Student Number: | | | | | | | | |

Teacher Name: ____________________

Penrith Selective High School

2025 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- Reference sheets are provided with this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 1–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–9)

- Attempt Questions 11–14
- Allow about 1 hours and 45 minutes for this section

	Multiple Choice	Q11	Q12	Q13	Q14	Total
Functions	4 /1	d, e /6			a /2	/9
Trigonometric Functions	1, 6 /2	b /1		a, b /6	b /1	/10
Calculus	5, 7, 8 /3	a, c /8	b, c /5		e /3	/19
Combinatorics	10 /1		a /2		d /3	/6
Proof			d /3			/3
Vectors	3, 9 /2			c /9	f /4	/15
Binomial Distribution	2 /1		e /5		c /2	/8
Total	/10	/15	/15	/15	/15	/70

This paper MUST NOT be removed from the examination room

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which expression is equivalent to $\cos(4\theta+\alpha)\cos(3\theta+\alpha)+\sin(4\theta+\alpha)\sin(3\theta+\alpha)$?

A. $\cos(7\theta+2\alpha)$

B. $\sin(7\theta+2\alpha)$

C. $\cos(\theta+2\alpha)$

D. $\cos\theta$

2 An eight-sided die is rolled 64 times and the number of times a 7 appears is noted. What is the mean and standard deviation of the binomial distribution formed?

A. $E(X)=10.7,\ \sigma=8.9$

B. $E(X)=10.7,\ \sigma=2.981$

C. $E(X)=8,\ \sigma=2.646$

D. $E(X)=8,\ \sigma=7$

3 A particle moves through the air according to the displacement vector $\underset{\sim}{r}=\begin{pmatrix} 3t \\ 4t-2t^2 \end{pmatrix}$, where $\underset{\sim}{r}$ is measured in metres. What horizontal distance will the particle travel before it hits the ground?

A. 2 m

B. 4 m

C. 6 m

D. 9 m

4 Let α, β, γ and δ be the roots of the equations $x^4+2x^3-5x^2+7x-11=0$.

Find $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}+\dfrac{1}{\delta}$.

A. $\dfrac{7}{11}$

B. $-\dfrac{7}{11}$

C. $-\dfrac{11}{7}$

D. $\dfrac{2}{11}$

5 Use the substitution $u=x^2+4x-32$ to find $\int (x+2)\sqrt[3]{x^2+4x-32}\ dx$.

A. $\dfrac{1}{2}\sqrt[3]{x^2+4x-32}+c$

B. $\dfrac{3}{8}\left(x^2+4x-32\right)^{\frac{4}{3}}+c$

C. $\dfrac{3}{4}\left(x^2+4x-32\right)^{\frac{3}{4}}+c$

D. $\dfrac{3}{2}\left(x^2+4x-32\right)^{\frac{4}{3}}+c$

6 Which expression is equal to $\dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$ if $t=\tan\dfrac{\theta}{2}$?

A. t

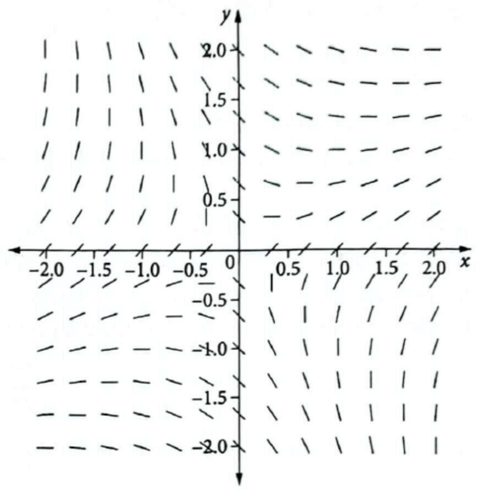
B. $\dfrac{1}{t}$

C. $2t$

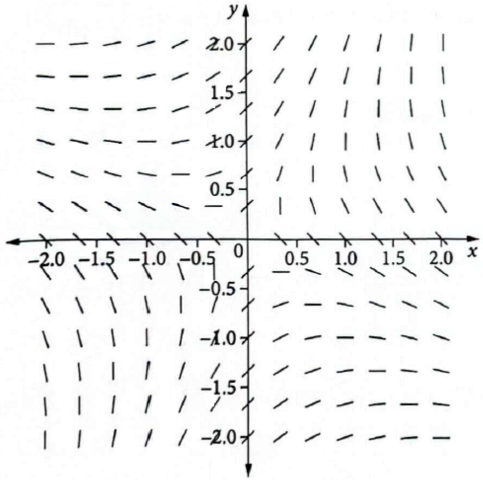
D. $\dfrac{1}{2t}$

7 Which of the following slope field best represents the differential equation $\frac{dy}{dx} = \frac{x+y}{x-y}$?

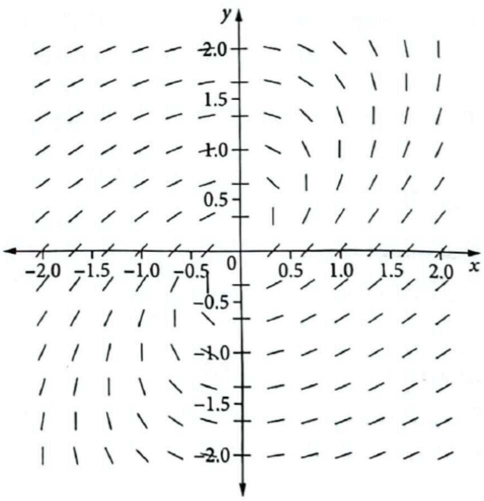
A.



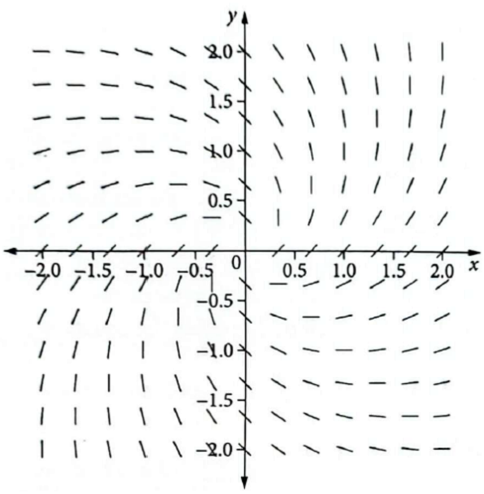
B.



C.



D.



8 The population $P(t)$ of a certain species changes over time according to the differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{10\ 000}\right)$ with an initial population $P(0) = 4000$, where t is the time in years.

Note: $\frac{1}{P} + \frac{1}{20\ 000 - P} = \frac{20\ 000}{P(20\ 000 - P)}$.

What is the population as $t \to \infty$?

A. 4 000

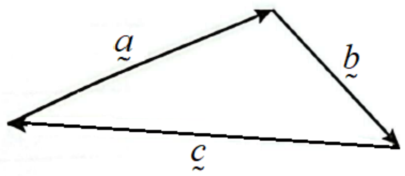
B. 5 000

C. 10 000

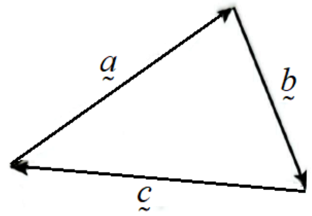
D. 20 000

9 In which diagram is $\left|proj_{\underset{\sim}{b}}\underset{\sim}{a}\right| > \left|\underset{\sim}{b}\right|$ (diagrams are drawn to scale)?

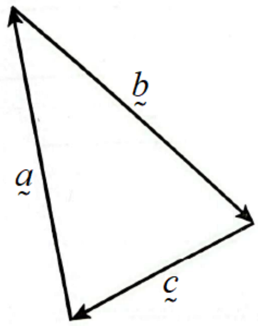
A.



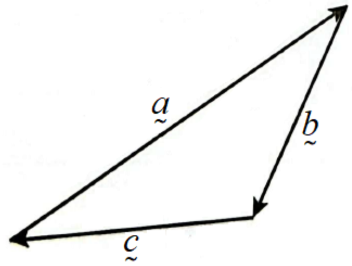
B.



C.



D.



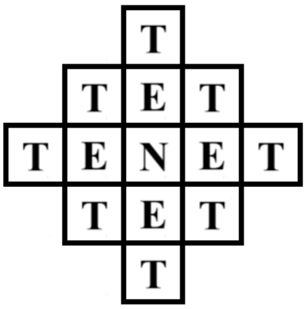
10 Starting from any letter '**T**' as shown in the diagram, the word **TENET** can be spelled by moving up, down, left, or right to an adjacent letter. Letters may be revisited (i.e., repetition is allowed). How many ways are there to spell **TENET**?

A. 64

B. 100

C. 144

D. 196



Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

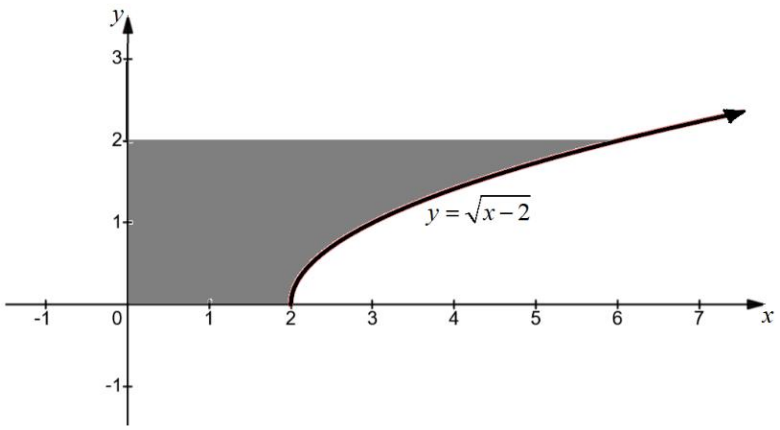
Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate Writing Booklet

(a) The shaded region bounded by the curve $y = \sqrt{x-2}$, the coordinate axes and the line $y = 2$ shown below is rotated about the y-axis. **3**

Calculate the volume of the solid generated.



(b) Show that $\cos\left(\tan^{-1} x\right) = \dfrac{1}{\sqrt{x^2+1}}$. **1**

(c) Eleanor placed an object in a freezer to cool. After t minutes, its temperature is T°C. The freezer is at a constant temperature of –8°C.

The object's temperature decreases according to the differential equation $\dfrac{dT}{dt} = k\left(T+8\right)$.

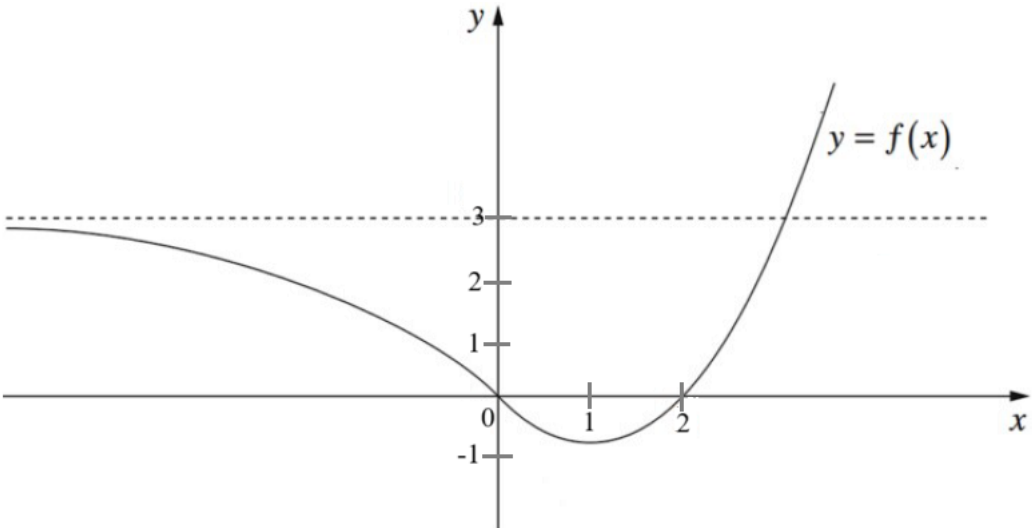
(i) If the object's initial temperature is 40°C, solve the differential equation for T in terms of k and t. **2**

(ii) If the object cools to 30°C in half an hour, show that $k = \dfrac{1}{30}\ln\left(\dfrac{19}{24}\right)$. **1**

(iii) When will the temperature of the object be 0°C correct to the nearest minute? **2**

(d) Solve $\dfrac{x^2+4}{x} < 5$. **3**

(e) The diagram shows the graph of $y = f(x)$, which has a horizontal asymptote at $y = 3$. **3**



A copy of this graph is provided on page 10 of this booklet.

On the graph provided on page 10, sketch the graph of $y = \left(f(x)\right)^2$.

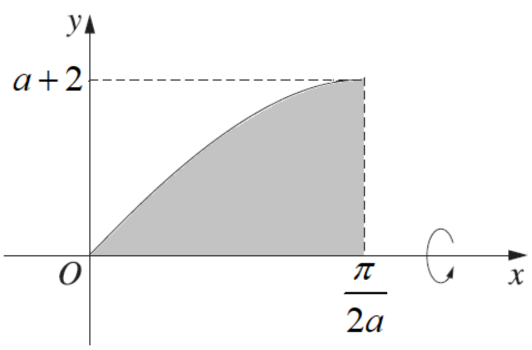
Use the full space provided for your sketch on page 10.

Question 12 (15 marks) Use a separate Writing Booklet

(a) Find an expression for the coefficient of $x^8 y^4$ in the expansion of $(3x+2y)^{12}$. **2**

(b) Differentiate $y = \ln\left(\sin^{-1} 4x\right)$. **2**

(c) A solid of revolution is to be found by rotating the region bounded by the x-axis and the curve $y=(a+2)\sin(ax)$, where $a>0$, between $x=0$ and $x=\dfrac{\pi}{2a}$ about the x-axis. See the diagram shown. **3**



Find the value of a for which the volume is $2\pi^2$ cubic units.

(d) Use the principle of mathematical induction to prove that for any integer $n \geq 1$, $3^{3n-1}+5^{3n-2}$ is divisible by 7. **3**

(e) A scientist captures and tags 30 trout fish before releasing them back into a lake. Several days later, he randomly catches a sample of 15 trout and finds that 2 of them are tagged.

(i) Show, using appropriate reasoning and calculations, that the total number of trout in the lake is approximately 225. **1**

(ii) Show that the probability of finding at most 3 tagged fish in a random catch of 15 fish is approximately 0.871. **2**

(iii) The scientist continues this experiment, taking 20 separate samples of 15 trout. What is the probability that exactly 4 of these samples contain more than 3 tagged fish? **2**

Question 13 (15 marks) Use a separate Writing Booklet

(a) Solve the equation $\sin 2x = 2\sin^2 x$ for $0 < x < \pi$. **2**

(b) By expressing $\sqrt{3}\sin x + 3\cos x$ in the form $A\sin(x+\alpha)$, solve $\sqrt{3}\sin x + 3\cos x = \sqrt{3}$, for $0 \le x \le 2\pi$. **4**

(c) A water cannon is placed at ground level on a flat horizontal surface. The cannon fires water at a fixed speed v m s^{-1}, but the angle of projection θ may vary. The water follows a parabolic path under the influence of gravity.

Assuming that the point of projection is the origin, the position of the water at time t is given by the vector function:

$$\underset{\sim}{r}(t) = (vt\cos\theta)\underset{\sim}{i} + \left(vt\sin\theta - \frac{1}{2}gt^2\right)\underset{\sim}{j}$$

where g m s^{-2} is the acceleration due to gravity. (Do NOT prove this.)

(i) Show that the water returns to ground level at a horizontal distance of $\dfrac{v^2\sin 2\theta}{g}$ metres from the cannon. **2**

Now suppose the cannon is aimed at a vertical wall that is 20 metres high and located 40 metres away (horizontally) from the cannon. It is known that when the angle of projection θ is 15°, the water just reaches the base of the wall

(ii) Show that $v^2 = 80g$. **1**

(iii) Show that the Cartesian equation of the path of the water is given by **2**

$$y = x\tan\theta - \frac{x^2\sec^2\theta}{160}.$$

(iv) Show that the water just clears the top of the wall if $\tan^2\theta - 4\tan\theta + 3 = 0$. **2**

(v) Hence, find all values of θ, to the nearest degree, for which the water just clears the top of the wall. **2**

Question 14 (15 marks) Use a separate Writing Booklet

(a) Find the Cartesian form of the equation for $x = -2 + 5\cos\theta$, $y = 4 + 5\sin\theta$ and describe its shape. **2**

(b) What is the range of the function $y = 4\sin^{-1}(4x) + 4\cos^{-1}(4x)$? **1**

(c) A binomial random variable X is defined such that $E(X) = 3.5$ and $Var(X) = 2.625$. **2**
Calculate $P(X = 5)$, correct to 4 decimal places.

(d) The eleven letters of the word **EXAMINATION** are written on eleven separate pieces of A5 paper.

(i) Find the number of arrangements that can be made using these eleven letters. **1**

(ii) Find the probability that the four letter word **EXAM** will appear in one of these eleven letter arrangement. **2**

(e) Consider the differential equation $\left(e^x + 1\right)\dfrac{dy}{dx} = e^x \sec y$. **3**

Using separation of variables, find the solution if its curve passes through the origin.

(f) Let A and B be points on the number plane where $\overrightarrow{AB}$ is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. **4**

Let C be another point so that the area of ΔABC is 5 units^2.

If $\left|\overrightarrow{AC}\right| = 2\sqrt{5}$, find all possible vectors $\overrightarrow{AC}$.

End of paper

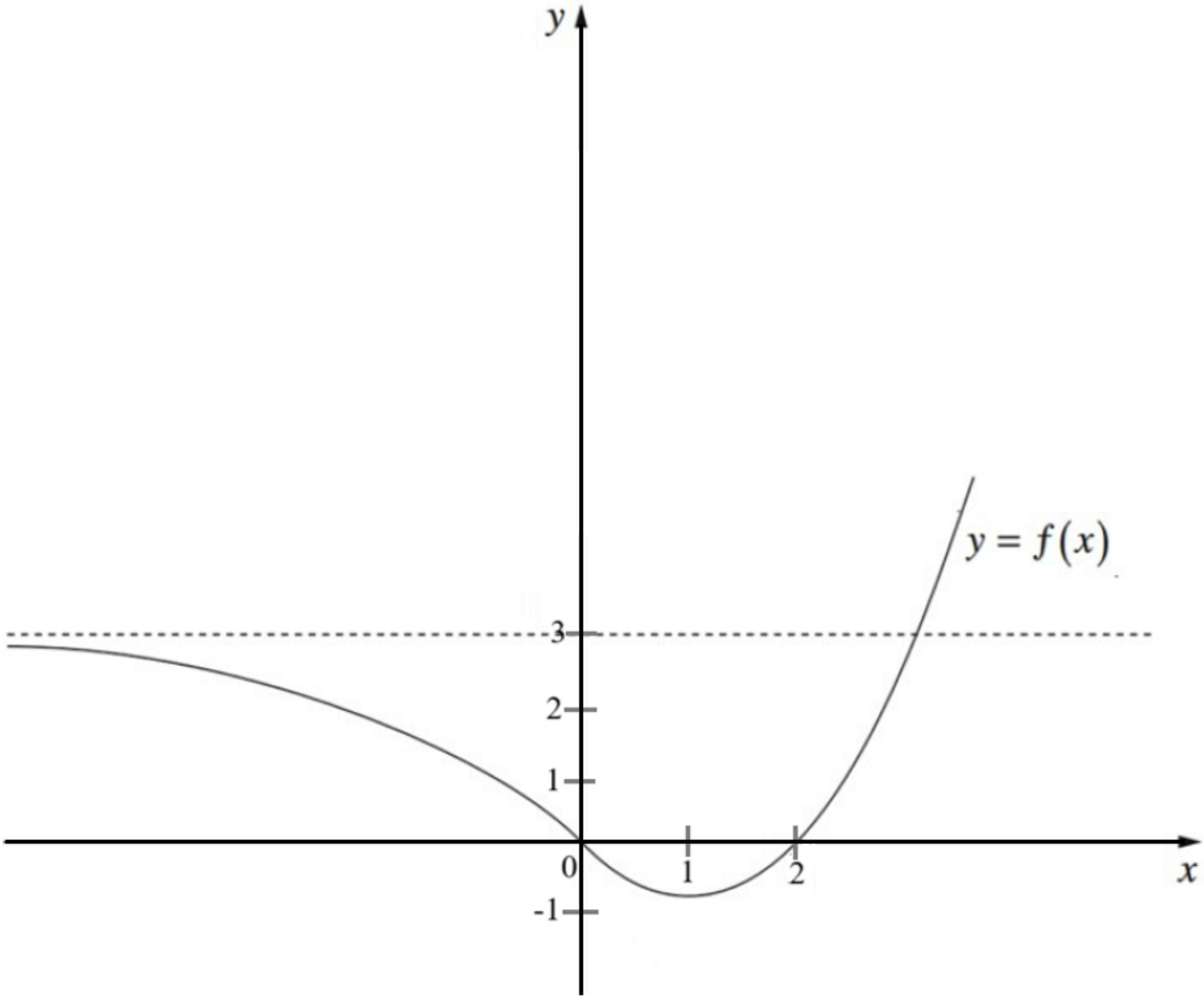
Student Number: | | | | | | | | |

Teacher Name: ____________________

Question 11 Detach this page and insert into your Question 11 writing booklet.

(e) The diagram shows the graph of $y = f(x)$, which has a horizontal asymptote at $y = 3$. **3**

Sketch the graph of $y = (f(x))^2$.



Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 Which expression is equivalent to $\cos(4\theta+\alpha)\cos(3\theta+\alpha)+\sin(4\theta+\alpha)\sin(3\theta+\alpha)$?

A. $\cos(7\theta+2\alpha)$

B. $\sin(7\theta+2\alpha)$

C. $\cos(\theta+2\alpha)$

D. $\cos\theta$

$\cos A \cos B + \sin A \sin B = \cos(A-B)$

$A = 4\theta+\alpha$

$B = 3\theta+\alpha$

$A-B=\theta$

2 An eight-sided die is rolled 64 times and the number of times a 7 appears is noted. What is the mean and standard deviation of the binomial distribution formed?

A. $E(X)=10.7,\ \sigma=8.9$

B. $E(X)=10.7,\ \sigma=2.981$

C. $E(X)=8,\ \sigma=2.646$

D. $E(X)=8,\ \sigma=7$

$P(7)=\frac{1}{8}$

$n=64$

$E(x)=np=64\times\frac{1}{8}=8$

$Var(x)=npq=64\times\frac{1}{8}\times\frac{7}{8}=7$

$\therefore \sigma=\sqrt{7}\approx 2.646$

3 A particle moves through the air according to the displacement vector $\underset{\sim}{r}=\begin{pmatrix}3t\\4t-2t^2\end{pmatrix}$, where $\underset{\sim}{r}$ is measured in metres. What horizontal distance will the particle travel before it hits the ground?

A. 2 m

B. 4 m

C. 6 m

D. 9 m

$x=3t$

$y=4t-2t^2$

let $y=0$

$4t-2t^2=0$

$2t(2-t)=0$

$\therefore t=0,\ t=2$

$\therefore x=3\times2=6$

4 Let α, β, γ and δ be the roots of the equations $x^4+2x^3-5x^2+7x-11=0$.

Find $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}$.

A. $\frac{7}{11}$

B. $-\frac{7}{11}$

C. $-\frac{11}{7}$

D. $\frac{2}{11}$

$a=1,\ b=2,\ c=-5,\ d=7,\ e=-11$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma}{\alpha\beta\gamma\delta}=\frac{\frac{-d}{a}}{\frac{e}{a}}$$

$$=\frac{-7}{-11}=\frac{7}{11}$$

5 Use the substitution $u=x^2+4x-32$ to find $\int(x+2)\sqrt[3]{x^2+4x-32}\,dx$.

A. $\frac{1}{2}\sqrt[3]{x^2+4x-32}+c$

B. $\frac{3}{8}\left(x^2+4x-32\right)^{\frac{4}{3}}+c$

C. $\frac{3}{4}\left(x^2+4x-32\right)^{\frac{3}{4}}+c$

D. $\frac{3}{2}\left(x^2+4x-32\right)^{\frac{4}{3}}+c$

$u=x^2+4x-32$

$du=(2x+4)\,dx$

$du=2(x+2)\,dx$

$\frac{du}{2}=(x+2)dx$

$I=\frac{1}{2}\int u^{\frac{1}{3}}\,du$

$=\frac{1}{2}\times\frac{3}{4}u^{\frac{4}{3}}+C$

6 Which expression is equal to $\frac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$ if $t=\tan\frac{\theta}{2}$?

A. t

B. $\frac{1}{t}$

C. $2t$

D. $\frac{1}{2t}$

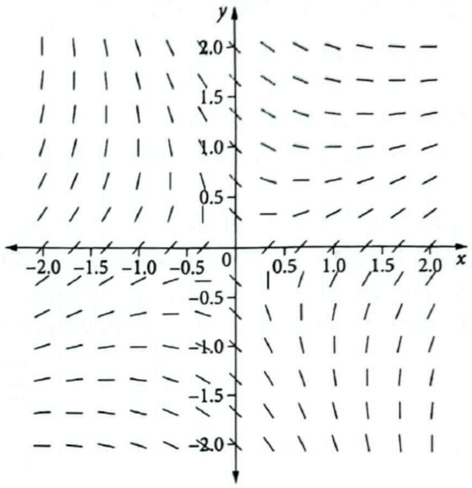
$$\frac{\left(1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}\right)(1+t^2)}{\left(1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}\right)(1+t^2)}$$

$$=\frac{1+\cancel{t^2}+2t+1-\cancel{t^2}}{\cancel{1}+t^2+2t-\cancel{1}+t^2}$$

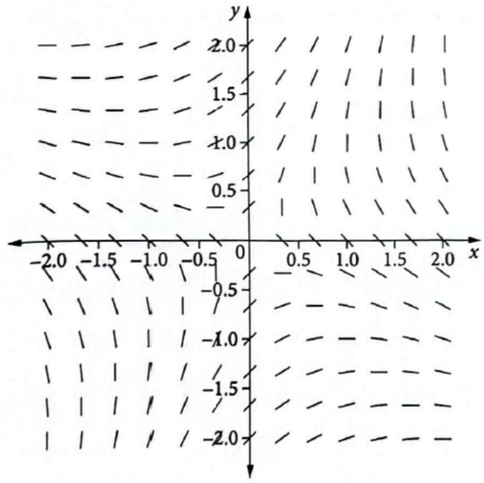
$$=\frac{2+2t}{2t^2+2t}=\frac{2\cancel{(1+t)}}{2t\cancel{(t+1)}}=\frac{1}{t}$$

7 Which of the following slope field best represents the differential equation $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$?

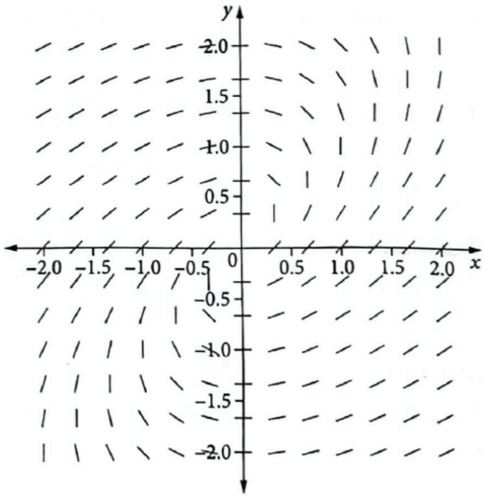
A.



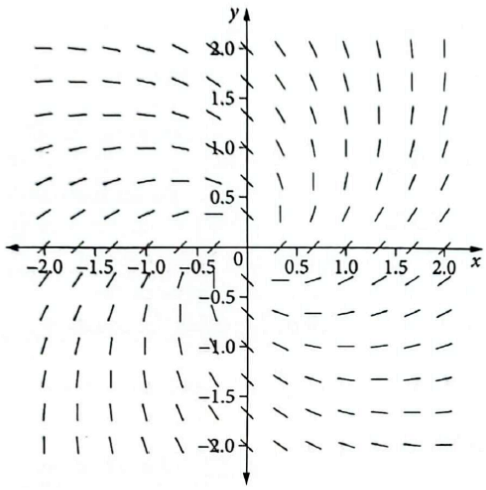
B.



C.



D.



8 The population $P(t)$ of a certain species changes over time according to the differential equation $\dfrac{dP}{dt} = P\left(2 - \dfrac{P}{10\ 000}\right)$ with an initial population $P(0) = 4000$, where t is the time in years.

Note: $\dfrac{1}{P} + \dfrac{1}{20\ 000 - P} = \dfrac{20\ 000}{P(20\ 000 - P)}$.

What is the population as $t \to \infty$?

A. 4 000

B. 5 000

C. 10 000

D. 20 000

$\dfrac{dP}{dt} = P\left(\dfrac{20000-P}{10000}\right)$

$\dfrac{dP}{P(20000-P)} = \dfrac{dt}{10000}$

$\dfrac{20000}{P(20000-P)}\,dP = 2\,dt$

$\left(\dfrac{1}{P} + \dfrac{1}{20000-P}\right) dP = 2\,dt$

$\ln|P| - \ln|20000-P| = 2t + C$

$\ln \dfrac{P}{20000-P} = 2t + C$

when $t = 0$, $P = 4000$

$\therefore c = \ln\frac{1}{4}$

$\ln \dfrac{4P}{20000-P} = 2t$

$\dfrac{20000-P}{4P} = e^{-2t}$

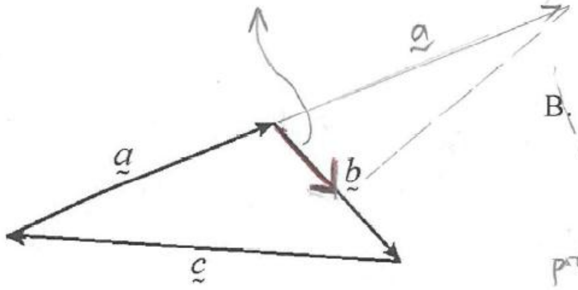
when $t \to \infty$

$20000 - P = 0$

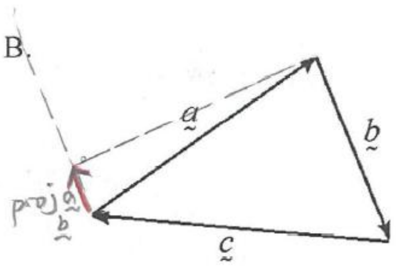
$\therefore P = 20000$

9 In which diagram is $\left|proj_{\underset{\sim}{b}}\underset{\sim}{a}\right| > |\underset{\sim}{b}|$ (diagrams are drawn to scale)?

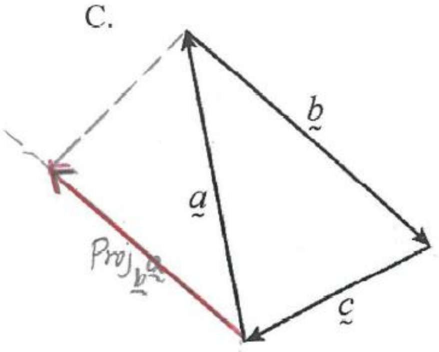
A.



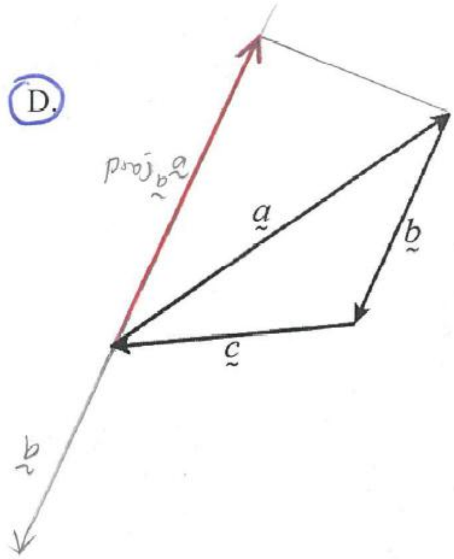
B.



C.



D.



10 Starting from any letter '**T**' as shown in the diagram, the word **TENET** can be spelled by moving up, down, left, or right to an adjacent letter. Letters may be revisited (i.e., repetition is allowed). How many ways are there to spell **TENET**?

A. 64

B. 100

C. 144

D. 196

		T		
	T	**E**	**T**	
T	**E**	**N**	**E**	**T**
	T	**E**	**T**	
		T		

There are 12 ways to spell TEN
and 12 ways to spell NET
∴ $12 \times 12 = 144$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

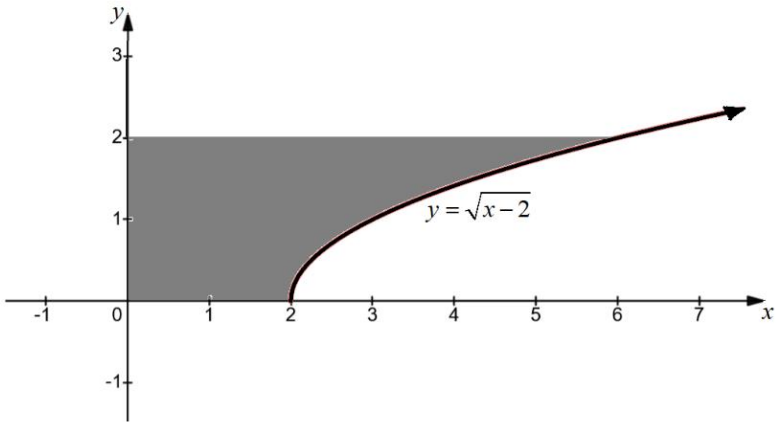
Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate Writing Booklet

(a) The shaded region bounded by the curve $y = \sqrt{x-2}$, the coordinate axes and the line $y = 2$ shown below is rotated about the y-axis. **3**

Calculate the volume of the solid generated.



$V = \pi \int_0^2 x^2 \, dy$

$y = \sqrt{x-2}$

$y^2 = x - 2$

$x = y^2 + 2$

$x^2 = (y^2+2)$

$= y^4 + 4y^2 + 4$

common error: some students found $y^2 = x-2$ and used this.

$V = \pi \int_0^2 y^4 + 4y^2 + 4 \, dy$ ✓

many students expanded $(y^2+2)^2$ as $y^4 + 2y^2 + 4$

$= \pi \left[\frac{y^5}{5} + \frac{4y^3}{3} + 4y \right]_0^2$

$= \pi \left[\left(\frac{(2)^5}{5} + \frac{4(2)^3}{3} + 4(2) \right) - (0 + 0 + 0) \right]$ ✓

$= \pi \left[\frac{32}{5} + \frac{32}{3} + 8 \right]$

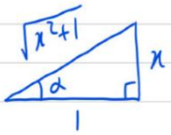
$= \frac{376\pi}{15} \; u^3$ ✓

(b) Show that $\cos\left(\tan^{-1} x\right) = \dfrac{1}{\sqrt{x^2+1}}$. **1**

$\cos(\tan^{-1} x) = \dfrac{1}{\sqrt{x^2+1}}$

let $\alpha = \tan^{-1} x$

$\therefore x = \tan\alpha$



$\therefore \cos\alpha = \dfrac{1}{\sqrt{x^2+1}}$

$\cos(\tan^{-1} x) = \dfrac{1}{\sqrt{x^2+1}}$

(c) Eleanor placed an object in a freezer to cool. After t minutes, its temperature is T°C. The freezer is at a constant temperature of –8°C.

The object's temperature decreases according to the differential equation $\dfrac{dT}{dt} = k(T+8)$.

(i) If the object's initial temperature is 40°C, solve the differential equation for T in terms of k and t. **2**

(ii) If the object cools to 30°C in half an hour, show that $k = \dfrac{1}{30}\ln\left(\dfrac{19}{24}\right)$. **1**

(iii) When will the temperature of the object be 0°C correct to the nearest minute? **2**

i) $\dfrac{dT}{T+8} = k\ dt$

$\ln|T+8| = kt + c$

when $t=0$, $T=40$

$\ln|48| = 0 + c$

$\therefore c = \ln|48|$

$\ln|T+8| = kt + \ln|48|$

$\ln|T+8| - \ln|48| = kt$

$\ln\left|\dfrac{T+8}{48}\right| = kt$

$e^{\ln\left|\frac{T+8}{48}\right|} = e^{kt}$

$\dfrac{T+8}{48} = e^{kt}$

$T+8 = 48e^{kt}$

$T = 48e^{kt} - 8$

ii) From part i)

$T = 48e^{kt} - 8$

sub $t=30$, $T=30$

$30 = 48e^{kt} - 8$

$38 = 48e^{kt}$

$e^{kt} = \dfrac{38}{48}$

$\ln e^{kt} = \ln\dfrac{38}{48}$

$kt = \ln\dfrac{38}{48}$

$30k = \ln\dfrac{38}{48}$

$k = \dfrac{1}{30}\ln\left(\dfrac{19}{24}\right)$

iii) $T = 48e^{kt} - 8$

when $T=0$, $t=?$

$0 = 48e^{kt} - 8$

$e^{kt} = \dfrac{8}{48}$

$kt = \ln\dfrac{1}{6}$

$t = \dfrac{\ln\frac{1}{6}}{\frac{1}{30}\ln\left(\frac{19}{24}\right)}$

$t \approx 230$ mins

(d) Solve $\dfrac{x^2+4}{x} < 5$. **3**

$\dfrac{x^2+4}{x} \times x^2 < 5 \times x^2$ ✓ $\times x^2$

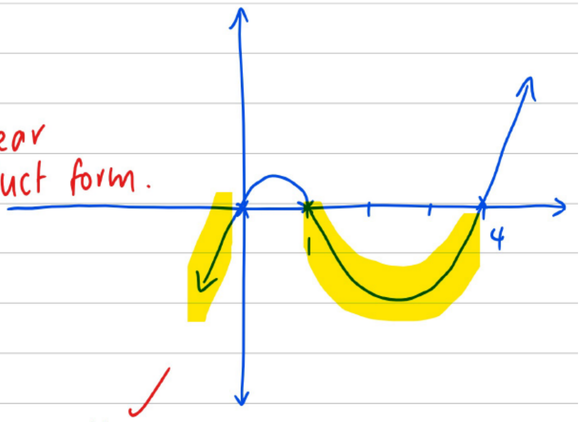
$(x^2+4)x < 5x^2$

$x^3+4x-5x^2 < 0$

$x^3-5x^2+4x < 0$

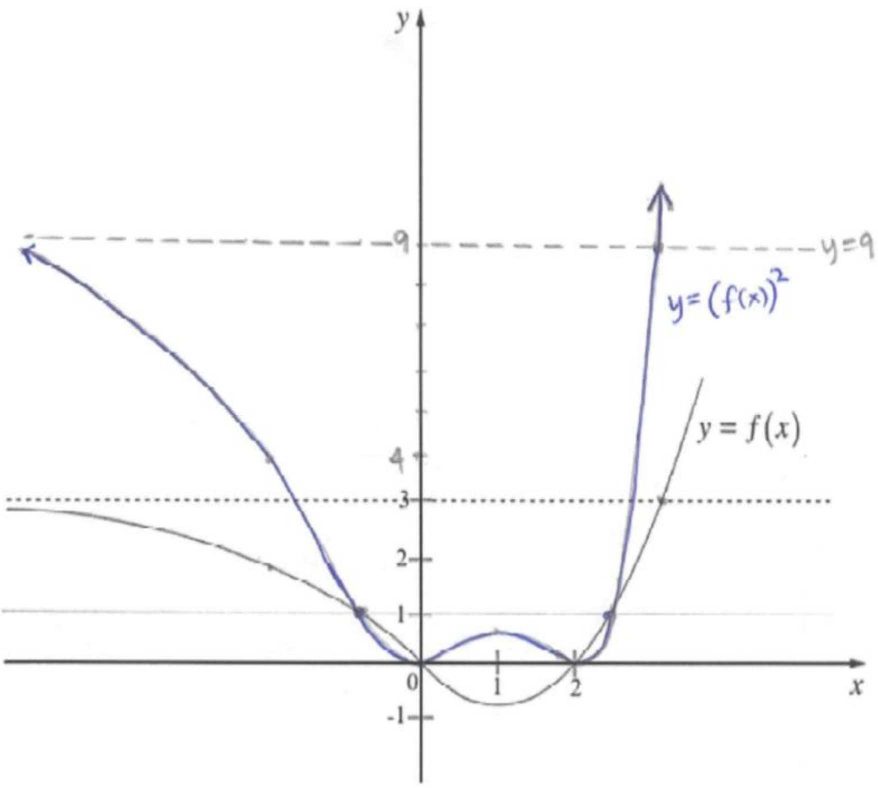
$x(x^2-5x+4) < 0$ ✓ linear product form.

$x(x-1)(x-4) < 0$



$x<0,\ 1<x<4$ ✓

(e) The diagram shows the graph of $y = f(x)$, which has a horizontal asymptote at $y = 3$. **3**



1 mark
- end points (limiting behaviour)

1 mark
- y values between 0 and 1
$(f(x))^2 < f(x)$

1 mark
domain
$0 \le x \le 2$
y value < 1.

Question 12 (15 marks) Use a separate Writing Booklet

(a) Find an expression for the coefficient of $x^8 y^4$ in the expansion of $(3x+2y)^{12}$. **2**

coefficient of x^8y^4 of $(3x+2y)^{12}$

${}^{12}C_r (3x)^r (2y)^{12-r}$

for $r = 8$:

${}^{12}C_8 (3)^8 (2)^4 x^8 y^4$

coefficient of x^8y^4

$= 51963120$

This question was done well.

(b) Differentiate $y = \ln\left(\sin^{-1} 4x\right)$. **2**

$y = \ln(\sin^{-1} 4x)$

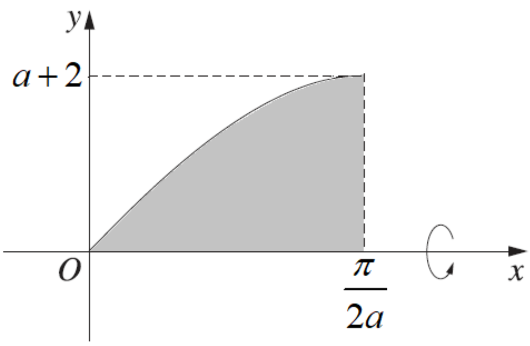
let $u = \sin^{-1} 4x$

$u' = \dfrac{4}{\sqrt{1-16x^2}}$

$\therefore y' = \dfrac{4}{(\sin^{-1}4x)\sqrt{1-16x^2}}$

This question was done well.

(c) A solid of revolution is to be found by rotating the region bounded by the x-axis and the curve $y = (a+2)\sin(ax)$, where $a > 0$, between $x = 0$ and $x = \dfrac{\pi}{2a}$ about the x-axis. See the diagram shown. **3**



Find the value of a for which the volume is $2\pi^2$ cubic units.

$V = \pi \int_0^{\frac{\pi}{2a}} y^2 \, dx$

where $y = (a+2)\sin(ax)$

$y^2 = (a+2)^2 \sin^2(ax)$ (1 mark)

$V = \pi \int_0^{\frac{\pi}{2a}} (a+2)^2 \sin^2(ax) \, dx$

(1 mark for using identity)

$V = \pi (a+2)^2 \int_0^{\frac{\pi}{2a}} \dfrac{1-\cos 2ax}{2} \, dx$

Given $V = 2\pi^2$

$2\pi^2 = \pi(a+2)^2 \left[\frac{1}{2}x - \dfrac{\sin 2ax}{4a}\right]_0^{\frac{\pi}{2a}}$

$2\pi = (a+2)^2 \left[\frac{\pi}{4a} - 0 - (0-0)\right]$

$2 = \dfrac{(a+2)^2}{4a}$

$(a+2)^2 = 8a$

$a^2 + 4a + 4 - 8a = 0$

$a^2 - 4a + 4 = 0$

$(a-2)^2 = 0$

$\therefore a = 2$

Some students got x = 2 with incorrect working so no mark is given.

(d) Use the principle of mathematical induction to prove that for any integer $n \geq 1$, $3^{3n-1} + 5^{3n-2}$ is divisible by 7. **3**

$3^{3n-1} + 5^{3n-2}$ is divisible by 7, $n \geq 1$

For $n = 1$

$3^{3-1} + 5^{3-2}$

$= 3^2 + 5$

$= 14$ is divisible by 7

Assume true for $n = k$,

i.e. $3^{3k-1} + 5^{3k-2} = 7M$, where $M \in \mathbb{Z}$

$\Rightarrow 3^{3k-1} = 7M - 5^{3k-2}$

Prove true for $n = k+1$

i.e. $3^{3(k+1)-1} + 5^{3(k+1)-2} = 7P$, $P \in \mathbb{Z}$

$$\begin{aligned}
LHS &= 3^{3k+2} + 5^{3k+1} \\
&= 3^{3k-1} \times 3^3 + 5^{3k-2} \times 5^3 \\
&= (7M - 5^{3k-2}) \times 27 + 5^{3k-2} \times 125 \quad \text{(from assumption)} \\
&= 7M \times 27 - 5^{3k-2}(27 - 125) \\
&= 7M \times 27 + 98 \times 5^{3k-2} \\
&= 7(27M + 14 \times 5^{3k-2}) \\
&= 7P, \ P \in \mathbb{Z}
\end{aligned}$$

$\therefore$ by the principle of mathematical induction,
$3^{3n-1} + 5^{3n-2}$ is divisible by 7 for all integer $n \geq 1$.

This question was mostly done well.
Some students did not include $M \in \mathbb{Z}$ or $P \in \mathbb{Z}$.
It was not penalised this time but make sure to do this on your HSC.

(e) A scientist captures and tags 30 trout fish before releasing them back into a lake. Several days later, he randomly catches a sample of 15 trout and finds that 2 of them are tagged.

(i) Show, using appropriate reasoning and calculations, that the total number of trout in the lake is approximately 225. **1**

(ii) Show that the probability of finding at most 3 tagged fish in a random catch of 15 fish is approximately 0.871. **2**

(iii) The scientist continues this experiment, taking 20 separate samples of 15 trout. What is the probability that exactly 4 of these samples contain more than 3 tagged fish? **2**

(i) Let n be the total number of trout fish in the lake

$$\frac{n}{30} = \frac{15}{2}$$

$$n = \frac{15}{2} \times 30 = 225$$

$\therefore$ there are 225 trout fish

(ii) $P(X \leq 3) = P(0) + P(1) + P(2) + P(3)$

$$= {}^{15}C_0 \left(\frac{2}{15}\right)^0 \left(\frac{13}{15}\right)^{15} + {}^{15}C_1 \left(\frac{2}{15}\right)^1 \left(\frac{13}{15}\right)^{14} + {}^{15}C_2 \left(\frac{2}{15}\right)^2 \left(\frac{13}{15}\right)^{13} + {}^{15}C_3 \left(\frac{2}{15}\right)^3 \left(\frac{13}{15}\right)^{12}$$

$$= 0.116891 + 0.269749 + 0.290499 + 0.193666$$

$$= 0.870805$$

$$\approx 0.871$$

(iii) $P(\text{more than 3 tagged fish}) = 1 - 0.871 = 0.129$

$X \sim Bin(20, 0.129)$

$$P(X=4) = {}^{20}C_4 \, (0.129)^4 \times (0.871)^{16}$$

$$= 0.147$$

This question was done well.

Question 13 (15 marks) Use a separate Writing Booklet

(a) Solve the equation $\sin 2x = 2\sin^2 x$ for $0 < x < \pi$. **2**

$\sin 2x = 2\sin^2 x$, $0 < x < \pi$

$2\sin x \cos x = 2\sin^2 x$

$2\sin x(\cos x - \sin x) = 0$

$\sin x = 0$

$x = 0, \pi, 2\pi, \ldots$

but $0 < x < \pi$

$\therefore$ no solution

$\cos x - \sin x = 0$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$

for $0 < x < \pi$

$\therefore \boxed{x = \frac{\pi}{4}}$ is the only solution.

Common errors: Students did not consider $\sin x = 0$. Students should not simply cancel $\sin x$ from both sides. Some students included 0 and π as part of their solutions.

(b) By expressing $\sqrt{3}\sin x + 3\cos x$ in the form $A\sin(x+\alpha)$, solve $\sqrt{3}\sin x + 3\cos x = \sqrt{3}$, for $0 \le x \le 2\pi$. **4**

$\sqrt{3}\sin x + 3\cos x = A\sin(x+\alpha)$

$= A(\sin x\cos\alpha + \cos x\sin\alpha)$

equating both sides:

$\sqrt{3} = A\cos\alpha \rightarrow ①$

$3 = A\sin\alpha \rightarrow ②$

$①^2 + ②^2$

$\sqrt{3}^2 + 3^2 = A^2(\cos^2\alpha + \sin^2\alpha)$

$12 = A^2$

$\therefore A = 2\sqrt{3}$

$② \div ①$

$\tan\alpha = \frac{3}{\sqrt{3}} = \sqrt{3}$

$\therefore \alpha = \frac{\pi}{3}$

$\therefore 2\sqrt{3}\sin\left(x + \frac{\pi}{3}\right) = \sqrt{3}$

$\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

$x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$

$\therefore x = -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}$

but $0 \le x \le 2\pi$

$\therefore \boxed{x = \frac{\pi}{2}, \frac{11\pi}{6}}$

Common errors: Students writing $\tan\alpha = \frac{\sqrt{3}}{3}$. Students included $\frac{-\pi}{6}$ and/or excluding $\frac{11\pi}{6}$ as final answers.

Students who stated $\frac{\pi}{3} \le x + \frac{\pi}{3} \le \frac{7\pi}{3}$ did much better in avoiding this mistake.

(c) A water cannon is placed at ground level on a flat horizontal surface. The cannon fires water at a fixed speed v m s^{-1}, but the angle of projection θ may vary. The water follows a parabolic path under the influence of gravity.

Assuming that the point of projection is the origin, the position of the water at time t is given by the vector function:

$$\underset{\sim}{r}(t) = (vt\cos\theta)\underset{\sim}{i} + \left(vt\sin\theta - \frac{1}{2}gt^2\right)\underset{\sim}{j}$$

where g m s^{-2} is the acceleration due to gravity. (Do NOT prove this.)

(i) Show that the water returns to ground level at a horizontal distance of $\dfrac{v^2\sin 2\theta}{g}$ metres from the cannon. **2**

at ground level, height = 0

$vt\sin\theta - \frac{1}{2}gt^2 = 0$

$t(v\sin\theta - \frac{1}{2}gt) = 0$

$t = 0$, $v\sin\theta - \frac{1}{2}gt = 0$

$v\sin\theta = \frac{1}{2}gt$

$\therefore t = \dfrac{2v\sin\theta}{g}$

For horizontal distance:

$vt\cos\theta$

$= v\left(\dfrac{2v\sin\theta}{g}\right)\cos\theta$

$= \dfrac{v^2(2\sin\theta\cos\theta)}{g}$

$= \dfrac{v^2\sin 2\theta}{g}$

Need to show this!

Now suppose the cannon is aimed at a vertical wall that is 20 metres high and located 40 metres away (horizontally) from the cannon. It is known that when the angle of projection θ is 15°, the water just reaches the base of the wall

(ii) Show that $v^2 = 80g$. **1**

When $\theta = 15°$, $x = 40$, $y = 20$

$40 = \dfrac{v^2\sin(2\times 15°)}{g}$

$v^2 = \dfrac{40g}{\sin 30°}$

$v^2 = 40g \div \frac{1}{2}$

$v^2 = 80g$

It is important that students show adequate working (2 lines are not sufficient) for a show question.

(iii) Show that the Cartesian equation of the path of the water is given by **2**

$$y = x\tan\theta - \frac{x^2\sec^2\theta}{160}.$$

$y = vt\sin\theta - \frac{1}{2}gt^2 \rightarrow (1)$

$x = vt\cos\theta \rightarrow (2)$

From (2)

$t = \frac{x}{v\cos\theta}$ Now, sub into (1)

$y = \cancel{v}\left(\frac{x}{\cancel{v}\cos\theta}\right)\sin\theta - \frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2$

$y = x\tan\theta - \frac{1}{2}g\frac{x^2}{v^2\cos^2\theta}$

sub $v^2 = 80g$

$y = x\tan\theta - \frac{\cancel{g}x^2}{2(80\cancel{g})\cos^2\theta}$

$\therefore y = x\tan\theta - \frac{x^2\sec^2\theta}{160}$

This part of the question was done well.

(iv) Show that the water just clears the top of the wall if $\tan^2\theta - 4\tan\theta + 3 = 0$. **2**

when $y = 20$, $x = 40$

using (iii)

$20 = 40\tan\theta - \frac{40^2\sec^2\theta}{160}$

$20 = 40\tan\theta - 10\sec^2\theta$

$20 = 40\tan\theta - 10(1 + \tan^2\theta)$

$20 = 40\tan\theta - 10 - 10\tan^2\theta$

$10\tan^2\theta - 40\tan\theta + 30 = 0$

$\therefore \tan^2\theta - 4\tan\theta + 3 = 0$

Students who tried to work backwards struggled to sufficiently “show” the result. Needed some middle line between substitution into the Cartesian equation and $y = 20$.

(v) Hence, find all values of θ, to the nearest degree, for which the water just clears the top of the wall. **2**

$\tan^2\theta - 4\tan\theta + 3 = 0$

$(\tan\theta - 3)(\tan\theta - 1) = 0$

$\tan\theta = 3$ or $\tan\theta = 1$

$\theta = 71°34'$ $\theta = 45°$

$\therefore$ the water just clears the top of the wall when $\theta = 45°$ or $72°$.

Some students gave reflex angles which are not valid values of θ.

Question 14 (15 marks) Use a separate Writing Booklet

(a) Find the Cartesian form of the equation for $x = -2 + 5\cos\theta$, $y = 4 + 5\sin\theta$ and describe its shape. **2**

$x = -2 + 5\cos\theta$ | $y = 4 + 5\sin\theta$

$\frac{x+2}{5} = \cos\theta$ | $\frac{y-4}{5} = \sin\theta$

As $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{x+2}{5}\right)^2 + \left(\frac{y-4}{5}\right)^2 = 1$$

$$(x+2)^2 + (y-4)^2 = 5^2$$

it is a circle with centre $(-2, 4)$ and radius $= 5$ units.

Some students forgot to provide description of the shape. Students who only mentioned that the shape is a circle were not awarded a mark.

(b) What is the range of the function $y = 4\sin^{-1}(4x) + 4\cos^{-1}(4x)$? **1**

Range of $y = 4\sin^{-1}(4x) + 4\cos^{-1}(4x)$

$y = 4\left(\sin^{-1}(4x) + \cos^{-1}(4x)\right)$

$y = 4 \times \frac{\pi}{2} = 2\pi$

$\therefore$ range $= 2\pi$

This question was not done well.

(c) A binomial random variable X is defined such that $E(X) = 3.5$ and $Var(X) = 2.625$. Calculate $P(X = 5)$, correct to 4 decimal places. **2**

Given: $E(X) = 3.5$, $Var(X) = 2.625$

$np = 3.5$

$np(1-p) = 2.625$

$1 - p = \frac{2.625}{3.5} = 0.75$

$\therefore p = 0.25$

$q = 0.75$

$n(0.25) = 3.5$

$n = 14$

$X \sim Bin(14, 0.25)$

$P(X=5) = {}^{14}C_5 (0.25)^5 (0.75)^9$

$= 0.1468$ (4 d.p.)

This question was done very well.

(d) The eleven letters of the word **EXAMINATION** are written on eleven separate pieces of A5 paper.

(i) Find the number of arrangements that can be made using these eleven letters. **1**

EXAMINATION

A = 2, I = 2, N = 2

no. of arrangments $= \dfrac{11!}{2!2!2!} = 4989600$

This question was done very well.

(ii) Find the probability that the four letter word **EXAM** will appear in one of these eleven letter arrangement. **2**

$\underbrace{\boxed{\text{EXAM}}}_{\text{1 unit}}\ \underbrace{\text{INATION}}_{\text{7 units}}$

no. of arrangements

$= \dfrac{8!}{2!2!} = 10080$

$P(\text{"EXAM"}) = \dfrac{10\,080}{4989600}$

$= \dfrac{1}{495}$ or 0.00202

1 mark for calculating the number of arrangements and 1 mark for calculating the probability. Unfortunately, some students forgot to calculate for the probability.

(e) Consider the differential equation $\left(e^x+1\right)\dfrac{dy}{dx} = e^x \sec y$. **3**

Using separation of variables, find the solution if its curve passes through the origin.

$(e^x+1)\dfrac{dy}{dx} = e^x \sec y$

$(e^x+1)\dfrac{dy}{dx} = \dfrac{e^x}{\cos y}$

$\int \cos y \, dy = \int \dfrac{e^x}{e^x+1} dx$

$\sin y = \ln|e^x+1| + C$

Now, $e^x+1>0$, $\therefore |e^x+1| = e^x+1$

So, $\sin y = \ln(e^x+1)+C$

Since the curve passes through $(0,0)$

$\sin 0 = \ln(e^0+1)+C$

$C = -\ln 2$

$\therefore \sin y = \ln(e^x+1) - \ln 2$

$\sin y = \ln\left(\dfrac{e^x+1}{2}\right)$

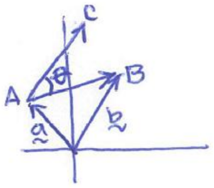
$y = \sin^{-1}\left(\ln\left(\dfrac{e^x+1}{2}\right)\right)$

Common errors: Students integrating incorrectly. But this question was mostly done quite well.

(f) Let A and B be points on the number plane where $\overrightarrow{AB}$ is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. **4**

Let C be another point so that the area of ΔABC is 5 units^2.

If $\left|\overrightarrow{AC}\right| = 2\sqrt{5}$, find all possible vectors $\overrightarrow{AC}$.



Given: $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$A_{\Delta ABC} = 5\,u^2$

$|\overrightarrow{AC}| = 2\sqrt{5}$

$\overrightarrow{AC} = ?$

$|\overrightarrow{AB}| = \sqrt{3^2+1^2}$

$= \sqrt{10}$

$\overrightarrow{AC} \cdot \overrightarrow{AB} = |\overrightarrow{AC}| \times |\overrightarrow{AB}| \cos\theta$

$= 2\sqrt{5} \times \sqrt{10}\, \cos\theta$

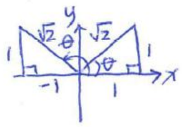
$= 10\sqrt{2}\, \cos\theta$

Area of $\Delta ABC = \frac{1}{2} |\overrightarrow{AC}| \times |\overrightarrow{AB}| \sin\theta$

$5 = \frac{1}{\cancel{2}} \times \cancel{2}\sqrt{5} \times \sqrt{10}\, \sin\theta$

$\frac{5}{\sqrt{50}} = \sin\theta$

$\sin\theta = \frac{1}{\sqrt{2}}$



$\therefore \cos\theta = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

$\therefore \overrightarrow{AC} \cdot \overrightarrow{AB} = 10\sqrt{2} \times \left(\pm\frac{1}{\sqrt{2}}\right) = \pm 10$

Common Error:

Students forgot that if $\sin\theta = \frac{1}{\sqrt{2}}$ then $\cos\theta = \pm\frac{1}{\sqrt{2}}$, hence, producing only 2 possible vectors instead of 4. This question was not done well.

let $\overrightarrow{AC} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 10$

$3x + y = 10 \rightarrow ①$

Also, $|\overrightarrow{AC}|^2 = (2\sqrt{5})^2$

$x^2 + y^2 = 20 \rightarrow ②$

From ① $y = 10 - 3x$

Sub into ②

$x^2 + (10-3x)^2 = 20$

$x^2 + 100 - 60x + 9x^2 = 20$

$10x^2 - 60x + 80 = 0$

$x^2 - 6x + 8 = 0$

$(x-4)(x-2) = 0$

$\therefore x = 4, 2$

$\Rightarrow y = -2, 4$

Likewise,

$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = -10$

$3x + y = -10 \rightarrow ③$

$x^2 + y^2 = 20 \rightarrow ④$

From ③ $y = -10 - 3x$

Sub into ④

$x^2 + (-10-3x)^2 = 20$

$x^2 + 100 + 60x + 9x^2 = 20$

$10x^2 + 60x + 80 = 0$

$x^2 + 6x + 8 = 0$

$(x+4)(x+2) = 0$

$\therefore x = -4, -2$

$\Rightarrow y = 2, -4$

$\therefore$ possible vectors $\overrightarrow{AC}$ are:

$\begin{pmatrix} 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

End of paper