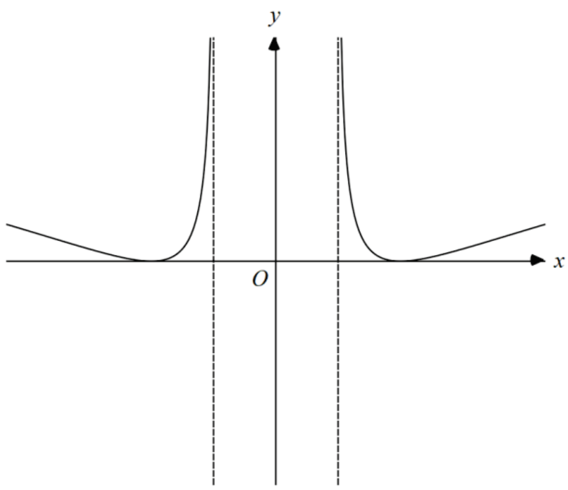


C.



D.



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At what rate is the radius changing when it is 4 mm?

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A. $\dfrac{1}{\sqrt{85}}$

B. $\dfrac{19}{\sqrt{85}}$

C. $\dfrac{1}{85}$

D. $\dfrac{1}{\sqrt{26}}$

8 Which expression is equivalent to $\int \frac{1}{\sqrt{7-4x^2}}\,dx$?

A. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{7}}\right)+C$

B. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{7}\right)+C$

C. $\frac{1}{\sqrt{7}}\sin^{-1}\left(\frac{2x}{\sqrt{7}}\right)+C$

D. $\frac{1}{7}\sin^{-1}\left(\frac{4x}{\sqrt{7}}\right)+C$

9 When expanded, which expression contains a non-zero constant term?

A. $\left(x^2-\frac{1}{\sqrt{x}}\right)^3$

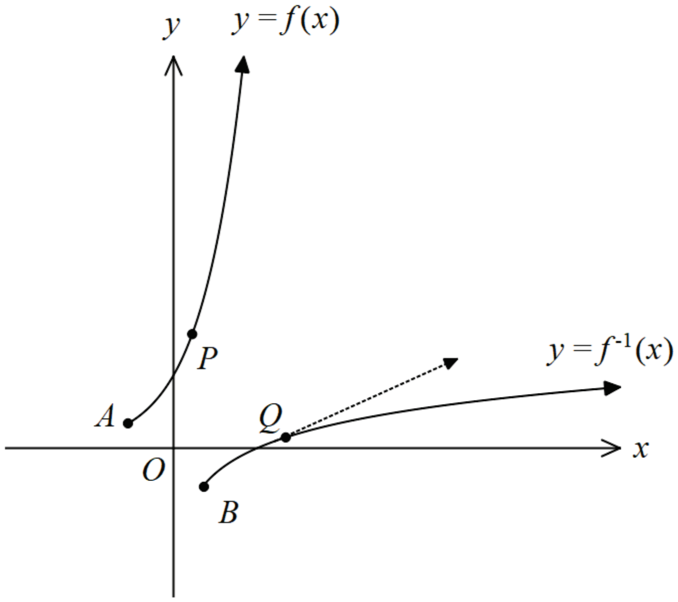
B. $\left(x^2-\frac{1}{\sqrt{x}}\right)^6$

C. $\left(x^2-\frac{1}{\sqrt{x}}\right)^8$

D. $\left(x^2-\frac{1}{\sqrt{x}}\right)^{10}$

10 Two particles move simultaneously along the curves $y = f(x)$ starting from A and $y = f^{-1}(x)$ starting from B respectively on the Cartesian plane as shown below.

When the particle on $y = f(x)$ reaches P, it is one unit vertically above the origin. At that moment, the particle on $y = f^{-1}(x)$ reaches point Q and is one unit to the right of the origin. The velocity vector of the particle at Q is also shown.



If the velocity vector of the particle at Q is $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, which of the following could represent the velocity vector at P?

A. $\begin{pmatrix} 2 \\ \sqrt{3} \end{pmatrix}$

B. $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$

C. $\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

D. $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

End of Section I

Question 11

(a) Solve $\dfrac{4x}{2-x} \geq 3$. $x \neq 2$ **3**

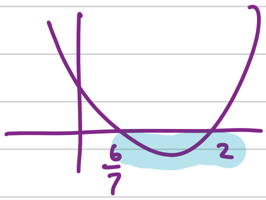
$4x(2-x) \geq 3(2-x)^2$

$3(2-x)^2 - 4x(2-x) \leq 0$

$(2-x)[3(2-x)-4x] \leq 0$

$(2-x)(6-7x) \leq 0$

$(x-2)(7x-6) \leq 0$



$\therefore \frac{6}{7} \leq x < 2$ as $x \neq 2$

(b) Find the derivative of $y = x\arctan(2x)$. **2**

$\frac{d}{dx}\left(x\tan^{-1}(2x)\right)$

$= \tan^{-1}(2x) + x\left(\frac{1}{1+4x^2} \times 2\right)$

$= \tan^{-1}(2x) + \frac{2x}{1+4x^2}$

(c) The parametric equations of a curve are given below. **2**

$$x = e^{-4t} \quad \text{and} \quad y = e^{12t} - 1$$

Find the Cartesian equation of this curve, stating any restrictions on the domain and range.

$x = e^{-4t}$ $\quad x > 0$

$y = e^{12t} - 1$ $\quad y > -1$

$\therefore e^{4t} = \frac{1}{x}$

$e^{12t} = \frac{1}{x^3}$

$\therefore y = \frac{1}{x^3} - 1, \quad x > 0, \; y > -1$

Question 11

(d) Two prefects sit around a circular table with six other students.

i) Find the number of possible seating arrangements. **1**

$n = 8$

i) Total $= (8-1)!$

$= 7!$

$= 5040$

ii) Find the number of seating arrangements if the prefects are not seated together. **1**

ii) Prefects together ways $= (7-1)! \times 2!$

$= 6! \times 2!$

Prefects not together $= 7! - 6! \times 2$

$= 3600$

(e) Use the substitution $u = e^x + 3$ to find $\int_0^{\ln 6} \frac{e^x}{(e^x+3)^{\frac{3}{2}}}\,dx$. **3**

$$\int_0^{\ln 6} \frac{e^x}{(e^x+3)^{\frac{3}{2}}}\,dx$$

let $u = e^x + 3$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

x	0	$\ln 6$
u	4	9

$$= \int_4^9 \frac{du}{u^{\frac{3}{2}}}$$

$$= \int_4^9 u^{-\frac{3}{2}}\,du$$

$$= -2\left[u^{-\frac{1}{2}}\right]_4^9$$

$$= -2\left[\frac{1}{\sqrt{9}} - \frac{1}{\sqrt{4}}\right]$$

$$= \frac{1}{3}$$

Question 11

(f) Using the substitution $t = \tan\theta$, solve $\sec 2\theta - 2\tan 2\theta = 1$ for $0 \le \theta \le \pi$. **3**

$\sec 2\theta - 2\tan 2\theta = 1 \qquad 0 \le \theta \le \pi$

let $t = \tan\theta$

$\therefore \cos 2\theta = \dfrac{1-t^2}{1+t^2} \qquad \tan 2\theta = \dfrac{2t}{1-t^2}$

$\therefore \dfrac{1+t^2}{1-t^2} - \dfrac{4t}{1-t^2} = 1$

$1+t^2-4t = 1-t^2$

$2t^2 - 4t = 0$

$2t(t-2) = 0$

$\therefore t = 0$ or $t = 2$

$\therefore \tan\theta = 0 \qquad \tan\theta = 2$

$\theta = 0, \pi \qquad \theta = \tan^{-1}(2)$

$= 1.107$ (3dp)

$\therefore \theta = 0, \pi, 1.107$ (3dp)

Question 12

(a) (i) Evaluate $(\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{c}$, given that $\underset{\sim}{a}=\begin{pmatrix}5\\-1\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}-4\\2\end{pmatrix}$ and $\underset{\sim}{c}=\begin{pmatrix}3\\7\end{pmatrix}$. **2**

$$\underset{\sim}{a}+\underset{\sim}{b} = \begin{pmatrix}5\\-1\end{pmatrix}+\begin{pmatrix}-4\\2\end{pmatrix}$$
$$= \begin{pmatrix}1\\1\end{pmatrix}$$

$$\therefore (\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{c} = \begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}3\\7\end{pmatrix}$$
$$= 3+7$$
$$= 10$$

(ii) Hence, find the size of the acute angle between $\underset{\sim}{a}+\underset{\sim}{b}$ and $\underset{\sim}{c}$. Give your answer to the nearest degree. **2**

Let θ be angle between vectors.

$$(\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{c} = |\underset{\sim}{a}\cdot\underset{\sim}{b}||\underset{\sim}{c}|\cos\theta$$

$$\therefore 10 = \left|\begin{pmatrix}1\\1\end{pmatrix}\right|\left|\begin{pmatrix}3\\7\end{pmatrix}\right|\cos\theta$$
$$= \sqrt{1^2+1^2}\times\sqrt{3^2+7^2}\cos\theta$$
$$= \sqrt{2}\times\sqrt{58}\cos\theta$$

$$\therefore \cos\theta = \frac{10}{\sqrt{116}}$$

$$\therefore \theta = 21.80\ldots$$

$$\therefore \theta = 22^\circ \text{ (nearest degree)}$$

Question 12

(b) Determine the coefficient of x^2 in the expansion of $(3-2x)(2+x)^4$. **3**

$$(3-2x)(2+x)^4 = 3(2+x)^4 - 2x(2+x)^4$$

Term of x^2 $= 3\times(\text{term of } x^2) - 2x\times(\text{term of } x)$

$= 3\times\binom{4}{2}2^2x^2 - 2x\times\binom{4}{3}2^3x$

$= 3\times6\times4x^2 - 2\times4\times8x^2$

$= 8x^2$

$\therefore$ Coefficient of x^2 is 8.

(c) Use mathematical induction to prove that, for integers $n \geq 1$, **3**

$$\ln\left(\frac{1}{1+2}\right)+\ln\left(\frac{2}{2+2}\right)+\ln\left(\frac{3}{3+2}\right)+\ldots+\ln\left(\frac{n}{n+2}\right)=\ln\left(\frac{2}{(n+1)(n+2)}\right)$$

Test $n=1$

LHS $= \ln\left(\frac{1}{1+2}\right) = \ln\left(\frac{1}{3}\right)$

RHS $= \ln\left(\frac{2}{(1+1)(1+2)}\right)$

$= \ln\left(\frac{2}{2\times3}\right)$

$= \ln\left(\frac{1}{3}\right)$

$=$ LHS

$\therefore$ True for $n=1$

Assume true for $n=k$, $k\in\mathbb{Z}$, $k\geq1$

ie $\ln\left(\frac{1}{1+2}\right)+\ln\left(\frac{2}{2+3}\right)+\ldots+\ln\left(\frac{k}{k+2}\right) = \ln\left(\frac{2}{(k+1)(k+2)}\right)$

Prove true for $n=k+1$

ie $\ln\left(\frac{1}{1+2}\right)+\ln\left(\frac{2}{2+3}\right)+\ldots+\ln\left(\frac{k}{k+2}\right)+\ln\left(\frac{k+1}{k+3}\right) = \ln\left(\frac{2}{(k+2)(k+3)}\right)$

LHS $= \ln\left(\frac{2}{(k+1)(k+2)}\right)+\ln\left(\frac{k+1}{k+3}\right)$ from assumption

$= \ln\left(\frac{2(k+1)}{(k+1)(k+2)(k+3)}\right)$

$= \ln\left(\frac{2}{(k+2)(k+3)}\right) = \ln\left(\frac{2}{((k+1)+1)((k+1)+2)}\right)$

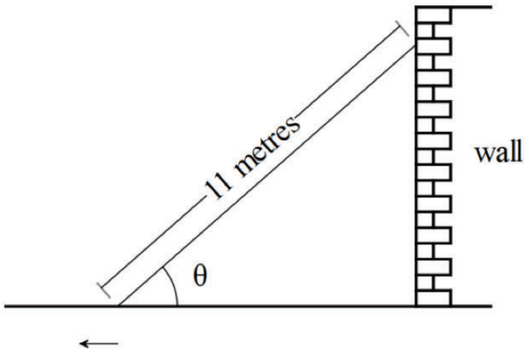
$=$ RHS

$\therefore$ True for $n=k+1$ when true for $n=k$

$\therefore$ The result is true by Mathematical Induction.

Question 12

(d) A 11 metre ladder rests with one end against a wall and the other end on horizontal ground, which is level with the base of the wall, and x metres from the bottom of the wall. The end that is in contact with the ground is sliding away from the wall at a constant rate of 0.1m/s. **3**



Find the rate at which the angle, θ, in radians, between the ground and the ladder is decreasing when the ladder is 6 metres away from the wall.

Let x be distance of ladder foot to wall

$\therefore \frac{dx}{dt} = 0.1$

Now $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

and $\cos\theta = \frac{x}{11} \Rightarrow x = 11\cos\theta$

$\therefore \frac{dx}{d\theta} = -11\sin\theta$

$\therefore \frac{d\theta}{dx} = \frac{-1}{11\sin\theta}$

So $\frac{d\theta}{dt} = \frac{-1}{11\sin\theta} \times 0.1$

When $x = 6$, $\cos\theta = \frac{6}{11}$

$\therefore \sin\theta = \frac{\sqrt{11^2-6^2}}{11}$

$\therefore \frac{d\theta}{dt} = \frac{-1}{11} \times \frac{11}{\sqrt{121-36}} \times 0.1 = -0.01085 \text{ rad/s}$

Question 12

(e) An ice-cream parlour offers five flavours of ice-cream. Patrons can choose to have one, two or three scoops of ice-cream, and can choose any of the flavours for each scoop. **2**

On a busy day, 500 patrons bought at least one scoop each from this ice-cream parlour.

Explain, using the pigeonhole principle, why at least 10 patrons bought the exact same combination of flavours.

Case 1: 1 scoop

ways $= {}^5C_1 = 5$

Case 2: 2 scoops

ways = (different flavours) + (same flavour)
$= {}^5C_2 + 5 = 15$

Case 3: 3 scoops

ways = (all unique) + (2 flavours same) + (all same)
$= {}^5C_3 + {}^5C_2 \times 2 + 5$
$= 10 + 20 + 5$
$= 35$

$\therefore$ Total ways $= 35 + 15 + 5 = 55$

$\frac{500}{55} = 9.09$

With 55 combinations, at most 495 patrons bought icecreams without exceeding 9 patrons per combination.

But there were 500 patrons, so at least one combination has 10 or more purchases.

That is, at least 10 patrons must have bought the exact same combination of flavours

Question 13

(a) (i) Show that $2\sin 5x \cos x = \sin 6x + \sin 4x$. **1**

$$\text{LHS} = 2\sin 5x\cos x$$
$$= 2 \times \left(\tfrac{1}{2}\left(\sin(5x+x) + \sin(5x-x)\right)\right) \quad \text{from reference sheet}$$
$$= \sin(6x) + \sin(4x)$$
$$= \text{RHS}$$

(ii) Hence, find $\displaystyle\int \frac{\sin 4x + \sin 6x}{\cos 4x + \cos 6x}\,dx$. **2**

$$\int \frac{\sin 4x + \sin 6x}{\cos 4x + \cos 6x}\,dx$$
$$= \int \frac{2\sin 5x\cos x}{2\cos 5x\cos x}\,dx \quad \text{Reference sheet}$$
$$= \int \frac{\sin 5x}{\cos 5x}\,dx$$
$$= -\frac{1}{5}\ln|\cos 5x| + c$$

(b) (i) Write $\sqrt{3}\cos x + \sin x$ in the form $R\cos(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

$$\sqrt{3}\cos x + \sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$$

Equating coefficients: $R\cos\alpha = \sqrt{3}$ (1)
$R\sin\alpha = 1$ (2)

$(1)^2 + (2)^2$

$$R^2\sin^2\alpha + R^2\cos^2\alpha = R^2$$
$$= (\sqrt{3})^2 + 1^2$$
$$= 4$$

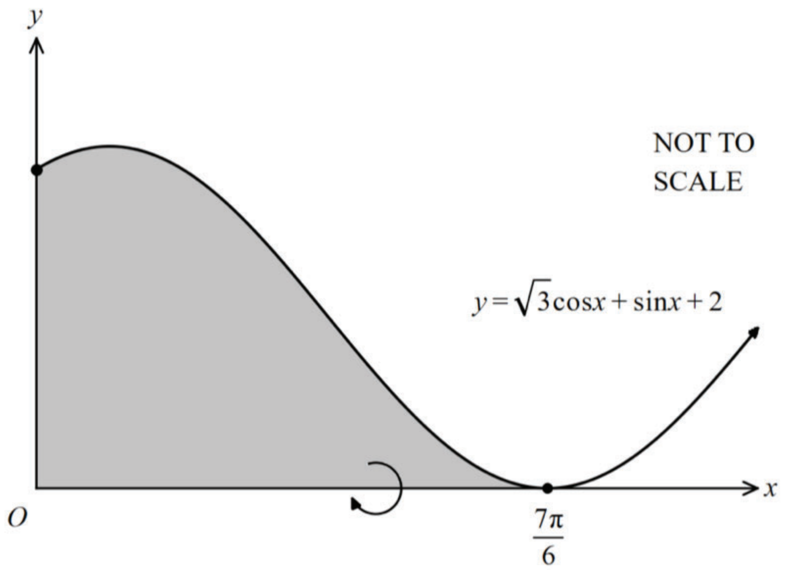
$\frac{(2)}{(1)}$: $\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore R = 2, \; R > 0$

$\alpha = \frac{\pi}{6}, \quad 0 < \alpha < \frac{\pi}{2}$

$$\therefore \sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right)$$

(ii) A region bounded by the curve $y = \sqrt{3}\cos x + \sin x + 2$ and the coordinate axes is shown below. The curve touches the x-axis at $x = \frac{7\pi}{6}$. **3**



Hence, or otherwise, find the volume of the solid formed when the shaded region is rotated about the x-axis.

$$V = \pi \int_0^{\frac{7\pi}{6}} (\sqrt{3}\cos x + \sin x + 2)^2 \, dx$$

$$= \pi \int_0^{\frac{7\pi}{6}} \left(2\cos\left(x - \frac{\pi}{6}\right) + 2\right)^2 dx \quad \text{from i)}$$

$$= 4\pi \int_0^{\frac{7\pi}{6}} \left(\cos^2\left(x - \frac{\pi}{6}\right) + 2\cos\left(x - \frac{\pi}{6}\right) + 1\right) dx$$

$$= 4\pi \int_0^{\frac{7\pi}{6}} \left(\frac{1 + \cos\left(2x - \frac{\pi}{3}\right)}{2} + 2\cos\left(x - \frac{\pi}{6}\right) + 1\right) dx$$

$$= 4\pi \left[\frac{3x}{2} + \frac{1}{4}\sin\left(2x - \frac{\pi}{3}\right) + 2\sin\left(x - \frac{\pi}{6}\right)\right]_0^{\frac{7\pi}{6}}$$

$$= 4\pi \left[\frac{3}{2} \times \frac{7\pi}{6} + \frac{1}{4}\sin\left(\frac{7\pi}{3} - \frac{\pi}{3}\right) + 2\sin\left(\frac{7\pi}{6} - \frac{\pi}{6}\right) - \left(0 + \frac{1}{4}\sin\left(\frac{-\pi}{3}\right) + 2\sin\left(\frac{-\pi}{6}\right)\right)\right]$$

$$= 4\pi \left(\frac{7\pi}{4} + \frac{1}{4}\sin(2\pi) + 2\sin(\pi) + \frac{1}{4}\sin\frac{\pi}{3} + 2\sin\frac{\pi}{6}\right)$$

$$= 4\pi \left(\frac{7\pi}{4} + 0 + 0 + \frac{\sqrt{3}}{8} + 1\right)$$

$$= \left(7\pi^2 + \frac{\sqrt{3}\pi}{2} + 4\pi\right) u^3$$

Question 13

(c) (i) Show that $\sec\left(2\sin^{-1}(x)\right)$ can be expressed as $\frac{1}{1-2x^2}$. **2**

Let $\sin^{-1}x = \alpha$ Note $-1 \le x \le 1$

$\therefore \sec(2\alpha) = \frac{1}{\cos 2\alpha}$

$= \frac{1}{1-2\sin^2\alpha}$

$= \frac{1}{1-2x^2}$

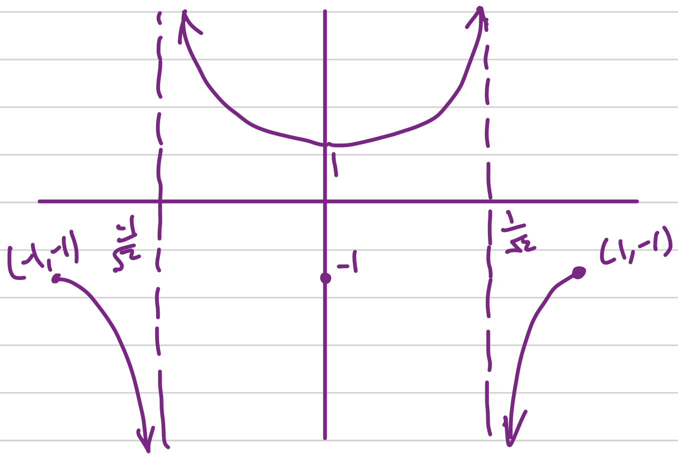
$=$ RHS

(ii) Hence, sketch the function $f(x) = \sec\left(2\sin^{-1}(x)\right)$. **2**

$f(x) = \sec(2\sin^{-1}x)$

$= \frac{1}{1-2x^2}$ from i)

D: $1-2x^2 \neq 0 \quad \therefore x \neq \frac{\pm 1}{\sqrt{2}}$

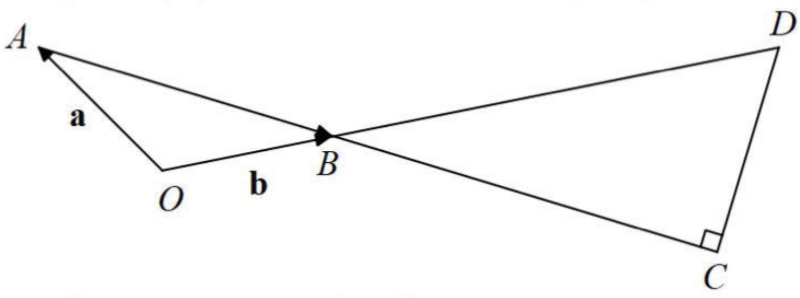


Question 13

(d) In the diagram below, the points A and B have respective position vectors $\mathbf{a}$ and $\mathbf{b}$, relative to a fixed point O. **4**

The vector $\overrightarrow{AB}$ is extended to a point C such that $\left|\overrightarrow{AB}\right| : \left|\overrightarrow{AC}\right| = 1:4$.

The vector $\overrightarrow{OB}$ is extended to point D such that $\left|\overrightarrow{OB}\right| : \left|\overrightarrow{OD}\right| = 1:k$, where k is a positive constant.



NOT TO SCALE

If the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular, show that

$$k = \frac{3|\mathbf{a}|^2 - 7\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2}{|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}}$$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{AC} = 4\overrightarrow{AB}$

$= 4(\mathbf{b} - \mathbf{a})$

$= 4\mathbf{b} - 4\mathbf{a}$

$\overrightarrow{OB} = \mathbf{b}$

$\overrightarrow{OD} = k\overrightarrow{OB}$

$= k\mathbf{b}$

Now $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= k\overrightarrow{OB} - (\overrightarrow{OA} + \overrightarrow{AC})$

$= k\mathbf{b} - (\mathbf{a} + 4\mathbf{b} - 4\mathbf{a})$

$= 3\mathbf{a} + (k-4)\mathbf{b}$

And since $\overrightarrow{AB} \perp \overrightarrow{CD} \Rightarrow \overrightarrow{AB}\cdot\overrightarrow{CD} = 0$

$\therefore \overrightarrow{AB}\cdot\overrightarrow{CD} = (3\mathbf{a} + (k-4)\mathbf{b})\cdot(\mathbf{b} - \mathbf{a}) = 0$

$3\mathbf{a}\cdot\mathbf{b} - 3\mathbf{a}\cdot\mathbf{a} + (k-4)\,\mathbf{b}\cdot(\mathbf{b} - \mathbf{a}) = 0$

$(k-4)\left[\mathbf{b}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a}\right] = 3|\mathbf{a}|^2 - 3\mathbf{a}\cdot\mathbf{b}$

$$\therefore k - 4 = \frac{3|\mathbf{a}|^2 - 3\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}}$$

$$\therefore k = \frac{3|\mathbf{a}|^2 - 3\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}} + 4$$

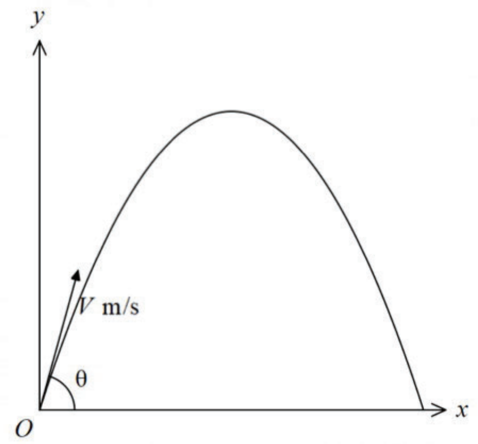
$$= \frac{3|\mathbf{a}|^2 - 3\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2 - 4\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}}$$

$$= \frac{3|\mathbf{a}|^2 - 7\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2}{|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}}$$

as required

Question 14

(a) A projectile is launched from the origin at ground level with an initial velocity V m/s at an angle θ above the horizontal.



The displacement vector of the particle t seconds after launch is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} Vt\cos\theta \\ -\frac{1}{2}gt^2 + Vt\sin\theta \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

(i) Show that the maximum height attained by the projectile is given by **3**

$$\frac{V^2\sin^2\theta}{2g}$$

$$\underset{\sim}{v}(t) = \begin{pmatrix} V\cos\theta \\ -gt + V\sin\theta \end{pmatrix}$$

For maximum height, $v_y(t) = 0$

$\therefore \; -gt + V\sin\theta = 0$

$t = \dfrac{V\sin\theta}{g}$ sub into $y = -\frac{1}{2}gt^2 + Vt\sin\theta$

$$\therefore y_{max} = -\frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2 + V\sin\theta\left(\frac{V\sin\theta}{g}\right)$$

$$= \frac{-V^2\sin^2\theta}{2g} + \frac{2V^2\sin^2\theta}{2g}$$

$$= \frac{V^2\sin^2\theta}{2g} \quad \text{as required.}$$

(ii) Hence, determine the set of values of θ for which the maximum height reached by the projectile is less than its range. **3**

Give your answers to the nearest degree.

For range, $y=0$

$\therefore \frac{-1}{2}gt^2 + Vt\sin\theta = 0$

$t\left(V\sin\theta - \frac{1}{2}gt\right) = 0$

$\therefore t=0$ or $t = \frac{2V\sin\theta}{g}$ but $t>0$

$\therefore$ Time of flight $= \frac{2V\sin\theta}{g}$ sub into $x = Vt\cos\theta$

$\therefore R = V\cos\theta\left(\frac{2V\sin\theta}{g}\right)$

$= \frac{2V^2\sin\theta\cos\theta}{g}$

We want $y_{max} < R$

$\therefore \frac{V^2\sin^2\theta}{2g} < \frac{2V^2\sin\theta\cos\theta}{g}$

$\therefore \sin^2\theta < 4\sin\theta\cos\theta$

$\therefore \sin^2\theta - 4\sin\theta\cos\theta < 0$

$\therefore \sin\theta(\sin\theta - 4\cos\theta) < 0$ as $0 < \theta < \frac{\pi}{2}$

Then $\sin\theta < 4\cos\theta$

$\tan\theta < 4$

$\therefore \theta < 76°$

(b) The function $f(x)=\dfrac{1+kx}{x^2-1}$ is continuous in the domain $-1<x<1$. **3**

Find the values of k such that the inverse function $f^{-1}(x)$ exists.

If $f^{-1}(x)$ exists, $f(x)$ must be a one-to-one function

Algebraically, we consider $f'(x)$ to check for conditions when $f(x)$ is monotonically increasing or monotonically decreasing.

$$f'(x) = \frac{k(x^2-1)-2x(1+kx)}{(x^2-1)^2}$$

$$= \frac{kx^2-k-2x-2x^2k}{(x^2-1)^2}$$

$$= \frac{-kx^2-2x-k}{(x^2-1)^2}$$

ie $kx^2+2x+k>0$ OR $kx^2+2x+k<0$

Considering the discriminant, $\Delta = 2^2-4k^2$
$= 4-4k^2$
$= 4(1-k^2)$

We have 2 cases:

$k>0$ and $\Delta \leq 0$ (positive definite)

$1-k^2 \leq 0$

$k^2 \geq 1$

$k \leq -1$ or $k \geq 1$ and $k>0$

$\therefore k \geq 1$

$k<0$ and $\Delta \leq 0$ (negative definite)

$\therefore k^2-1 \geq 0$

$(k \leq -1$ or $k \geq 1)$ and $k \leq 0$

$\therefore k \leq -1$

$\therefore k \leq -1$ or $k \geq 1$ for $f^{-1}(x)$ to exist as $f(x)$ will be monotonically increasing / decreasing

Question 14

(c) The equation $px^3 + qx + r = 0$ has a double root.

(i) Show that $27pr^2 + 4q^3 = 0$. **3**

Let $p(x) = px^3 + qx + r$

If $p(x) = 0$ has double root, $p'(x) = 0$ must contain a root candidate

$p'(x) = 3px^2 + q$

$\therefore x^2 = \frac{-q}{3p}$

So if $px^3 + qx = r$

$x(px^2 + q) = r$

$x^2\left(p\left(\frac{-q}{3p}\right) + q\right)^2 = r^2$

$\left(\frac{-q}{3p}\right)\left(\frac{-q}{3} + q\right)^2 = r^2$

$\therefore \left(\frac{-q}{3p}\right)\left(\frac{2q}{3}\right)^2 = r^2$

$\therefore \frac{-4q^3}{27p} = r^2$

$\therefore 27r^2p + 4q^3 = 0$

as required

<u>Alternatively</u>

Let roots be α, α, β

Sum of roots: $2\alpha + \beta = 0$

$\therefore \beta = -2\alpha$

Products two at a time: $\alpha^2 + 2\alpha\beta = \frac{q}{p}$

$\therefore \alpha^2 - 4\alpha^2 = \frac{q}{p}$

$\therefore \alpha^2 = -\frac{q}{3p}$

Product of roots: $\alpha^2\beta = \frac{-r}{p}$

$-2\alpha^3 = -\frac{r}{p}$

$4\alpha^6 = \frac{r^2}{p^2}$

$\therefore 4\left(\frac{-q}{3p}\right)^3 = \frac{r^2}{p^2}$

$\therefore \frac{-4q^3}{27p^3} = \frac{r^2}{p^2}$

$\therefore 27pr^2 + 4q^3 = 0$

Question 14

(ii) Hence, find the exact value of the y coordinates of the stationary points on the curve $y = 2x^3 - 3x - 1$ without the use of calculus. **2**

Suppose that the stationary points occur at $y = k$ for two values of x.

Then $2x^3 - 3x - 1 = k$ must have a double root.

ie $2x^3 - 3x - (1+k) = 0$

From i), if $px^3 + qx + r = 0$ has a double root then $27pr^2 + 4q^3 = 0$

then $27(2)(-(1+k))^2 + 4(-3)^3 = 0$

$\therefore 54(1+k)^2 - 108 = 0$

$(1+k)^2 = 2$

$\therefore 1+k = \pm\sqrt{2}$

$\therefore k = -1+\sqrt{2}$ or $k = -1-\sqrt{2}$

$\therefore$ y-coordinates of stationary points are $-1 \pm \sqrt{2}$.