Name:…………………………………..

Student Number

Teacher:………………………………..

Pymble Ladies' College

Mathematics Extension 1

HSC Trial Examination

Term 3 2024

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks 70

SECTION 1 – 10 marks (pages 1-4)

- Attempt Questions 1-10
- Allow about 15 minutes for this section
- Answer each question on the multiple-choice answer sheet provided in the answer booklet.

SECTION II – 60 marks (pages 5-11)

- Attempt Questions 11-14
- Allow about 1 hours and 45 minutes for this section
- Answer each question in the appropriate space in the Answer Booklet. Extra writing pages are included at the end of each question.

SECTION I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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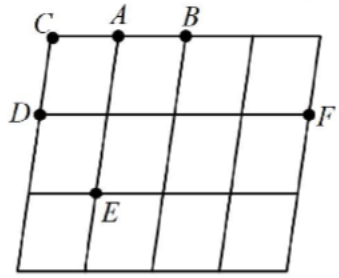
1. The grid shown is made up of identical parallelograms.

Let $\underset{\sim}{a} = \overrightarrow{AB}$ and $\underset{\sim}{c} = \overrightarrow{CD}$. What is the vector $\overrightarrow{EF}$ is equal to?



(A) $\underset{\sim}{a} + 3\underset{\sim}{c}$

(B) $-3\underset{\sim}{a} + \underset{\sim}{c}$

(C) $-3\underset{\sim}{a} - \underset{\sim}{c}$

(D) $3\underset{\sim}{a} - \underset{\sim}{c}$

2. Four female and four male students are to be seated around a circular table.
In how many ways can this be done if the males and females must alternate?

(A) $4! \times 4!$

(B) $3! \times 4!$

(C) $3! \times 3!$

(D) $2 \times 3! \times 3!$

3. Which of the following is equivalent to $\dfrac{d}{dx}\left(2\sin^{-1}\dfrac{x}{2}\right)$?

(A) $\dfrac{1}{\sqrt{1-x^2}}$

(B) $\dfrac{2}{\sqrt{1-x^2}}$

(C) $\dfrac{2}{\sqrt{4-x^2}}$

(D) $\dfrac{1}{2\sqrt{4-x^2}}$

4. Which expression is identical to $2\sin 3x \sin 5x$?

(A) $-\cos 2x - \cos 8x$

(B) $\cos 8x - \cos 2x$

(C) $\cos 2x - \cos 8x$

(D) $\cos 2x + \cos 8x$

5. If $\sin A = t$ and $\cos B = t$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, then what is $\cos(B + A)$ equal to?

(A) 0

(B) $\sqrt{1-t^2}$

(C) $1-2t^2$

(D) $-2t\sqrt{1-t^2}$

6. Let R be the region between the graphs of $y = 1$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$. Which expression gives the volume of the solid obtained by revolving R about the x-axis?

(A) $2\pi \int_0^{\frac{\pi}{2}} x \sin x \, dx$

(B) $\pi \int_0^{\frac{\pi}{2}} (1 - \sin x)^2 \, dx$

(C) $\pi \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$

(D) $\pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

7. A spherical balloon is being inflated at a constant rate of $200\pi\,\text{cm}^3\text{s}^{-1}$. At what rate is the radius of the balloon increasing when the radius is 10 cm?

(A) 0.25 cm s^{-1}

(B) 0.5 cm s^{-1}

(C) 1 cm s^{-1}

(D) 2 cm s^{-1}

8. Which of the following is the domain of $y = 5\cos^{-1}\left(\dfrac{2-x}{3}\right)$?

(A) $x \in [1,5]$

(B) $x \in [-1,5]$

(C) $x \in [-5,1]$

(D) $x \in [-5,-1]$

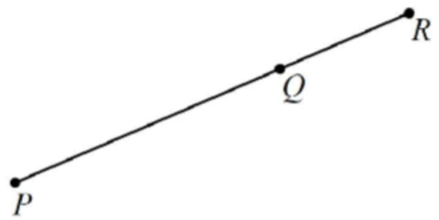
9. PQR is a straight line and $PQ = 2QR$.
If $\overrightarrow{OQ} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\overrightarrow{OR} = \underset{\sim}{i} + 3\underset{\sim}{j}$, where O is the origin, then what is $\overrightarrow{OP}$?

(A) $-\underset{\sim}{i} + 8\underset{\sim}{j}$

(B) $7\underset{\sim}{i} - 12\underset{\sim}{j}$

(C) $4\underset{\sim}{i} - 10\underset{\sim}{j}$

(D) $-4\underset{\sim}{i} + 10\underset{\sim}{j}$



10. The inverse function of $f(x) = \ln(x-1)$ is $g(x)$.
Which one of these statements must be true for all x in the domain of $g(x)$?

(A) $g(x) < 0$

(B) $g'(x) < 0$

(C) $g''(x) > 0$

(D) $g''(x) < 0$

SECTION II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section.

Answer these questions in the Answer Book provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) Find the term independent of x in the expansion of $\left(3x^2 + \frac{2}{x}\right)^{12}$. **2**

(b) Mrs Munro needs to decide the order in which to schedule 8 exams. **2**
Two of these exams are Mathematics Extension 1 and Mathematics Extension 2.

Find the number of different ways that Mrs Munro can schedule the 8 exams so that Mathematics Extension 1 and Mathematics Extension 2 are NOT consecutive.

(c) Solve $\frac{x^2+10}{x} \leq 7$. **3**

(d) The polynomial $P(x) = 8x^4 - 38x^3 + 9x^2 + ax + b$ has a double root at $x = 3$. **3**
Find the values of a and b, where a and b are real numbers.

(e) Evaluate $\int_0^2 \frac{dx}{\sqrt{16-x^2}}$. **2**

(f) Use mathematical induction to prove that $8n^3 - 2n$ is divisible by 3 for all integers $n \geq 1$. **3**

End of Question 11

Question 12 (14 marks) **Marks**

(a) By using the substitution $t = \tan\dfrac{x}{2}$, solve **3**

$$\cos x - \sqrt{3}\sin x + 1 = 0 \quad \text{for } 0 \le x \le 2\pi.$$

(b) Use the substitution $u = x - 4$ to find the following integral: **3**

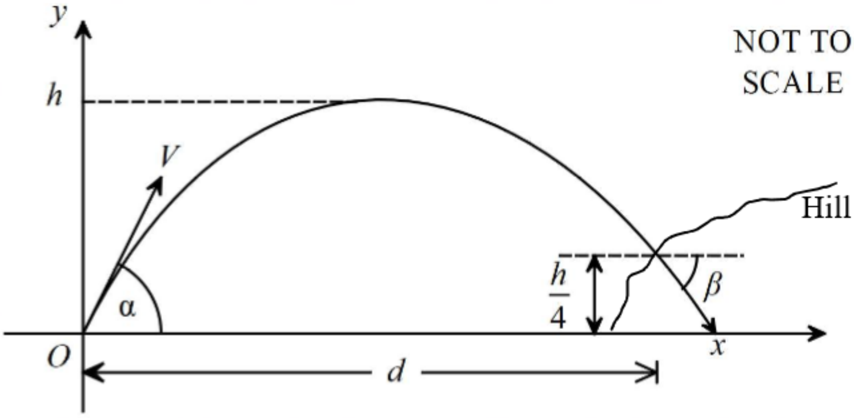
$$\int x\sqrt{x-4}\,dx$$

(c) When the polynomial $P(x)$ is divided by $9x^2 - 1$ the remainder is $3x + 7$. **2**

What is the remainder when $P(x)$ is divided by $3x + 1$?

Question 12 continues on page 7

(d) Julie kicks a soccer ball from the origin O, which is on level ground, with velocity V ms^{-1}, at an angle of α to the horizontal. The ball rises to a maximum height h and lands on a hill at distance d metres from the origin, with a height of $\frac{h}{4}$ metres, making an angle of β with the horizontal as shown.



Use the axes as shown and assume there is no air resistance.

The position vector of the ball, t seconds after being kicked, where g is acceleration due to gravity, is given by

$$\underset{\sim}{r}(t) = (Vt\cos\alpha)\underset{\sim}{i} + \left(Vt\sin\alpha - \frac{g}{2}t^2\right)\underset{\sim}{j}$$

DO NOT prove this

(i) Show that the maximum height h reached by the ball is **2**

$$h = \frac{V^2\sin^2\alpha}{2g}.$$

(ii) Show that the time taken for the ball to land on the hill is **3**

$$t = \frac{\left(2+\sqrt{3}\right)V\sin\alpha}{2g} \text{ seconds.}$$

(iii) Calculate the horizontal distance d travelled by the ball. **1**

End of Question 12

Question 13 (15 marks) **Marks**

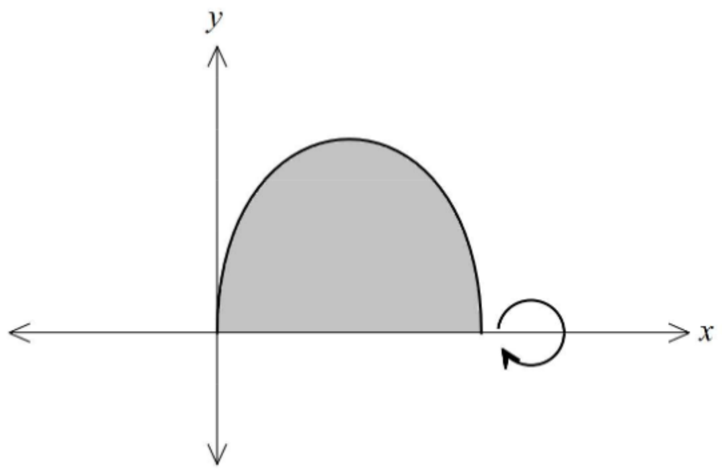
(a) Given the equation $f(x) = x\sin^{-1}\left(\frac{x}{2}\right), \quad -2 \le x \le 2$

(i) show that $f(x)$ is an even function **1**

(ii) sketch the graph of $y = f(x)$ showing all features including intercept(s) and endpoints. **2**

(b) After t minutes the temperature $T°C$ of water in a jug is given by $T = 20 + 80e^{-0.2t}$. What is the rate at which the water is cooling when its temperature has fallen to half its initial value? **2**

(c) Part of the graph of $y = \sqrt{\cos(3x)\sin(2x)}$ is shown in the diagram.



(i) By solving $\cos(3x)\sin(2x) = 0$, show that the smallest positive solution is $x = \frac{\pi}{6}$. **2**

(ii) Hence, find the volume of the solid of revolution formed when the shaded region is rotated around the x-axis. **3**

Question 13 continues on page 9

Question 13 continued. **Marks**

(d) (i) Sketch the curve $y = -\tan^{-1} x$ and label the point where $x = 1$. **2**

(ii) Find the area bounded by the curve, the x-axis and the line $x = 1$. **3**

End of Question 13

Question 14 (16 marks) **Marks**

(a) A bug moves such that its acceleration is given by $\frac{dv}{dt} = \sqrt{v+1}\ ms^{-2}$. **3**

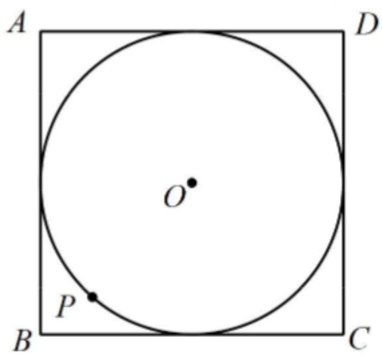
Initially the bug is at rest. Find its velocity after 1 second.

(b) A population of penguins on an island satisfies $\frac{dP}{dt} = 0.001P(400 - P)$ where P is the number of penguins and t is measured in years. Initially there are 50 penguins.

(i) What is the carrying capacity of the island? **1**

(ii) Given that $\frac{1}{P(400-P)} = \frac{1}{400}\left(\frac{1}{P} + \frac{1}{400-P}\right)$, calculate when the population of penguins will reach 50% of the carrying capacity. **3**

(c) In the figure, the circle has centre O and radius r. The circle is inscribed in a square $ABCD$, and P is any point on the circle.

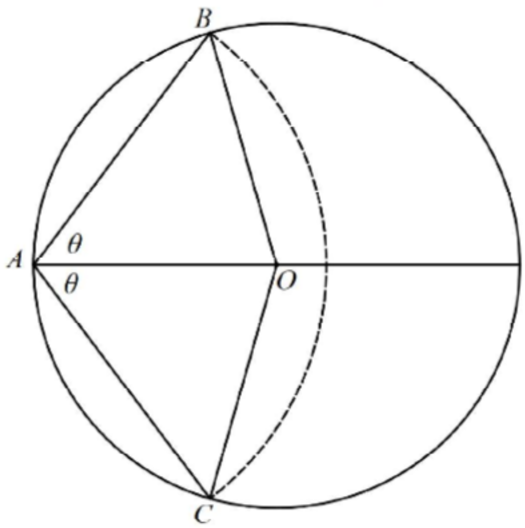


(i) Show that $\overrightarrow{AP} \cdot \overrightarrow{AP} = 3r^2 - 2\overrightarrow{OP} \cdot \overrightarrow{OA}$. **2**

(ii) Hence find $AP^2 + BP^2 + CP^2 + DP^2$ in terms of r. **2**

Question 14 continues on page 11

(d) A is the point on the circumference of a circle with centre O and radius a. With A as the centre, an arc of radius r is drawn which meets the circle at two points B and C, and $r < 2a$. Arc length $BC = \ell$ and $\angle BAC = 2\theta$.



(i) Show that $r = 2a\cos\theta$ and $\ell = 4a\theta\cos\theta$. **2**

(ii) Hence, show that ℓ is a maximum when $\theta = \cot\theta$. **3**

END OF PAPER

Name:......................................	
Student Number:	
Teacher's Name:..........................	

Mathematics Extension 1
HSC Trial Examination
2024

Section I – Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ○ B ● C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.

correct

A ● B ● C ○ D ○

1	A ○	B ○	C ○	D ●
2	A ○	B ●	C ○	D ○
3	A ○	B ○	C ●	D ○
4	A ○	B ○	C ●	D ○
5	A ○	B ○	C ○	D ●
6	A ○	B ○	C ○	D ●
7	A ○	B ●	C ○	D ○
8	A ○	B ●	C ○	D ○
9	A ○	B ●	C ○	D ○
10	A ○	B ○	C ●	D ○

SECTION II 60 marks
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is provided at the end of each question.

Question 11 (16 marks) **Marks**

(a) **2**

$${}^{12}C_r\, 3^{12-r} \left(x^2\right)^{12-r} \left(\frac{2}{x}\right)^r \quad *$$

$$= {}^{12}C_r\, 3^{12-r}\, x^{24-2r} \times 2^r \times x^{-r}$$

$\therefore\ 24 - 3r = 0$

$r = 8$

$\therefore$ Term independent of x is ${}^{12}C_8\, 3^4 \times 2^8$

$= 10264320$

2	Correct
1	Initial expression *

(b) **2**

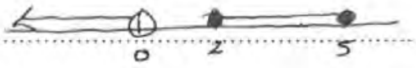
Number of ways $= 8! - 7! \times 2!$

$= 30240$

2	Correct
1	$8!$ correct

(c) **3**

$$\frac{x^2+10}{x} \le 7$$



Consider

$x^2 + 10 = 7x$

$x^2 - 7x + 10 = 0$

$(x-5)(x-2) = 0$

$x = 2, 5$

Test $x = 1$: $\frac{1+10}{1} \le 7$ False

$x = -1$: $\frac{1+10}{-1} \le 7$ True

$x = 3$: $\frac{9+10}{3} \le 7$ True

$x = 6$: $\frac{36+10}{6} \le 7$ False

$\therefore\ x < 0$ or $2 \le x \le 5$

$\frac{x^2+10}{x} \le 7 \quad \times x^2 \quad \boxed{x \ne 0}$

$x(x^2+10) \le 7x^2$

$x(x^2+10) - 7x^2 \le 0$

$x\left[x^2+10-7x\right] \le 0$

$x(x-5)(x-2) \le 0$

$\boxed{x < 0 \text{ or } 2 \le x \le 5}$

add lines

3	Correct
2	One inequality Correct
1	The beginning of a correct approach

Question 11 continues on page 2

Question 11 continued. **Marks**

(d) $P(x) = 8x^4 - 38x^3 + 9x^2 + ax + b$ double root $x=3$ **3**

$P'(x) = 32x^3 - 114x^2 + 18x + a$

$P'(3) = 0 \quad \therefore \quad 32\times3^3 - 114\times3^2 + 18\times3 + a = 0$

$a = 108$

$P(3) = 0 \quad 8\times3^4 - 38\times3^3 + 9\times3^2 + 108\times3 + b = 0$

$b = -27$

3	Correct
2.	Differentiated and one value correct
1	Just differentiated

(e) $\displaystyle\int_0^2 \frac{dx}{\sqrt{16 - x^2}}$ $= \pi/6$ ~~3~~ 2

$= \left[\sin^{-1}\frac{x}{4}\right]_0^2$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$

2.	Correct
1	one error

Question 11 continued. **Marks**

(f) Let the statement be $8n^3 - 2n$ is divisible by 3 $\forall\ n \geq 1$ **3**

When $n = 1$ $\quad 8 \times 1^3 - 2 \times 1$

$= 6$ which is divisible by 3

Assume the statement is true for $n = k$

i.e $8k^3 - 2k = 3P$ where $P \in \mathbb{Z}$ $k \geq 1$

3	Correct
2	Set up and part of argument
1	Just set up and conclusion

RTP true for $n = k + 1$

i.e $8(k+1)^3 - 2(k+1)$ is divisible by 3

Now

$8(k+1)^3 - 2(k+1)$

$= 8(k^3 + 3k^2 + 3k + 1) - 2k - 2$

$= 8k^3 + 24k^2 + 22k + 6$

From assumption $8k^3 = 3P + 2k$, substituting

$= 3P + 2k + 24k^2 + 22k + 6$

$= 3P + 24k^2 + 24k + 6$

$= 3(P + 8k^2 + 8k + 2)$

Since $k \in \mathbb{Z}^+$ and $P \in \mathbb{Z}$, $P + 8k^2 + 8k + 2 \in \mathbb{Z}$

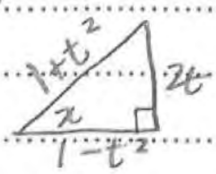
$\therefore 8(k+1)^3 - 2(k+1)$ is divisible by 3

Hence, since the statement is true for $n = 1$ it is true for $n = 1 + 1 = 2$ and so on by the principle of mathematical induction

End of Question 11

Question 12

12 a) $t = \tan \frac{x}{2}$



$\cos x - \sqrt{3} \sin x + 1 = 0 \qquad 0 \leq x \leq 2\pi$

$\frac{1-t^2}{1+t^2} - \frac{2\sqrt{3}t}{1+t^2} + 1 = 0 \qquad 0 \leq \frac{x}{2} \leq \pi$

$1 - t^2 - 2\sqrt{3}t + 1 + t^2 = 0$

$2 - 2\sqrt{3}t = 0$

$2\sqrt{3}t = 2$

$t = \frac{1}{\sqrt{3}}$

$\tan \frac{x}{2} = \frac{1}{\sqrt{3}}$

$\frac{x}{2} = \frac{\pi}{6}$

$x = \frac{\pi}{3}$

check: $x = \pi$ $\quad LHS = \cos \pi - \sqrt{3} \sin \pi + 1$

$= -1 - \sqrt{3}(0) + 1$

$= 0$

$= RHS \quad \therefore x = \pi$ soln

soln: $\boxed{x = \frac{\pi}{3}, \pi}$

[1] $\frac{1-t^2}{1+t^2} - \frac{2\sqrt{3}t}{1+t^2} + 1 = 0$

applies t-formula

[2] $t = \frac{1}{\sqrt{3}}$

$\tan \frac{x}{2} = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{3}$

[3] includes $x = \pi$ as soln.

Question 12

12 b) $I = \int x\sqrt{x-4}\,dx$

$u = x - 4$

$\frac{du}{dx} = 1$

$du = dx$

$x = u + 4$

$I = \int (u+4)\sqrt{u}\,du$

$= \int u^{\frac{1}{2}}(u+4)\,du$

$= \int u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\,du$

$= \frac{2}{5}u^{\frac{5}{2}} + 4.\frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{5}(x-4)^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}} + c$

$= \frac{2}{5}\sqrt{(x-4)^5} + \frac{8}{3}\sqrt{(x-4)^3} + c$

$\boxed{1}$ $\frac{du}{dx} = 1$ or ~~$x = u + 4$~~

$\int u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\,du$ or equivalent

$\boxed{2}$ $I = \frac{2}{5}u^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}} + C$

$\boxed{3}$ $I = \frac{2}{5}(x-4)^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}} + C$

12 (c) $P(x) = (9x^2-1).Q(x) + (3x+7)$

$P(x) = (3x+1)(3x-1).Q(x) + (3x+7)$

so $P(-\frac{1}{3}) = 0 + 3(-\frac{1}{3}) + 7$

$= 6$

remainder $= 6$

$\boxed{1}$ correct polynomial

$\boxed{2}$ uses $P(-\frac{1}{3})$ and correct soln.

Question 12

12 (d) $r(t) = (vt\cos\alpha)\underset{\sim}{i} + \left(vt\sin\alpha - \frac{g}{2}t^2\right)\underset{\sim}{j}$

i) $v(t) = (v\cos\alpha)\underset{\sim}{i} + (v\sin\alpha - gt)\underset{\sim}{j}$

max height at $\dot{y} = 0$

$v\sin\alpha - gt = 0$

$t = \frac{v\sin\alpha}{g}$

sub $\rightarrow$ $y = v\left(\frac{v\sin\alpha}{g}\right)\sin\alpha - \frac{g}{2}\left(\frac{v^2\sin^2\alpha}{g^2}\right)$

$h = \frac{v^2\sin^2\alpha}{g} - \frac{v^2\sin^2\alpha}{2g}$

$h = \frac{2v^2\sin^2\alpha - v^2\sin^2\alpha}{2g}$

$\boxed{h = \frac{v^2\sin^2\alpha}{2g}}$

[1]
- $v(t) = v\cos\alpha\, i + (v\sin\alpha - gt)\underset{\sim}{j}$
- and $v\sin\alpha - gt = 0$
- $t = \frac{v\sin\alpha}{g}$

[2] substitutes t correctly into g and simplifies

12 d) (iii) horizontal distance (x)

at $t = \frac{(2+\sqrt{3})v\sin\alpha}{2g}$

$d = \frac{v\cos\alpha\,(2+\sqrt{3})v\sin\alpha}{2g}$

$d = \frac{v^2\, 2\sin\alpha\cos\alpha\,(2+\sqrt{3})}{4g}$

$d = \frac{v^2\sin 2\alpha\,(2+\sqrt{3})}{4g}$

[1] dist $= v\cos\alpha\left[\frac{(2+\sqrt{3})v\sin}{2g}\right.$

and some correct simplification.

12

If you use this space, clearly indicate which question you are answering.

(d)

(ii) $\frac{h}{4} = \frac{v^2 \sin^2\alpha}{8g}$

sub: $\frac{v^2\sin^2\alpha}{8g} = vt\sin\alpha - \frac{g}{2}t^2$

$\frac{g}{2}t^2 - vt\sin\alpha + \frac{v^2\sin^2\alpha}{8g} = 0$

$t = \frac{v\sin\alpha \pm \sqrt{v^2\sin^2\alpha - 4\left(\frac{g}{2}\right)\cdot\frac{v^2\sin^2\alpha}{8g}}}{g}$

$t = \frac{v\sin\alpha \pm \sqrt{v^2\sin^2\alpha - \frac{v^2\sin^2\alpha}{4}}}{g}$

$t = \frac{v\sin\alpha \pm \sqrt{\frac{3}{4}v^2\sin^2\alpha}}{g}$

$t = \frac{v\sin\alpha \pm \frac{\sqrt{3}}{2}v\sin\alpha}{g} \times \frac{2}{2}$

$= \frac{2v\sin\alpha + \sqrt{3}v\sin\alpha}{2g}$

$t = \frac{(2 \pm \sqrt{3})v\sin\alpha}{2g}$ $t > 0$

need $t > \frac{v\sin\alpha}{g}$, use t-value on way down so bigger value.

so $\boxed{t = \frac{(2+\sqrt{3})v\sin\alpha}{2g}}$

[1]
- $\frac{h}{4} = \frac{1}{4}\left(\frac{v^2\sin^2\alpha}{2g}\right)$
- $4g^2t^2 - 8gv\sin\alpha t + v^2\sin^2\alpha = 0$
- $gt^2 - 2vt\sin\alpha + \frac{v^2\sin^2\alpha}{4g} = 0$
- $\frac{g}{2}t^2 - vt\sin\alpha + \frac{v^2\sin^2\alpha}{8g} =$ (or equivalent)
- $vt\sin\alpha - \frac{g}{2}t^2 = \frac{h}{4}$

[2] develops quadratic and sub into quad formula

[3] selects correct t from quadratic eqn.

Question 13 (15 marks) **Marks**

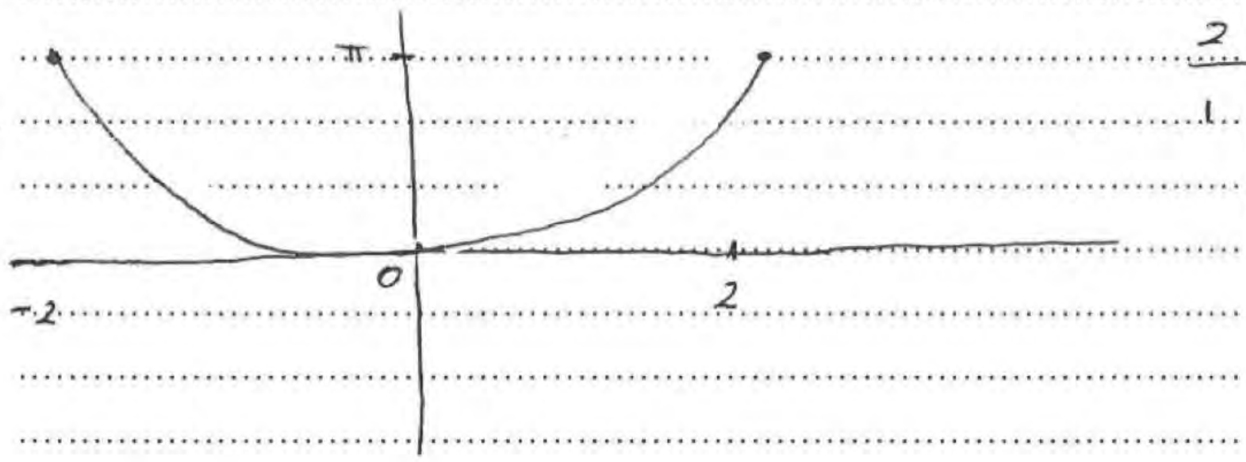
(a)

(i) $f(-x) = -x\sin^{-1}\left(\frac{-x}{2}\right)$ since $f(x) = f(-x)$ 1 r/w

$= x\sin^{-1}\left(\frac{x}{2}\right)$ the f^n is even

$f(x) = x\sin^{-1}\left(\frac{x}{2}\right)$

(ii) Change to a blank set of axes with lines 2



increase to a page

2	Correct (w) $(2, \pi)$
1	Partially correct

$f(2) = 2\sin^{-1} 1$ $f(0) = 0$ $f'(x) = x \times \frac{1/2}{\sqrt{1-(x/2)^2}} + \sin^{-1}\left(\frac{x}{2}\right)$

$= \pi$

$= \frac{2x}{2\sqrt{4-x^2}} + \sin^{-1}\left(\frac{x}{2}\right)$

$= \frac{x}{\sqrt{4-x^2}} + \sin^{-1}\left(\frac{x}{2}\right)$

$f'(1) > 0$ $\therefore (0,0)$

$f'(-1) < 0$ is a min

(b) 2

When $t = 0$ $T = 20 + 80e^0$

$= 100^\circ$

$\frac{dT}{dt} = 80 \times (-0.2)e^{-0.2t}$

$= -0.2 \times 80e^{-0.2t}$

$= -0.2(T - 20)$

When $T = 50^\circ$ $\frac{dT}{dt} = -0.2(50^\circ - 20^\circ)$

$= -6^\circ$

$\therefore$ The water is cooling at $6^\circ c/min$.

2	Correct
1	Finding derivative and intial temp

Question 13 continues on page 12

Question 13 continued **Marks**

(c)
(i)

$\cos 3x = 0$ or $\sin 2x = 0$

$3x = \pi/2, 3\pi/2, \ldots$ $\qquad 2x = 0, \pi, 2\pi \ldots$ 2 | multiple solutions ~~for both~~, showing smallest

$x = \pi/6, \pi/2, \ldots$ $\qquad x = 0, \pi/2, \pi \ldots$ 1 | $\cos 3x = 0$ and $\sin 2x = 0$

$\therefore$ smallest positive solⁿ is $x = \pi/6$

(ii) **3**

$$V = \pi \int_0^{\pi/6} \left(\sqrt{\cos(3x)\sin(2x)}\right)^2 dx$$

3 | Collect

$$= \pi \int_0^{\pi/6} \cos 3x \sin 2x \; dx$$

2 | correct integral expression

$$= \frac{\pi}{2} \int_0^{\pi/6} (\sin 5x - \sin x)\, dx \quad *$$

1 | Getting to $*$

$$= \frac{\pi}{2}\left[-\frac{\cos 5x}{5} + \cos x\right]_0^{\pi/6}$$

$$= \frac{\pi}{2}\left\{-\frac{1}{5}\cos\frac{5\pi}{6} + \cos\frac{\pi}{6} - \left(-\frac{1}{5}\cos 0 + \cos 0\right)\right\}$$

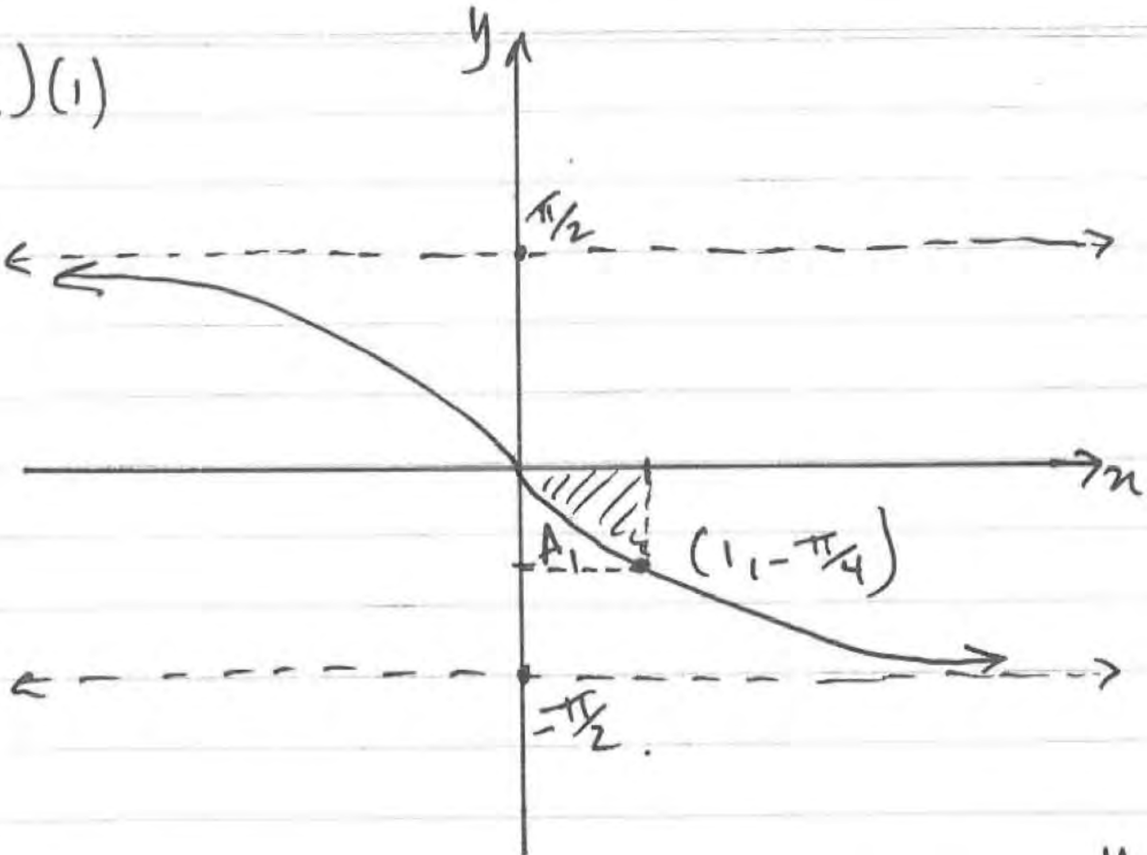
$$= \frac{\pi}{2}\left\{\frac{\sqrt{3}}{10} + \frac{\sqrt{3}}{2} + \frac{1}{5} - 1\right\}$$

$$= \frac{\pi}{2}\left\{\frac{6\sqrt{3}}{10} - \frac{4}{5}\right\}$$

$$= \pi\left\{\frac{3\sqrt{3}}{10} - \frac{2}{5}\right\} \text{ units}^3$$

Question 13 continues on page 13

(d)(i)



2	graph with pt $(1,-\frac{\pi}{4})$ with scale correct
1	correct shape

$y = -\tan^{-1}(x)$

$x = \tan(-y)$

$= -\tan y$

(ii) $A = \frac{\pi}{4} - \int_{-\frac{\pi}{4}}^{0} -\tan y \, dy$

$= \frac{\pi}{4} - \int_{-\frac{\pi}{4}}^{0} \frac{-\sin y}{\cos y}{}^{*} \, dy$

$= \frac{\pi}{4} - \Big[\ln|\cos y|\Big]_{-\pi/4}^{0}$

$= \frac{\pi}{4} - \left[\ln|\cos 0| - \ln\left|\cos(-\tfrac{\pi}{4})\right|\right]$

$= \frac{\pi}{4} - \left[\ln 1 - \ln\frac{1}{\sqrt{2}}\right]$

$= \frac{\pi}{4} - \left[0 + \ln\sqrt{2}\right]$

$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)$ units^2

3	correct evaluation *
2	progress to * to calculate A
1	$\frac{\pi}{4} - A_1$

AND FYI ⇒ $\ln\frac{1}{\sqrt{2}}$ is a negative number!

NOTE: Common error were incorrectly using absolute value brackets for A_1, and consequently having incorrect sign.

Question 14 (16 marks) **Marks**

(a) 3

When $t=0$ $v=0$

$\frac{dv}{dt} = (v+1)^{1/2}$

$\frac{dv}{(v+1)^{1/2}} = dt$

$\int \frac{1}{\sqrt{v+1}}\,dv = t + c$

$2(v+1)^{1/2} = t + c$ *

when $t=0$ $v=0$ $\therefore$ $c=2$

$\therefore$ $2\sqrt{v+1} = t+2$

when $t=1$ $\sqrt{v+1} = \frac{3}{2}$

$v+1 = \frac{9}{4}$

$v = \frac{5}{4}$ m/s

3	Correct
2	working in progress
1	Getting to *

(b) 1 r/w

(i) $P = 400$ penguins

Question 14 continues on page 18

Question 14
Part (b) continued

Marks

(ii) $\frac{dP}{dt} = 0.001P(400-P)$ 3

$\int \frac{dP}{P(400-P)} = \int 0.001\,dt$

$\frac{1}{400}\int\left(\frac{1}{P} + \frac{1}{400-P}\right) dP = 0.001t + C$

$\frac{1}{400}\left[\ln|P| - \ln|400-P|\right] = 0.001t + C$ $**$

When $t=0$ $P=50$

$\frac{1}{400}\left[\ln\frac{50}{350}\right] = C$

$C = \frac{1}{400} \times -\ln 7$

$= \frac{-\ln 7}{400}$

$\frac{1}{400}\ln\left|\frac{P}{400-P}\right| = 0.001t - \frac{\ln 7}{400}$

$\ln\left|\frac{P}{400-P}\right| = 0.4t - \ln 7$ $*$

When $P = 200$

$0.4t - \ln 7 = \ln\left|\frac{200}{200}\right|$

$0.4t = \ln 7$

$t = 4.86477373$ years.

$\doteqdot$ 4 years 10 months.

3	correct
2	Gets to $*$ or equivalent.
1	Gets to $**$ or equivalent

Question 14 continues on page 19

(c)

(i) **2**

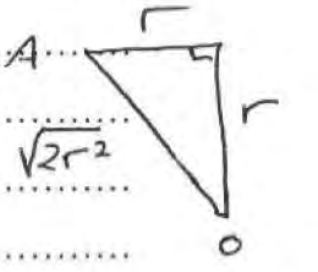
Side length of the square is $2r$

$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$\overrightarrow{AP} \cdot \overrightarrow{AP} = (\overrightarrow{AO} + \overrightarrow{OP}) \cdot (\overrightarrow{AO} + \overrightarrow{OP})$

$= \overrightarrow{AO} \cdot \overrightarrow{AO} + 2\overrightarrow{AO} \cdot \overrightarrow{OP} + \overrightarrow{OP} \cdot \overrightarrow{OP}$

$*\quad = |\overrightarrow{AO}|^2 + |\overrightarrow{OP}|^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OP}$



$= \left(\sqrt{2r^2}\right)^2 + r^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OP}$

$= 2r^2 + r^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OP}$

$= 3r^2 - 2\overrightarrow{OP} \cdot \overrightarrow{OA}$ As req'd

2	Correct
1	Arrives at *

Question 14 continues on page 20

Question 14 **Marks**
Part (c) continued

(ii) 2

$AP^2 = \vec{AP} \cdot \vec{AP} = 3r^2 - 2\vec{OP} \cdot \vec{OA}$

Similarly

$BP^2 = 3r^2 - 2\vec{OP} \cdot \vec{OB}$

$CP^2 = 3r^2 - 2\vec{OP} \cdot \vec{OC}$

$DP^2 = 3r^2 - 2\vec{OP} \cdot \vec{OD}$

Adding

$AP^2 + BP^2 + CP^2 + DP^2 = 4 \times 3r^2 - 2\vec{OP}\,(\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD})$

$= 12r^2 - 2\vec{OP} \cdot \underset{\sim}{0}$

$= 12r^2$

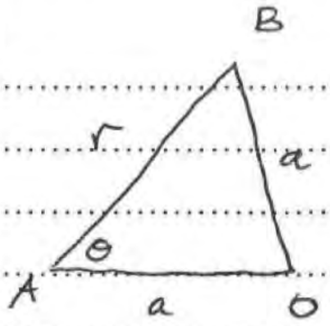
2	Correct
1	Set up and link to i)

Question 14 continues on page 21

Question 14 continued **Marks**

(d)

(i) **2**



$AB = r$

$AO = a$

$OB = a$

$$\cos\theta = \frac{a^2 + r^2 - a^2}{2ar}$$

$$= \frac{r^2}{2ar}$$

$$\cos\theta = \frac{r}{2a}$$

$$\therefore\ r = 2a\cos\theta$$

2	Both Correct
1	Correct

$$l = r \times 2\theta$$

$$= 2a\cos\theta \times 2\theta$$

$$l = 4a\theta\cos\theta$$

Question 14 continued **Marks**

Part (d) continued.

(ii) **3**

$\frac{dl}{d\theta} = 4a\theta \times -\sin\theta + \cos\theta \times 4a$

$= -4a\theta\sin\theta + 4a\cos\theta$

$\frac{d^2l}{d\theta^2} = -4a\theta\cos\theta + \sin\theta \times -4a - 4a\sin\theta$

$= -4a\theta\cos\theta - 8a\sin\theta$

When $\frac{dl}{d\theta} = 0$ $4a\theta\sin\theta = 4a\cos\theta$

$\theta\tan\theta = 1$

$\theta = \frac{1}{\tan\theta}$

$= \cot\theta$

When $\theta = \cot\theta$ $\frac{d^2l}{d\theta^2} = -4a\cot\theta\cos\theta - 8a\sin\theta$

Since $0 < \theta < \pi/2$ $\cot\theta, \sin\theta \cos\theta > 0$

$\therefore \frac{d^2l}{d\theta^2} = -4a(\cot\theta\cos\theta + 2\sin\theta) < 0$

$\therefore \theta = \cot\theta$ will produce a maximum l

3	Correct
2	Finds derivatives Shows $\theta = \cot\theta$ is a stationary pt
1	Finds derivatives

END OF PAPER