Name:…………………………………..

Student Number

Teacher:………………………………..

Pymble Ladies' College

Mathematics Extension 1

HSC Trial Examination

Term 3 2025

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks 70

SECTION 1 – 10 marks (pages 1-5)

- Attempt Questions 1-10
- Allow about 15 minutes for this section
- Answer each question on the multiple-choice answer sheet provided in the answer booklet.

SECTION II – 60 marks (pages 7-14)

- Attempt Questions 11-14
- Allow about 1 hours and 45 minutes for this section
- Answer each question in the appropriate space in the Answer Booklet. Extra writing pages are included at the end of each question.

SECTION I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. What are the values of x for which $(x+3)(x-1)(2-x)>0$?

(A) $x<-3$ or $1<x<2$

(B) $x>-3$ or $1<x<2$

(C) $-3<x<1$ or $x>2$

(D) $-3<x<1$

2. If $x+a$ is a factor of $4x^5-11ax^3+x^4$, $a\neq 0$, what is the value of a?

(A) -2

(B) 3

(C) 2

(D) -3

3. What is the area of the region enclosed by the graphs of $y=4-x^2$ and $y=3$?

(A) $\frac{2}{3}$ square units

(B) $\frac{4}{3}$ square units

(C) $\frac{7}{3}$ square units

(D) $\frac{8}{3}$ square units

4. $\overrightarrow{AB} = \underset{\sim}{i} + 3\underset{\sim}{j}$ and $\overrightarrow{BC} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$. What is the magnitude of $\overrightarrow{AC}$?

(A) $\sqrt{10}$

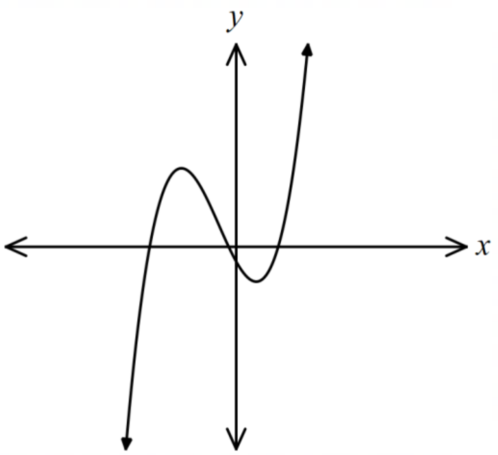
(B) $\sqrt{24}$

(C) $\sqrt{26}$

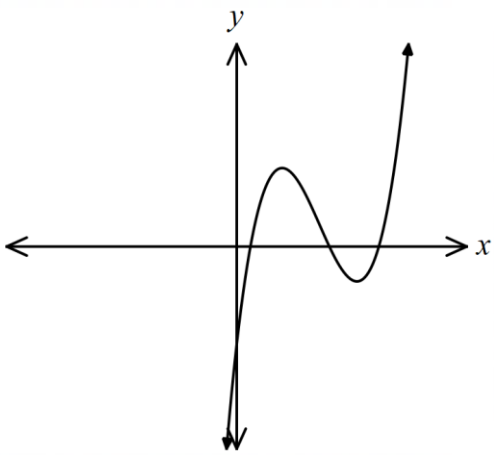
(D) $\sqrt{34}$

5. Consider the polynomial $p(x) = ax^3 + bx^2 + cx + d$, where $a < 0$, $b < 0$, $a \neq 0$.
Which of the following curves represents $y = p(x)$?

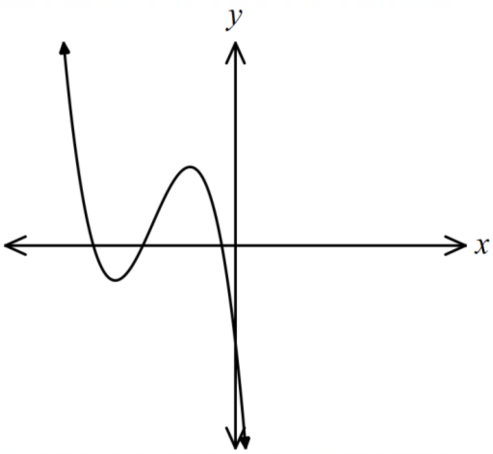
(A)



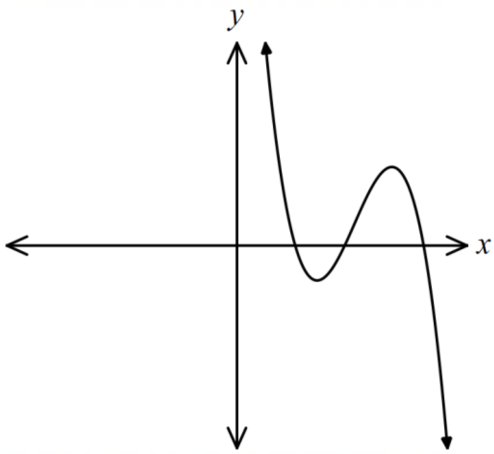
(B)



(C)



(D)



6. Which of the following is the derivative of $\arcsin\left(\frac{x}{3}\right)$?

(A) $\frac{1}{3\sqrt{1-x^2}}$

(B) $\frac{1}{\sqrt{9-x^2}}$

(C) $\frac{1}{3\sqrt{1-9x^2}}$

(D) $-\frac{1}{3}\cos\left(\frac{x}{3}\right)$

7. How many distinct arrangements are possible from the letters of the word NECESSITIES?

(A) 50 400

(B) 554 400

(C) 1 663 200

(D) 39 916 800

8. Given the function $f(x) = x + e^x,\ x \geq 0$.
What is the gradient of the inverse function $y = f^{-1}(x)$ at $(1, 0)$?

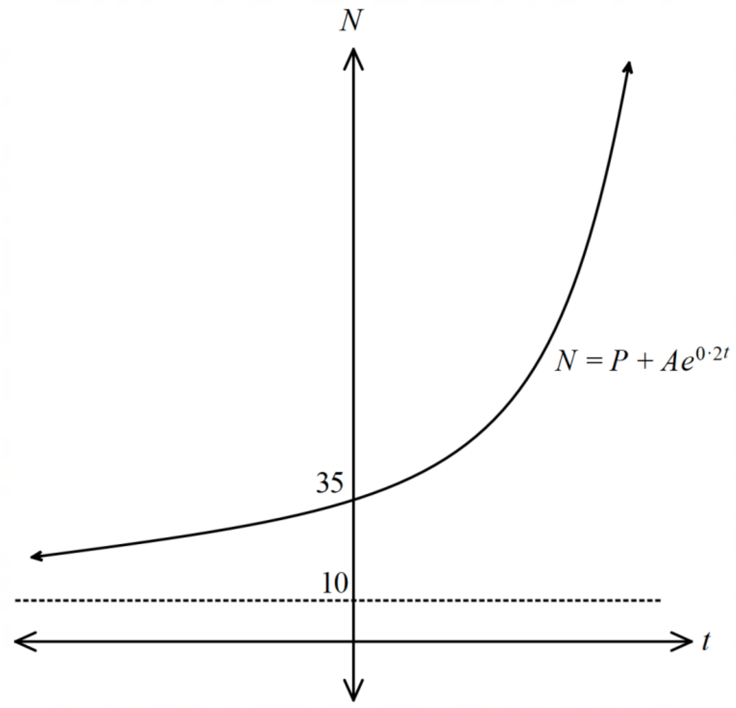
(A) 2

(B) $1+\frac{1}{e}$

(C) $\frac{1}{2}$

(D) $\frac{e}{e+1}$

9. The quantity N increases exponentially over time according to the equation $N = P + Ae^{0.2t}$. The graph below shows the change in N over time (t).



Which equation describes the rate of change of N with respect to t?

(A) $\dfrac{dN}{dt} = 0.2(N - 10)$

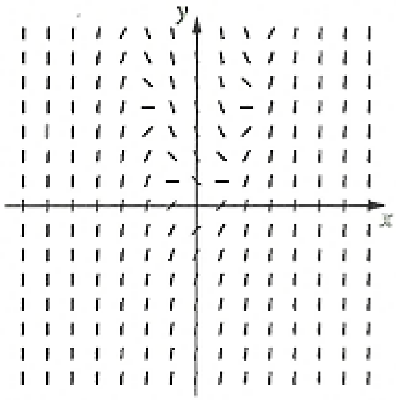
(B) $\dfrac{dN}{dt} = 0.2(N - 25)$

(C) $\dfrac{dN}{dt} = 10(N - 0.2)$

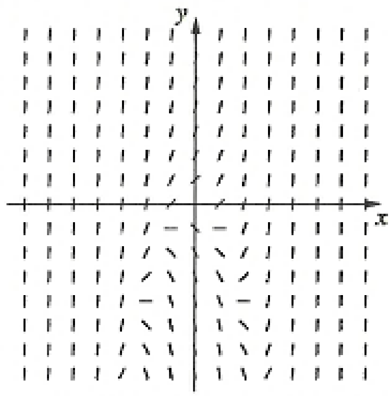
(D) $\dfrac{dN}{dt} = 25(N - 10)$

10. Which of the following slope fields represents the differential equation $\frac{dy}{dx} = x^2 - y$?

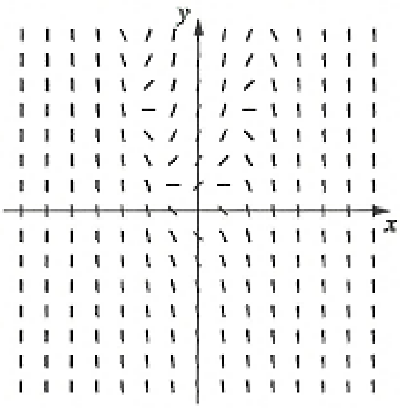
(A)



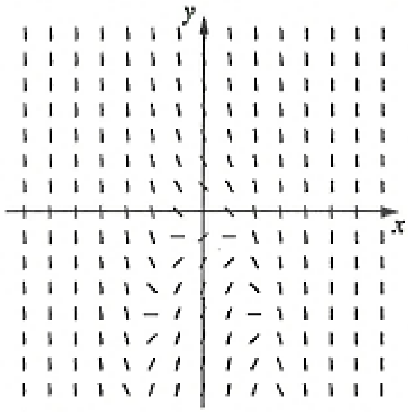
(B)



(C)



(D)



SECTION II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section.

Answer these questions in the Answer Book provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) **Marks**

(a) Given $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$, find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. **2**

(b) Solve $\dfrac{1}{2-x} \geq -1$. **2**

(c) Using the substitution $u = 3x^2 + 1$, find $\int x \cos\left(3x^2 + 1\right) dx$. **2**

(d) Find the coefficient of the term in x^6 in the expansion of $\left(\dfrac{3x^2}{2} + \dfrac{1}{5}\right)^6$. **2**

(e) The sum of two of the roots of the equation $2x^3 + 6x^2 + kx - 150 = 0$ is zero. Find the value of k. **2**

(f) Show that $\cot\left(A + B\right) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$. **2**

(g) Find the cartesian equation of the curve with the parametric equations given. **2**

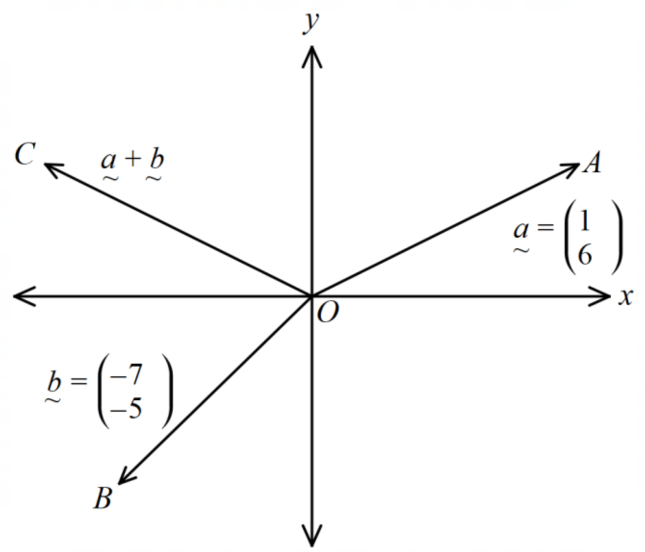
$$\begin{cases} x = 2\cos\theta \\ y = \sqrt{3}\sin\theta \qquad 0 \leq \theta \leq 2\pi \end{cases}$$

End of Question 11

Question 12 (15 marks) **Marks**

(a) Write $\sqrt{3}\cos x - 3\sin x$ in the format $R\cos(x+A)$ where $R>0$ and $0<A<\dfrac{\pi}{2}$. **2**

(b) The diagram shows the vectors $\underset{\sim}{a}=\begin{pmatrix}1\\6\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}-7\\-5\end{pmatrix}$ and the resultant vector $\underset{\sim}{a}+\underset{\sim}{b}$.



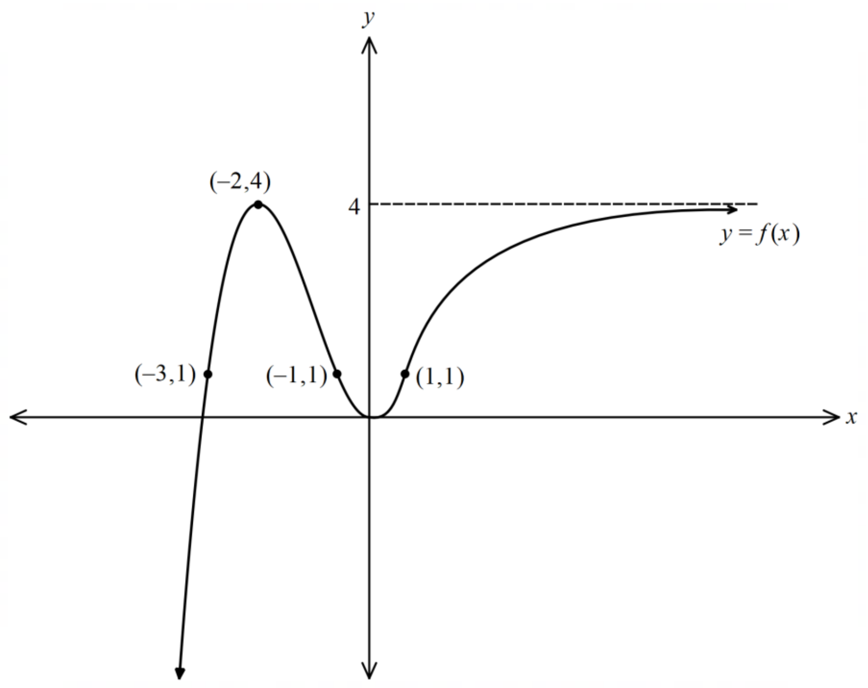
(i) Show that the triangle AOC formed by the vectors is an isosceles triangle. **2**

(ii) Find the size of $\angle AOC$. **2**

Question 12 continues on page 9

Question 12 continued.

(c) The diagram of $y = f(x)$ is shown. **2**



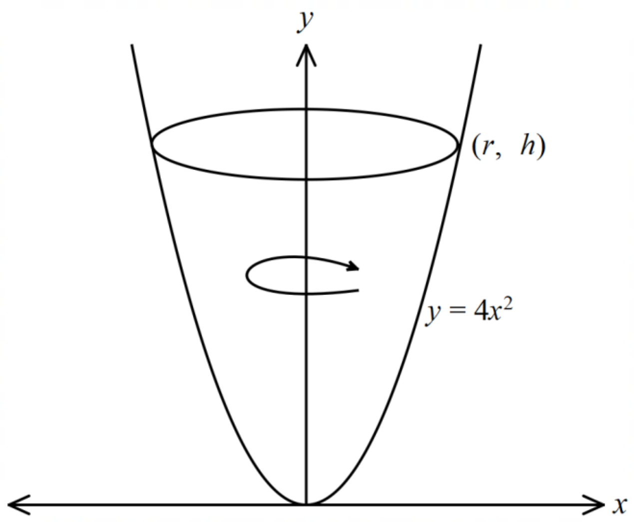
On the diagram provided in the answer booklet, sketch the graph of $y = \sqrt{f(x)}$, clearly showing all turning points, intercepts, asymptotes and points of intersection with $y = f(x)$.

(d) If the roots of the equation $x^3 + 3x^2 - 6x - 8 = 0$ are in arithmetic sequence, find all the roots. **2**

Question 12 continues on page 10

Question 12 continued. **Marks**

(e) (i) The internal surface of a water reservoir is created by rotating the parabola about the y axis between $y = 0$ and $y = h$. **2**



Show that the volume of the water in the reservoir is given by $V = 2\pi r^4$.

(ii) The variable h represents the depth of the water in the reservoir at time t hours and r metres is the radius of the upper surface of the water at that time. **1**

The radius of the surface is reducing at a constant rate such that $\dfrac{dr}{dt} = -2$.

Show that the rate of change of the depth is given by $\dfrac{dh}{dt} = -16r$.

(iii) Find the rate of change of the volume when the radius is 5 metres. **2**

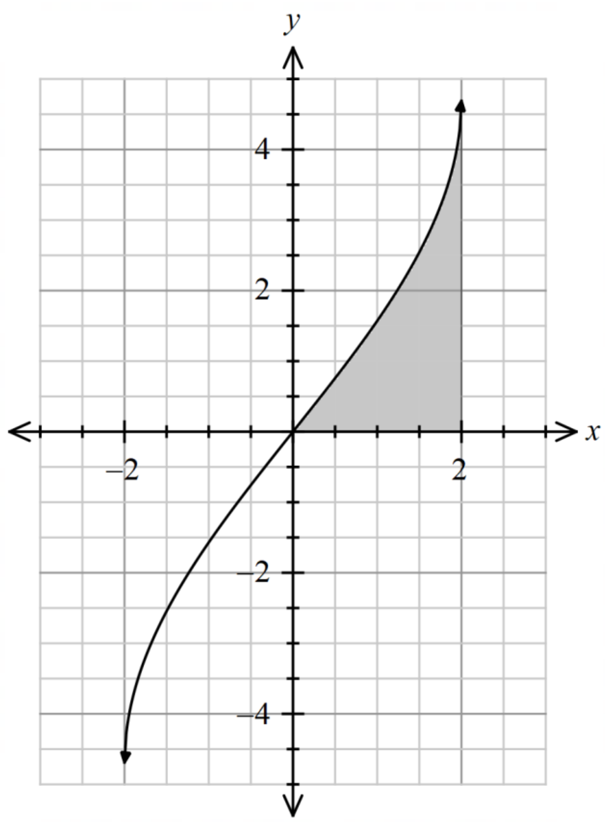
End of Question 12

Question 13 (16 marks) **Marks**

(a) Evaluate $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{\sin\left(\tan^{-1} x\right)}{1+x^2}\,dx$ by using the substitution $u = \tan^{-1} x$. **3**

(b) Use mathematical induction to prove that $9^n - 4^n$ is divisible by 5 for all integers $n \geq 1$. **3**

(c) Find the shaded area between the curve $y = 3\sin^{-1}\left(\dfrac{x}{2}\right)$ and the x-axis for $x \in [0, 2]$. **3**



Question 13 continues on page 10

Question 13 continued. **Marks**

(d) A beehive initially had a population of 100 bees. **4**
The population growth of the colony is given by the logistic equation

$$\frac{dP}{dt} = 0.2P\left(\frac{M-P}{M}\right),$$

where t is the time in months and M is the carrying capacity.

After 12 months the population of the hive was 400 bees.

Use the fact that $\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$ to show that the carrying capacity, M, is approximately 570 bees.

(e) It is known that $\cos 3\theta = 4\cos\theta\cos\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{3}+\theta\right)$. **3**

Using this result, or otherwise, solve $\sqrt{2}\cos\theta\left(\cos^2\theta - 3\sin^2\theta\right) = 1$ for $0 \le \theta \le 2\pi$.

End of Question 13

Question 14 (15 marks) **Marks**

(a) Given that $\frac{dy}{dx} = \frac{y-1}{x+2}$ and $y(6) = 3,$ find the value of $y(1)$. **3**

(b) Two circular tables seat 6 and 4 people respectively. In how many ways can 10 people be seated at the two tables if Taylor and Travis are to sit together at the bigger table? **2**

(c) (i) Solve for $\tan 4\alpha = 1$ for $0 \le \alpha \le \pi$. **1**

(ii) Given $\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$ and by letting $x = \tan\alpha,$ **2**

solve the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0.$

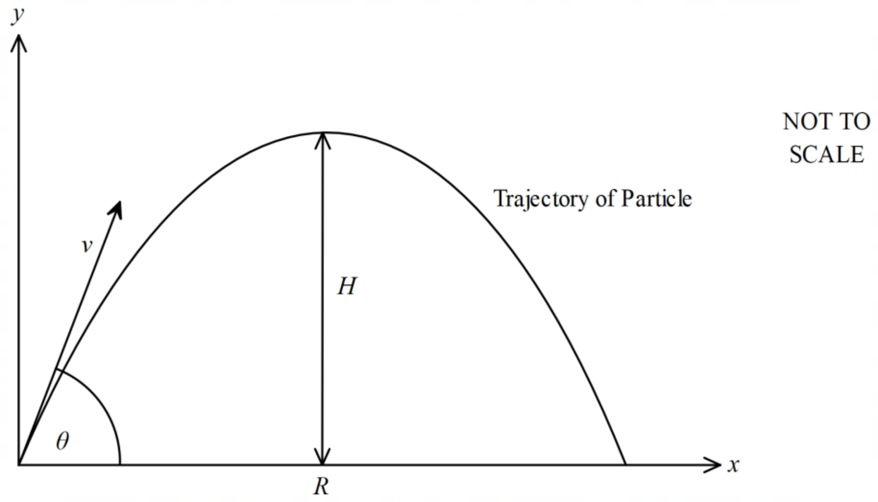
(iii) Hence, show that $\tan^2\frac{\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{9\pi}{16} + \tan^2\frac{13\pi}{16} = 28.$ **2**

Question 14 continues on page 12

Question 14 continued. **Marks**

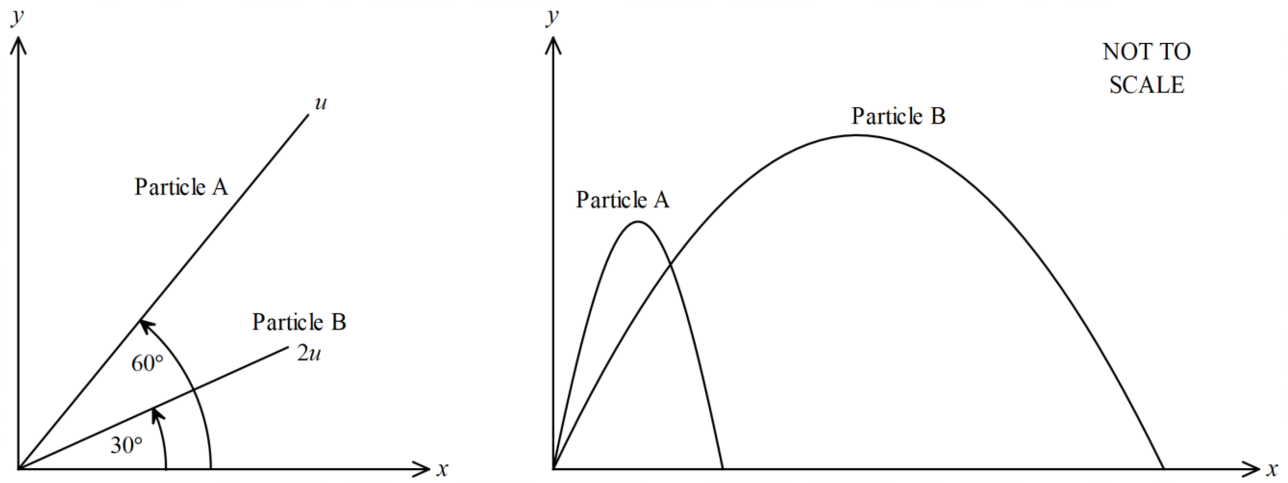
(d) The position vector of a particle projected from the origin at an initial velocity of v m/s at an angle of $\theta°$ to the horizontal is given by:

$$r(t) = \begin{pmatrix} vt\cos\theta \\ vt\sin\theta - \dfrac{gt^2}{2} \end{pmatrix}$$



(i) Show that the time of flight of the particle is given by $t = \dfrac{2v\sin\theta}{g}$ and the horizontal range by $R = \dfrac{v^2\sin 2\theta}{g}$. **3**

(ii) Particle A is projected from the origin at a velocity of u m/s at an angle of $60°$ to the horizontal. **2**
For particle B, the initial velocity is doubled to $2u$ m/s and the angle of elevation is halved to $30°$.



Show that the range of Particle B is 4 times that of Particle A.

END OF PAPER

Name:……………………………………

Student Number:

Teacher's Name:…………………………

Mathematics Extension 1
HSC Trial Examination
Term 3 2025

ANSWER BOOKLET

- **Put your name, Student Number and Teacher's name on this Booklet.**
- For Questions 1-10 use the Multiple-Choice Answer Sheet provided.
- Answer Questions 11-14 in the spaces allocated for each question as these spaces provide guidance for the expected length of your response.
- Write using a non-erasable black or blue pen. Black is preferred.
- Extra writing space is provided at the end of each question.
- Calculators approved by NESA may be used.

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Name:......................................	
Student Number:	
Teacher's Name:...........................	

Mathematics Extension 1
HSC Trial Examination
2025

Section I – Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ○ B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

1	A ●	B ○	C ○	D ○
2	A ○	B ●	C ○	D ○
3	A ○	B ●	C ○	D ○
4	A ●	B ○	C ○	D ○
5	A ○	B ○	C ●	D ○
6	A ○	B ●	C ○	D ○
7	A ○	B ●	C ○	D ○
8	A ○	B ○	C ●	D ○
9	A ●	B ○	C ○	D ○
10	A ●	B ○	C ○	D ○

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SECTION II

60 marks
Your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is provided at the end of each question.

Question 11 (14 marks) **Marks**

(a) **2**

$$\text{Proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|^2}\underset{\sim}{b}$$

$$= \frac{1-3}{(\sqrt{2})^2}(\underset{\sim}{i}-\underset{\sim}{j})$$

$$= -(\underset{\sim}{i}-\underset{\sim}{j})$$

$$= -\underset{\sim}{i}+\underset{\sim}{j}$$

1 – progress
2 – correct solutions.

(b) **2**

$$\frac{1}{2-x} \geq -1 \quad (x \neq 2).$$

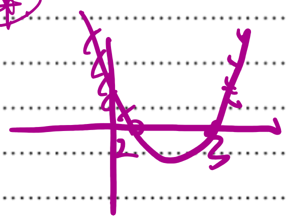
$$2-x \geq -(2-x)^2.$$

$$(2-x)+(2-x)^2 \geq 0.$$

$$(2-x)(1+2-x) \geq 0.$$

$$(2-x)(3-x) \geq 0. \quad (*)$$

$$(x-3)(x-2) \geq 0.$$



$\therefore\ x<2$ or $x \geq 3$

1 → get to *
2 – Correct solutions

Question 11 continues on page 2

Question 11 continued. **Marks**

(c) **2**

$\int x\cos(3x^2+1)\,dx$

$= \int \cos(u)\,\frac{du}{6}$

$= \frac{1}{6}(\sin u) + C.$

$= \frac{1}{6}\sin(3x^2+1) + C.$

let $u = 3x^2+1$

$\frac{du}{dx} = 6x.$

$\frac{du}{6} = x\,dx.$

1 — progress

2 — Correct solutions

(d) **2**

$\left(\frac{3x^2}{2} + \frac{1}{\sqrt{5}}\right)^6$

$= \sum_{r=0}^{6} {}^6C_r \left(\frac{3}{2}x^2\right)^{6-r} \left(\frac{1}{\sqrt{5}}\right)^r$

$= \sum_{r=0}^{6} {}^6C_r \left(\frac{3}{2}\right)^{6-r} \left(\frac{1}{\sqrt{5}}\right)^r x^{12-2r}$

let $x^{12-2r} = x^6$

$12-2r = 6$

$r = 3.$

$\therefore$ Coef of $x^6 = {}^6C_3 \left(\frac{3}{2}\right)^3 \left(\frac{1}{\sqrt{5}}\right)^3$

$= \frac{27}{50}$ $(= 0.54).$

1 — expansion of binomial

2 — correct solutions.

Question 11 continues on page 3

Question 11 continued. **Marks**

(e) **2**

Let the roots be α, $-\alpha$ and β of $2x^3+6x^2+kx-150=0$.

Sum of roots $= \alpha + (-\alpha) + \beta = \frac{-6}{2}$

$\beta = -3$

$2(-3)^3 + 6(-3)^2 - 3k - 150 = 0$

$-3k = 150$

$k = -50$

2	finding k correctly
1	finding one root

Question 11 continues on page 4

Question 11 continued. **Marks**

(f) **2**

$LHS = \cot(A+B)$

$= \dfrac{\cos(A+B)}{\sin(A+B)}$

$= \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \sin B \cos A}$

$= \dfrac{\dfrac{\cos A \cos B - \sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B + \sin B \cos A}{\sin A \sin B}}$

$= \dfrac{\dfrac{\cos A \cos B}{\sin A \sin B} - \dfrac{\sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\sin B \cos A}{\sin A \sin B}}$

$= \dfrac{\dfrac{\cos A}{\sin A} \times \dfrac{\cos B}{\sin B} - 1}{\dfrac{\cos B}{\sin B} + \dfrac{\cos A}{\sin A}}$

$= \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

$= RHS$

2	fully correct
1	Correct process of breaking up LHS or RHS

Question 11 continues on page 5

Question 11 continued. **Marks**

(g) **2**

$$\begin{cases} x = 2\cos\theta \Rightarrow \cos\theta = \frac{x}{2} \\ y = \sqrt{3}\sin\theta \Rightarrow \sin\theta = \frac{y}{\sqrt{3}} \end{cases}$$

Since $\sin^2\theta + \cos^2\theta = 1$,

so $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

2	Simplifying to answer
1	Correctly using $\sin^2\theta + \cos^2\theta = 1$ in terms of x and y

End of Question 11

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If you use this space, clearly indicate which question you are answering.

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Question 12 (15 marks) **Marks**

(a) 2

$\sqrt{3}\cos x - 3\sin x = R\cos(x + A)$

$R^2 = (\sqrt{3})^2 + 3^2$

$R = \sqrt{12}$

$= 2\sqrt{3}$

$\tan A = \frac{3}{\sqrt{3}} = \sqrt{3}$

$A = \frac{\pi}{3}$

$\therefore \sqrt{3}\cos x - 3\sin x = 2\sqrt{3}\cos\left(x + \frac{\pi}{3}\right)$

2	correct answer
1	getting R OR A correctly

Question 12 continues on page 10

Question 12 continued **Marks**

(b)

(i) 2

$\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$
$= \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

$|\overrightarrow{OC}| = \sqrt{(-6)^2 + 1^2}$
$= \sqrt{37}$

$\overrightarrow{OA} = \underset{\sim}{a}$
$= \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

$|\overrightarrow{OA}| = \sqrt{1^2 + 6^2}$
$= \sqrt{37}$

Since $|\overrightarrow{OC}| = |\overrightarrow{OA}|$,
$\therefore \triangle AOC$ is an isosceles triangle.

2	getting $|\overrightarrow{OC}| = |\overrightarrow{OA}|$
1	found $\overrightarrow{OC}$ and $|\overrightarrow{OA}|$ correctly

(ii) 2

$\cos \angle AOC = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{c}}{|\underset{\sim}{a}||\underset{\sim}{c}|}$
$= \dfrac{(1)(-6) + (6)(1)}{\sqrt{37} \times \sqrt{37}}$
$= \dfrac{0}{37}$
$= 0$

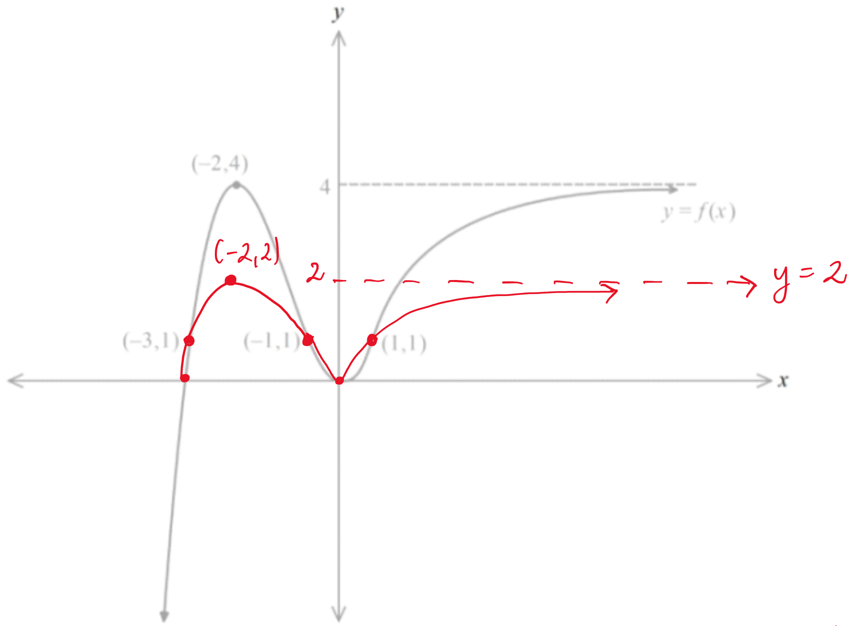
Since $\cos \angle AOC = 0$,
$\angle AOC = \frac{\pi}{2}$.

2	getting the angle correctly
1	getting $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

Question 12 continues on page 11

Question 12 continued. **Marks**

(c) **2**



$\boxed{2}$ marks - everything correct. Accept only the error where (-2,4) misaligned with y=2 asymptote

$\boxed{1}$ mark - everything correct but extended graph further to left incorrect
- correct shape but fail to pass through y=1.
- correct shape but fail to note asymptote y=2
- correct shape but fail to note (-2,2).
- correct shape but

$\boxed{0}$ mark - incorrect shape
- made more than 2 errors
- only drew one portion of graph.

Question 12 continues on page 12

Question 12 continued. **Marks**

(d) Let roots be $\alpha - d, \alpha, \alpha + d$ **2**

$$\alpha - d + \alpha + \alpha + d = -3$$
$$3\alpha = -3$$
$$\alpha = -1 \quad \boxed{*}$$

$$\alpha(\alpha - d)(\alpha + d) = 8$$
$$(1 - d^2) = 8$$
$$d^2 = 9$$
$$d = \pm 3$$

$\therefore$ Roots are $-4, -1, 2$ $\quad \boxed{**}$

$\boxed{1}$ mark get to $\boxed{*}$ or equivalent working out.

$\boxed{2}$ marks get $\boxed{**}$ with correct working out.

Question 12 continues on page 13

Question 12 continued. **Marks**

(e) (i) **2**

$V = \pi \int_0^h \frac{y}{4}\, dy$

$= \frac{\pi}{4}\left[\frac{y^2}{2}\right]_0^h$

$= \frac{\pi}{8}\left[h^2 - 0^2\right]$

$= \frac{\pi h^2}{8}$ $\boxed{*}$

But $h = 4r^2$

$V = \frac{\pi(4r^2)^2}{8}$

$\therefore V = 2\pi r^4$ $\boxed{**}$

$\boxed{1}$ mark *

$\boxed{2}$ marks ** with correct working out.

Question 12 continues on page 14

Question 12 (part e) continued. **Marks**

(ii) **1**

$h = 4r^2$

$\frac{dh}{dr} = 8r$

$\frac{dh}{dt} = \frac{dh}{dr} \times \frac{dr}{dt}$

$= 8r \times -2$

$= -16r$

(iii) **2**

$V = 2\pi r^4$

$\frac{dV}{dr} = 8\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$= 8\pi r^3 \times -2$

$= -16\pi r^3$

$\boxed{*}$

1 mark

when $r = 5$

$\frac{dV}{dt} = -16\pi(5)^3$

$= -2000\pi$

$\boxed{**}$

2 marks

End of Question 12

If you use this space, clearly indicate which question you are answering.

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Question 13 (16 marks) **Marks**

(a) 3

$$\int_0^{\frac{1}{\sqrt{3}}} \left[\frac{\sin(\tan^{-1} x)}{1+x^2}\right] dx$$

$$= \int_0^{\frac{\pi}{6}} (\sin u)\, du$$

$$= \left[-\cos u\right]_0^{\frac{\pi}{6}}$$

$$= -\cos\frac{\pi}{6} + \cos 0$$

$$= \frac{-\sqrt{3}}{2} + 1$$

$u = \tan^{-1} x$

$du = \frac{1}{1+x^2}\, dx$

When $x = \frac{1}{\sqrt{3}}$

$u = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

When $x = 0$,

$u = \tan^{-1} 0 = 0$

$\star$

3	fully correct
2	Putting $\star$ together as an integral with one variable
1	finding $\star$ correctly

Question 13 continues on page 18

Question 13 continued. **Marks**

(b) **3**

Let the statement be $(9^n - 4^n)$ is divisible by 5, for $n \geq 1$, $n \in \mathbb{Z}$.

For $n=1$, $9^n - 4^n$
$= 9 - 4$
$= 5$ which is divisible by 5.
$\therefore$ The statement is true for $n=1$.

Assume the statement is true for $n=k$, $k \geq 1$.
ie $9^k - 4^k = 5M$, $M \in \mathbb{Z}$.
To prove it's true for $n = k+1$.
ie $9^{k+1} - 4^{k+1} = 5N$, $N \in \mathbb{Z}$.
$LHS = 9^{k+1} - 4^{k+1}$
$= 9 \cdot 9^k - 4 \cdot 4^k$
$= 9 \cdot 9^k - 9 \cdot 4^k + 5 \cdot 4^k$
$= 9(9^k - 4^k) + 5 \cdot 4^k$
$= 9(5M) + 5 \cdot 4^k$ (from assumption)
$= 5(9M + 4^k)$.
Since $M, k \in \mathbb{Z}$, $\therefore (9M + 4^k) \in \mathbb{Z}$.
$\therefore LHS = 5(9 + 4^k)$
$= 5N$ where $N = 9M + 4^k$, $N \in \mathbb{Z}$.
$\therefore$ The statement is true for $n = k+1$ if it's true for $n=k$.

Since the statement is true for $n=1$, it's true for $n = 1+1 = 2$, $n = 2+1 = 3$ & so on. By Mathematical induction, it's true for all $n \geq 1$, $n \in \mathbb{Z}$.

1 — Correct proof for $n=1$.

2 — Good set up for the assumption, conclusion. working towards the proof.

3 — Correct solutions.

Question 13 continues on page 19

Question 13 continued. **Marks**

(c) **3**

$y = 3\sin^{-1}\left(\frac{x}{2}\right)$. At $x = -2$, $y = -\frac{3\pi}{2}$

At $x = 2$, $y = \frac{3\pi}{2}$.

$\frac{y}{3} = \sin^{-1}\left(\frac{x}{2}\right)$

$\frac{x}{2} = \sin\left(\frac{y}{3}\right)$

$x = 2\sin\left(\frac{y}{3}\right)$

$\left(-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}\right)$.

$\text{Area} = 2 \times \left(\frac{3\pi}{2}\right) - \int_0^{\frac{3\pi}{2}} 2\sin\left(\frac{y}{3}\right) dy$.

$= 3\pi - (2\times 3)\left[-\cos\left(\frac{y}{3}\right)\right]_0^{\frac{3\pi}{2}}$

$= 3\pi + 6\left(\cos\frac{\pi}{2} - \cos(0)\right)$.

$= (3\pi - 6)$ unit^2.

1 — Convert to $x = 2\sin\left(\frac{y}{3}\right)$

2 — Progress to areas

3 — correct solutions.

Question 13 continues on page 20

Question 13 continued.

(d) 4

$$\frac{dP}{dt} = 0.2P\left(\frac{m-P}{m}\right)$$

$$\int \frac{m}{P(m-P)}\,dP = \int 0.2\,dt$$ A

$$\int \frac{1}{P} + \frac{1}{m-P}\,dP = \int 0.2\,dt$$

$$\ln|P| - \ln|m-P| = 0.2t + C$$

$$\ln\left|\frac{P}{m-P}\right| = 0.2t + C$$

$$e^{0.2t+C} = \left|\frac{P}{m-P}\right|$$

$$e^{0.2t} \cdot e^{C} = \left|\frac{P}{m-P}\right|$$

let $e^C = A$, A pos or neg

$$Ae^{0.2t} = \left|\frac{P}{m-P}\right| \quad \boxed{1}$$

$\boxed{t=0,\ P=100}$

$$Ae^{0} = \left|\frac{100}{m-100}\right|$$

$$A = \frac{100}{m-100},\ A > 0 \quad \boxed{2}$$

$\boxed{1}$ $$Ae^{0.2t} = \frac{P}{m-P},\ A > 0$$

$\boxed{t=12\ \ P=400}$

$$Ae^{2.4} = \frac{400}{m-400}$$

$\boxed{2}$ $$\frac{100}{m-100}e^{2.4} = \frac{400}{m-400}$$

$$\frac{e^{2.4}}{m-100} = \frac{4}{m-400}$$

$$me^{2.4} - 400e^{2.4} = 4m - 400$$

$$m(e^{2.4} - 4) = 400e^{2.4} - 400$$

$$m = \frac{400e^{2.4} - 400}{e^{2.4} - 4}$$

$= 570.862\ldots$

≈ 570 bees

Question 13 continues on page 21

Question 13 part (d) continued. **Marks**

Question 13 continues on page 22

Question 13 continued. **Marks**

(e) **3**

$\cos\left(\frac{\pi}{3}-\theta\right) = \cos\frac{\pi}{3}\cdot\cos\theta + \sin\frac{\pi}{3}\cdot\sin\theta$
$= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$

$\cos\left(\frac{\pi}{3}+\theta\right) = \cos\frac{\pi}{3}\cdot\cos\theta - \sin\frac{\pi}{3}\cdot\sin\theta$
$= \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$

so $\cos\left(\frac{\pi}{3}-\theta\right)\cdot\cos\left(\frac{\pi}{3}+\theta\right)$
$= \frac{1}{4}\cos^2\theta - \frac{3}{4}\sin\theta$

so $4\cos\left(\frac{\pi}{3}-\theta\right)\cdot\cos\left(\frac{\pi}{3}+\theta\right)$
$= (\cos^2\theta - 3\sin\theta)$

so $4\cos\theta\cdot\cos\left(\frac{\pi}{3}-\theta\right)\cdot\cos\left(\frac{\pi}{3}+\theta\right)$
$= \cos\theta(\cos^2\theta - 3\sin\theta)$

so $\boxed{1}$ $\cos\theta(\cos^2\theta - 3\sin\theta) = \cos 3\theta$

$\boxed{2}$ $\rightarrow \cos\theta(\cos^2\theta - 3\sin\theta) = \frac{1}{\sqrt{2}}$

So $\cos 3\theta = \frac{1}{\sqrt{2}}$ K

$3\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}; \frac{17\pi}{4}, \frac{23\pi}{4}$

$\theta = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{15\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

$\boxed{\text{also}}$

$\cos 3\theta = 4\cos\theta\cdot\left[\cos\left(\frac{\pi}{3}-\theta\right)\cdot\cos\left(\frac{\pi}{3}+\theta\right)\right]$
$= 4\cos\theta\cdot\frac{1}{2}\left[\cos(2\theta) + \cos\left(\frac{2\pi}{3}\right)\right]$
$= 2\cos\theta\cdot\left(2\cos^2\theta - 1 - \frac{1}{2}\right)$
$= 2\cos\theta\left(2\cos^2\theta - \frac{3}{2}(\cos^2\theta + \sin^2\theta)\right)$
$= 2\cos\theta\left(\frac{1}{2}\cos^2\theta - \frac{3}{2}\sin^2\theta\right)$

$\cos 3\theta = \cos\theta(\cos^2\theta - 3\sin^2\theta)$

so $\sqrt{2}\cos 3\theta = 1$

$\boxed{1}$ expands $\cos\left(\frac{\pi}{3}-\theta\right)$ or $\cos\left(\frac{\pi}{3}+\theta\right)$
- attempts other identities to solve

$\boxed{2}$ $\cos\left(\frac{\pi}{3}-\theta\right)\cdot\cos\left(\frac{\pi}{\theta}+\theta\right)$
$= \frac{1}{4}\cos^2\theta - \frac{3}{4}\sin\theta$

$\boxed{3}$ $\cos 3\theta = \frac{1}{\sqrt{2}}$
and $\theta = \frac{\pi}{12}\ldots$ etc)

End of Question 13

If you use this space, clearly indicate which question you are answering.

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Question 14 (15 marks) **Marks**

(a) 3

$$\frac{dy}{dx} = \frac{y-1}{x+2}$$

$$\int \frac{dy}{y-1} = \int \frac{dx}{x+2}$$

$$\ln|y-1| = \ln|x+2| + c$$

$$y(6) = 3$$

$$\ln 2 = \ln 8 + c$$

$$c = \ln \frac{1}{4}$$

$$\ln|y-1| - \ln|x+2| = \ln\frac{1}{4}$$

$$\therefore \ln\left|\frac{y-1}{x+2}\right| = \ln\frac{1}{4}$$

$$\left|\frac{y-1}{x+2}\right| = \frac{1}{4}$$

for $y(1)$

$$\therefore \left|\frac{y-1}{3}\right| = \frac{1}{4}$$

$$|y-1| = \frac{3}{4}$$

$$y-1 = -\frac{3}{4} \text{ or } y-1 = \frac{3}{4}$$

$$\therefore \boxed{y = \frac{1}{4} \text{ or } y = \frac{7}{4}}$$

or

$$\int_3^y \frac{1}{y-1}\,dy = \int_6^1 \frac{1}{x+2}\,dx$$

$$[\ln|y-1|]_3^y = [\ln|x+2|]_6^1$$

$$\ln|y-1| - \ln 2 = \ln 3 - \ln 8$$

$$\ln|y-1| = \ln 3 - \ln 8 + \ln 2$$

$$\ln|y-1| = \ln\frac{3}{4}$$

1 $\ln|y-1| = \ln|x+2| + c$

2 $\left|\frac{y-1}{x+2}\right| = \frac{1}{4}$

attempts to use $y(1)$ to solve $|y-1| = \frac{3}{4}$

3 $y = \frac{1}{4}$ or $y = \frac{7}{4}$

or

1 $[\ln|y-1|]_3^y = [\ln|x+2|]_6^1$

2 $\ln|y-1| = \ln\frac{3}{4}$

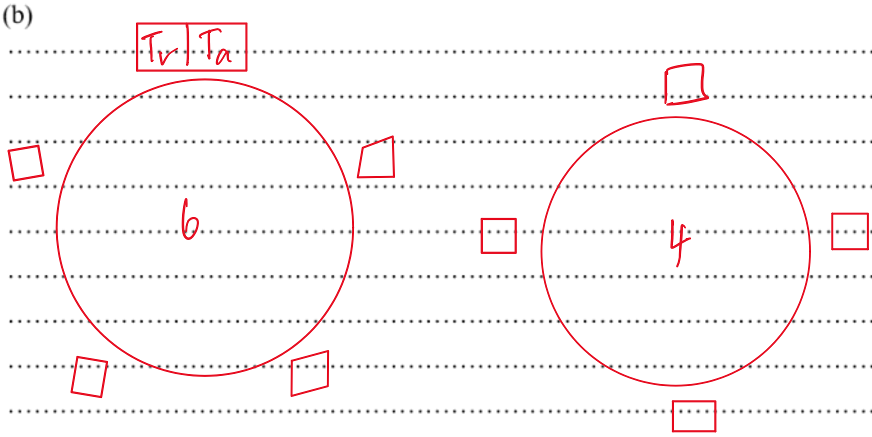
att's to solve $|y-1| = \frac{3}{4}$

3 $y = \frac{1}{4}$ or $y = \frac{7}{4}$

Question 14 continues on page 26

Question 14 continued. **Marks**

(b) **2**



$= {}^8C_4 \times 2! \times 4! \times 3!$

$= 3360 \times 6$

$= 20160$

$\boxed{0}$ mark — more than 3 errors.

Question 14 continues on page 27

$\boxed{1}$ mark — forgot to treat Tr and Ta seated together $\therefore 50400$

— forgot to include $2!$ $\therefore 10080$

— one mistake eg. ${}^8C_4 \times 2! \times 4! \times 4! = 80640$

$\boxed{2}$ marks — correct answer

Question 14 continued. **Marks**

(c) (i) 1

$\tan 4\alpha = 1 \qquad 0 \le \alpha \le \pi$

$0 \le 4\alpha \le 4\pi$

$4\alpha = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$\alpha = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$

1 | correct solution

(ii) 2

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$1 - 6x^2 + x^4 = 4x - 4x^3$

$1 = \dfrac{4x - 4x^3}{1 - 6x^2 + x^4}$

for $x = \tan\alpha$

$1 = \tan 4\alpha$

from (1) $\alpha = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$

$\therefore x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$.

Question 14 continues on page 28

1	uses identity to relate to (1), $\alpha =$
2	uses substitution $x = \tan\alpha$ to get sol.

Marks

(iii) sums/products of roots **2**

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$= (\text{sum of roots})^2 - 2(\text{sum of roots two at a time})$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$

$= (-4)^2 - 2(-6)$

$= 16 + 12$

$= 28$

$\therefore \tan^2\frac{\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{9\pi}{16} + \tan^2\frac{13\pi}{16} = 28$

1	correct $\left(-\frac{b}{a}\right)$ or $\left(\frac{c}{a}\right)$ (identifying sums of roots)
2	correctly showing

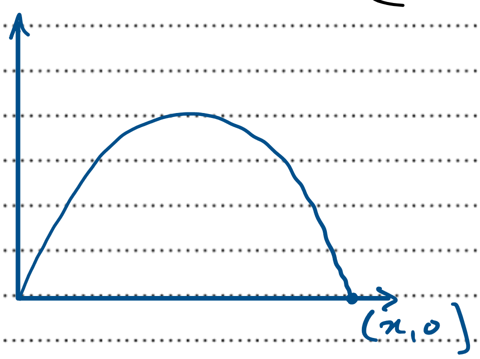
Note: Students should be identifying a sum/products of roots question and automatically finding $-\frac{b}{a}$, $\frac{c}{a}$ etc as required.

Question 14 continues on page 29

Question 14 continued. **Marks**

(d)

(i) $r(t) = Vt\cos\theta \, \underset{\sim}{i} + \left(Vt\sin\theta - \frac{gt^2}{2}\right)\underline{j}$ 3



Flight: $0 = Vt\sin\theta - \frac{gt^2}{2}$

$0 = 2Vt\sin\theta - gt^2$

$0 = t(2V\sin\theta - gt)$

$t = 0$ (initial time) $\therefore t = \frac{2V\sin\theta}{g}$

Range: $x = Vt\cos\theta$

when $t = \frac{2V\sin\theta}{g}$

$x = V\left[\frac{2V\sin\theta}{g}\right]\cos\theta$

$= \frac{V^2 . 2\sin\theta\cos\theta}{g}$

$\therefore x = \frac{V^2\sin 2\theta}{g}$

1	one correct
2	both correct.

note: For a quadratic equation (in t) you shouldn't be dividing by the variable as you lose a solution. A justification for not taking $t=0$ should be provided.

Question 14 continues on page 30

Question 14 part (d) continued. **Marks**

(ii) A: $v = u$ $\theta = 60°$ B: $v = 2u$ $\theta = 30°$ **2**

$x_A = \dfrac{u^2 . \sin[2.60°]}{g}$

$= \dfrac{u^2 \sin 120°}{g}$

$= \dfrac{u^2 . \frac{\sqrt{3}}{2}}{g}$

$= \dfrac{u^2\sqrt{3}}{2g}$

$x_B = \dfrac{(2u)^2 . \sin[2.(30)]}{g}$

$= \dfrac{4u^2 . \sin 60°}{g}$

$= \dfrac{4u^2 . \frac{\sqrt{3}}{2}}{g}$

$= \dfrac{2u^2\sqrt{3}}{g}$

$= 4 . \dfrac{u^2\sqrt{3}}{2g}$

$= 4x_A$

$\therefore$ 4 times the range of A.

1	$\dfrac{u^2\sqrt{3}}{2g}$ OR $\dfrac{2u^2\sqrt{3}}{g}$
2	correct

NOTE: Many students not using the expression for the range from part (i), and doing a lot of work to derive it again with the provided information. More attention needs to be paid to linking subsequent parts of questions.

END OF PAPER