SHORE SCHOOL

Sydney Church of England Grammar School

2003

TRIAL HSC EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using a black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of the paper
- This question paper is to be handed in at the end of the examination

Total marks - 84

- Attempt Questions 1-7
- Start each part in a new booklet
- Part A - Questions 1,2,3,4
- Part B - Questions 5,6,7
- All questions are of equal value
- All necessary working should be shown in every question

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination.

PART A

Question 1 (Start a new work booklet) **Marks**

a. i. Factorise completely $x^4 + x^3 - 2x^2 - 2x$ 1

ii. Hence sketch $y = x^4 + x^3 - 2x^2 - 2x$ for $-2 \le x \le 2$ 2

b. Find $\int \sin^2 3x \, dx$ 3

c. Calculate to the nearest minute the acute angle between the lines $x + y = 4$ and $3x - 4y = 2$ 3

d. Express $\dfrac{3^{501} + 3^{500}}{4}$ as a power of 3 1

e. Use the table of standard integrals to show that $\int_0^a \dfrac{1}{\sqrt{x^2 + a^2}} dx$ is independent of a 2

Question 2

a. Evaluate $\int_{-1}^{0} x\sqrt{1+x} \, dx$ using the substitution $u = 1 + x$ 3

b. The point $(a, 0)$ divides the interval AB joining the points A(9, 12) and B(1, b) in the ratio 2:1. Find a and b. 3

c. i. Show that the Cartesian equation of the curve with parametric equations $x = 6t$, $y = 3t^2$ is $12y = x^2$ 1

ii. Show that, if the tangents at the points $P(6p, 3p^2)$ and $Q(6q, 3q^2)$, intersect on the y-axis, then $p^2 = q^2$ 2

d. Solve $|x^2 - 5| = 5x + 9$ 3

Question 3 **Marks**

a. i. If $f(x) = e^{x+2}$, find the inverse function $f^{-1}(x)$ 1

ii. On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ showing all important features. 2

b. Find the term independent of x in the expansion $(2x - \frac{1}{x^2})^9$ 3

c. Assume the identity $(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$. By differentiating both sides, and making a suitable substitution for x, show that $n.2^{n-1} = \sum_{k=1}^{n} k.{}^nC_k$ 2

d. A particle moves in a straight line such that $v^2 = -9x^2 + 54x - 45$ where v is the velocity in metres/sec and x is the displacement, in metres, from the origin O.

(i) Show that the particle is moving in simple harmonic motion 2

(ii) What is the centre of oscillation? 1

(iii) What is the period of the motion? 1

Question 4 **Marks**

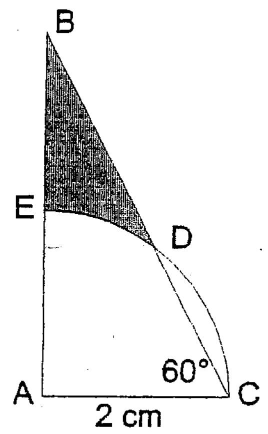
a. A golf ball is hit towards a tree 60 metres away and standing in the same horizontal plane as the ball. The tree is 20 metres high. The initial velocity of the ball is 30 metres per second. 6

Find the range of angles, in which the ball must be hit to clear the tree. Assume that $g = 10\ ms^{-2}$

b. A triangle ABC is right angled at A. $\angle C$ is 60° and AC is 2 cm long. 3

A circular arc, centre A and radius 2 cm, cuts BC at D and AB at E, as shown.

Show that the area of the shaded portion BED is $\left(\sqrt{3} - \frac{\pi}{3}\right) cm^2$



c. i. Show that the function $f(x) = xe^x - 1$ has a zero between $x = 0$ and $x = 1$ 1

ii. Using $x = 0.5$ as a first approximation, use Newton's Method once to obtain a second approximation to the zero. 2

PART B

Question 5 (Start a new work booklet) **Marks**

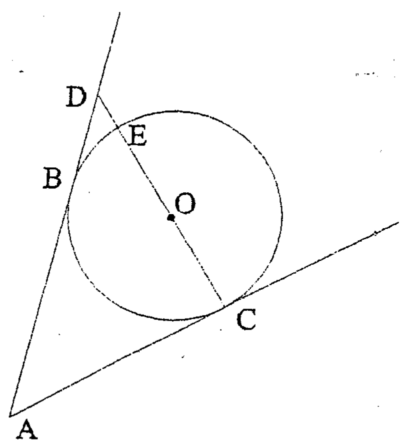
a. Use the derivative of $y = (x-2)e^{-x}$ to evaluate $\int_0^2 2e^{-x}(x - 3)\,dx$ **3**

b. At time t minutes, the temperature T°C of a body in a room of constant temperature 20°C is decreasing according to the equation
$\frac{dT}{dt} = -k(T - 20)$ for some constant $k > 0$.

i. Verify that $T = 20 + Ae^{-kt}$, where A is a constant, is a solution of the equation. **1**

ii. The initial temperature of the body is 90°C and it falls to 70°C after 10 minutes. Find the temperature after a further 5 minutes. **4**

c. The diagram shows two tangents, AB and AC, drawn from a common point, A, to a circle, centre O. **4**



The diameter CE produced cuts the tangent AB at the point D.

Copy the diagram into your workbook.

Show that $E\hat{D}A + 2 \times D\hat{E}B = 270°$

Marks

Question 6

a. A particle, initially at $x = 1.5$, has velocity given by $v = \sqrt{9 - 4x^2}$.

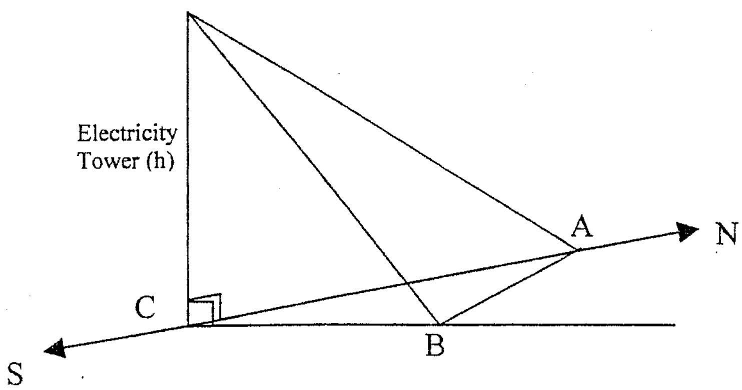
Derive an expression for x in terms of time, t **3**

b. Use the principle of mathematical Induction to prove that $3^{2n} + 7$ is divisible by 8, for positive integers $n \geq 1$ **4**

c. A person walks on level ground, in a northerly direction, away from an electricity tower. When she arrives at a point A, the angle of elevation to the top of the tower is 23^0. Another person walks on level ground on a bearing of 032^0T from the same tower, until he reaches point B, and notices that the angle of elevation is 17^0.

It is known that points A and B are 55m apart. Let h be the height of the tower and assume that the tower, base C, is perpendicular to the ground.

(i) Copy the diagram below onto your page/booklet and clearly mark on it all the information given. **1**



NOT TO SCALE

(ii) Find the distance from A to C and from B to C (Leave answer in terms of h). **2**

(iii) Hence, or otherwise, find the height h of the tower to the nearest metre. **2**

Question 7 **Marks**

a. i. Show by differentiation that $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right]=\frac{d}{dx}\left(\sin^{-1}x\right)$ **3**

ii. What is the significance of the result in part i. ? **1**

b. Evaluate $\sin\left[\cos^{-1}\frac{2}{3}+\tan^{-1}\left(-\frac{3}{4}\right)\right]$ giving its exact value. **3**

c. i. Find all solutions to $\frac{x}{1-x^2}\geq 0$ **2**

ii. Show that $\sec x\tan x\equiv\frac{\sin x}{1-\sin^2 x}$ **1**

iii. Hence, or otherwise, find all solutions to $\sec x\tan x\geq 0$ for $0\leq x\leq 2\pi$. **2**

THE END

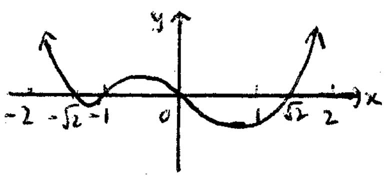
2003 Maths Extension 1 Trial Solutions

(1)(a)(i) $\text{Exp} = x(x^3+x^2-2x-2)$

$= x(x^2(x+1)-2(x+1))$

$= \underline{x(x+1)(x+\sqrt{2})(x-\sqrt{2})}$

(ii)



(b) $\int \sin^2 3x\,dx = \frac{1}{2}\int(1-\cos 6x)\,dx$

$= \underline{\frac{1}{2}x - \frac{1}{12}\sin 6x + C}$

(c) $x+y=4 \Rightarrow m_1=-1$

$3x-4y=2 \Rightarrow m_2 = \frac{3}{4}$

$\tan\theta = \left|\dfrac{\frac{3}{4}-(-1)}{1+\frac{3}{4}\times(-1)}\right|$

$= 7$

$\theta = \underline{81°52'}$

(d) $\dfrac{3^{501}+3^{500}}{4} = 3^{500}\left(\frac{3+1}{4}\right)$

$= \underline{3^{500}}$

(e) $\int_0^a \dfrac{1}{\sqrt{x^2+a^2}}\,dx = \left[\ln\left(x+\sqrt{x^2+a^2}\right)\right]_0^a$

$= \ln(a(1+\sqrt{2})) - \ln a$

$= \underline{\ln(1+\sqrt{2})}$

(2)(a) $I = \int_{-1}^{0} x\sqrt{1+x}\,dx$, $u=1+x$, $du=dx$, $x=-1, u=0$, $x=0, u=1$

$= \int_0^1 (u-1)\sqrt{u}\,du$

$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{5}-\frac{2}{3}-(0-0)$

$= \underline{-\frac{4}{15}}$

(b) $a = \dfrac{1\times 9+2\times 1}{3}$, $0 = \dfrac{1\times 12+2\times b}{3}$

$\underline{a = 3\frac{2}{3}\ ,\ b=-6}$

(c)(i) $x=6t \Rightarrow t=\frac{x}{6}$

$y=3t^2 = 3\left(\frac{x}{6}\right)^2 = \frac{x^2}{12}$

$\therefore\ \underline{x^2=12y}$

(ii) $y'=\frac{x}{6}$. At P, $y'=p$

Tang at P $y-3p^2 = p(x-6p)$

$\therefore\ \underline{y=px-3p^2}$

y-int when $x=0 \Rightarrow y=-3p^2$

Similarly for tang at Q $y=-3q^2$

As points coincide $-3p^2=-3q^2$

$\Rightarrow \underline{p^2=q^2}$

(d) $|x^2-5| = 5x+9$

If $|x|\geq\sqrt{5}$, $x^2-5=5x+9$

$x^2-5x-14=0$

$(x-7)(x+2)=0$

$\underline{x=7 \text{ or } -2}$

Accept $\underline{x=7}$ only.

If $|x|<\sqrt{5}$, $-(x^2-5)=5x+9$

$0=x^2+5x+4$

$0=(x+1)(x+4)$

$\underline{x=-1 \text{ or } -4}$

Accept $\underline{x=-1}$ only

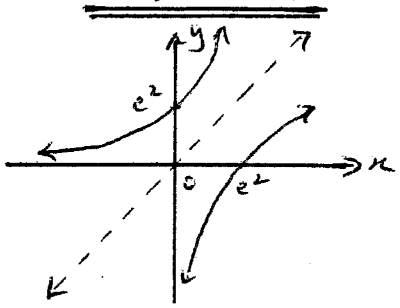
$\therefore\ \underline{x=7 \text{ or } -1}$

(3)(a)(i) $f(x)=e^{x+2}=y$

Inv $\Rightarrow x=e^{y+2} \Rightarrow y=\ln x - 2$

$\therefore\ \underline{f^{-1}(x)=\ln x-2}$

(ii)



(b) $\text{const} = {}^9C_3\,(2x)^6\left(-\frac{1}{x^2}\right)^3$

$= 84\times 64x^6 \times -\frac{1}{x^6}$

$= \underline{-5376}$

(c) $(1+x)^n = {}^nC_0+{}^nC_1x+{}^nC_2x^2+\cdots+{}^nC_nx^n$

$n(1+x)^{n-1} = {}^nC_1+2\,{}^nC_2x+\cdots+n\,{}^nC_nx^{n-1}$

Let $x=1$

$\underline{n.2^{n-1}} = {}^nC_1+2\,{}^nC_2+\cdots+n\,{}^nC_n$

$= \sum_{k=1}^{n} k.{}^nC_k$

(d) $v^2=-9x^2+54x-45$

(i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= -9x+27$

$= \underline{-9(x-3)}\quad \therefore$ SHM

(ii) Centre of motion $\underline{x=3}$

(iii) $n=3$, Period $= \underline{\frac{2\pi}{3}}$

(4)(a)

$\ddot{x}=0$ $\qquad \ddot{y}=-10$

$\dot{x}=30\cos\theta$ $\qquad \dot{y}=-10t+30\sin\theta$

$\underline{x=30t\cos\theta}$ $\qquad \underline{y=-5t^2+30t\sin\theta}$

$x=60 \Rightarrow \underline{t=2\sec\theta}$ and $y\geq 20$

$\Rightarrow -5(2\sec\theta)^2+30(2\sec\theta)\sin\theta \geq 20$

$-20(\tan^2\theta+1)+60\tan\theta-20\geq 0$

$\tan^2\theta-3\tan\theta+2\leq 0$

$(\tan\theta-1)(\tan\theta-2)\leq 0$

$\therefore\ 1\leq\tan\theta\leq 2$

$\underline{45°\leq\theta\leq 63°26'}$

(b) $AB=2\sqrt{3}$, $\angle DAC=60°$

Area $= \frac{1}{2}\times 2\times 2\sqrt{3} - \frac{1}{4}\pi\times 2^2 + \frac{1}{2}\times 2^2\left(\frac{\pi}{3}-\sin\frac{\pi}{3}\right)$

$= 2\sqrt{3}-\pi+\frac{2\pi}{3}-\sqrt{3}$

$= \underline{\sqrt{3}-\frac{\pi}{3}}\ \text{cm}^2$

(4)(c) $f(x) = xe^x - 1$

(i) $f(0) = -1$, $f(1) = e - 1 > 0$

sign change $\Rightarrow$ root between

(ii) $f'(x) = xe^x + e^x$

$$x_2 = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{-0.1756}{2.4731} = \underline{0.571} \quad (3\text{ d.p.})$$

B

(5)(a) $y = (x-2)e^{-x}$

$$y' = (x-2)(-e^{-x}) + e^{-x}(1) = \underline{(3-x)e^{-x}}$$

$$\int_0^2 2e^{-x}(x-3)\,dx = -2\int_0^2 (3-x)e^{-x}\,dx = -2\left[(x-2)e^{-x}\right]_0^2 = -2(0 - -2) = \underline{-4}$$

(b)(i) R.T.P $\frac{dT}{dt} = -k(T-20)$

$$LHS = \frac{d}{dt}(20 + Ae^{-kt}) = -Ake^{-kt} = -k(T-20) = \underline{RHS}$$

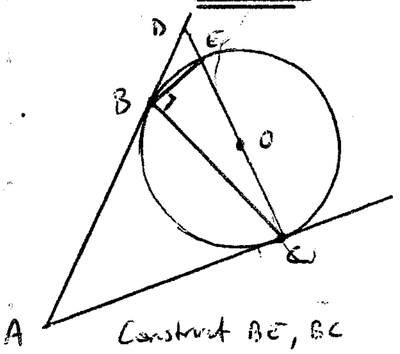
(ii) $t=0, T=90 \Rightarrow A = 70$

$t = 10, T = 70 \Rightarrow 70 = 20 + 70e^{-10k}$

$\Rightarrow k = -\frac{1}{10}\ln\left(\frac{5}{7}\right) = \underline{0.0336}$

$t = 15 \Rightarrow T = 20 + 70e^{-0.0336\times 15} = \underline{62.3°C}$

(c)



Construct BE, BC

$\angle EBC = 90°$ ($\angle$ in a semi circle)

$\angle ECB = \angle DEB - 90°$ (Ext $\angle$ of ΔEBC)

$\angle DBE = \angle ECB$ ($\angle$ between tang & chord equals $\angle$ in alt seg)

Now

$\angle EDA + \angle DEB + \angle DBE = 180°$ ($\angle$ sum ΔDBE)

$\angle EDA + \angle DEB + (\angle DEB - 90°) = 180°$

$\underline{\angle EDA + 2\times\angle DEB = 270°}$

(6) $v = \sqrt{9 - 4x^2}$

(a) $\frac{dt}{dx} = \frac{1}{\sqrt{9-4x^2}}$

$t = \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

$t = 0, x = 1.5 \Rightarrow \underline{c = -\frac{\pi}{4}}$

$2t = \sin^{-1}\frac{2x}{3} - \frac{\pi}{2}$

$\underline{x = \frac{3}{2}\sin\left(2t + \frac{\pi}{2}\right)}$

(b) R.T.P $3^{2n} + 7$ divis by 8

$n = 1$; $3^2 + 7 = 16$ ✓ divis by 8

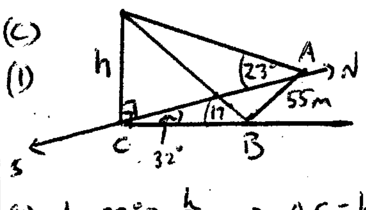
Assume true for $n = k$ (M integer) ie $3^{2k} + 7 = 8M$ or $3^{2k} = 8M - 7$

Now $3^{2(k+1)} + 7 = 3^2(8M - 7) + 7 = 72M - 56 = \underline{8(9M - 7)}$

So if true for $n = k$, also for $n = k+1$

True for $n = 1$ ∴ $n = 1 + 1 = 2$ and so on etc.

(c) (i)



(ii) $\tan 23° = \frac{h}{AC} \Rightarrow AC = h\cot 23°$

$\tan 17° = \frac{h}{BC} \Rightarrow BC = h\cot 17°$

(iii) $55^2 = AC^2 + BC^2 - 2\cdot AC\cdot BC\cos 32°$

$55^2 = h^2(\cot^2 23° + \cot^2 17° - 2\cot 23°\cot 17°\cos 32°)$

$$h = \sqrt{\frac{3025}{3.179}} = \underline{31\ m}$$

(7)(a)(i) $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right]$

$$= \frac{1}{1 + \frac{x^2}{1-x^2}} \times \frac{\sqrt{1-x^2} - x\times\frac{1}{2}\times\frac{1}{\sqrt{1-x^2}}}{1-x^2}$$

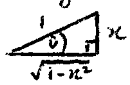
$$= \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{1 - x^2 + x^2}$$

$$= \frac{1 - x^2 + x^2}{\sqrt{1-x^2}}$$

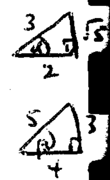
$$= \frac{1}{\sqrt{1-x^2}}$$

$$= \underline{\frac{d}{dx}(\sin^{-1}x)}$$

(ii) The two functions differ by a constant. The constant is in fact 0, so they are the same function



(b) Let $\alpha = \cos^{-1}\left(\frac{2}{3}\right)$, $\beta = \tan^{-1}(-$



$\sin(\alpha + \beta) =$

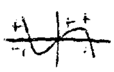
$\sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{\sqrt{5}}{3}\times\frac{4}{5} + \frac{2}{3}\times -\frac{3}{5} = \underline{\frac{4\sqrt{5} - 6}{15}}$$

(c)(i) $\frac{x}{1-x^2} \geq 0$, $x \neq \pm 1$

$(1 - x^2)x \geq 0$

$(1-x)(1+x)x \geq 0$



$\underline{x < -1,\ 0 \leq x < 1}$

(ii) RTP $\sec x\tan x = \frac{\sin x}{1 - \sin^2 x}$

$$RHS = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x\tan x = L$$

(iii) $\sec x\tan x \geq 0$, $0 \leq x \leq 2\pi$

$\Rightarrow \frac{\sin x}{1 - \sin^2 x} \geq 0$

$\Rightarrow \sin x < -1$ or $0 \leq \sin x < 1$

No solns ∴ $\underline{0 \leq x \leq \pi, x \neq}$