



SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2014

**TRIAL HIGHER SCHOOL
CERTIFICATE
EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 hours
- Write using black or blue pen.
Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question if full marks are to be awarded.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 70 Marks

Section I – 10 Marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 Marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

Examiner: *J. Chen*
R. Boros

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

1. If $(7, b)$ divides $(3, -4)$ and $(9, -7)$ internally in the ratio $a:1$, find the values of a and b .

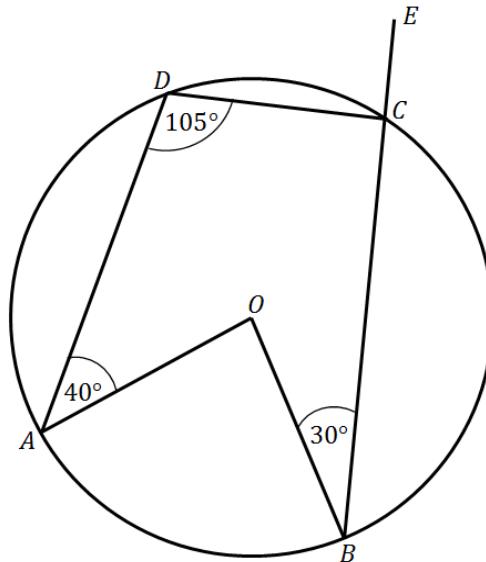
(A) $a = \frac{1}{2}$, $b = -\frac{23}{3}$

(B) $a = 2$, $b = -\frac{23}{3}$

(C) $a = \frac{1}{2}$, $b = -6$

(D) $a = 2$, $b = -6$

2. In the diagram below, O is the centre of the circle $ABCD$. BCE is a straight line. If $\angle ADC = 105^\circ$, $\angle OBC = 30^\circ$ and $\angle OAD = 40^\circ$, then $\angle DCE =$



(A) 75°

(B) 80°

(C) 85°

(D) 90°

3. $\alpha 3 \beta$ is a 3-digit number, where α and β are integers from 1 to 9 inclusive. Find the probability that the 3-digit number is divisible by 5.

(A) $\frac{1}{10}$

(B) $\frac{9}{50}$

(C) $\frac{1}{9}$

(D) $\frac{1}{5}$

4. Let $b > 1$ and $c > 1$. If $a = \log_c \sqrt{b}$, then $a^{-1} =$

(A) $\log_b c^2$

(B) $2 \log_c b$

(C) $\log_c \frac{1}{\sqrt{b}}$

(D) $\log_{\frac{1}{c}} \frac{1}{\sqrt{b}}$

5.

$$\frac{d}{dx}(x \sin^{-1} x) =$$

(A) $\sin^{-1} x - \frac{x}{\sqrt{1-x^2}}$

(B) $\sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$

(C) $\cos^{-1} x + \frac{x}{\sqrt{1-x^2}}$

(D) $\cos^{-1} x - \frac{x}{\sqrt{1-x^2}}$

6. It is given that α and β are roots of the equation $x^2 + 1 = 6x$, then $\alpha - \beta =$

(A) $-4\sqrt{2}$

(B) $4\sqrt{2}$

(C) $\pm 4\sqrt{2}$

(D) 32

7. If ${}^nP_2 = 56$, then

(A) $n = -7$

(B) $n = 8$

(C) $n = 11$

(D) $n = 112$

8. The minimum value of $\frac{1}{\sin^2 x - 2}$ is

(A) $-\frac{1}{2}$

(B) -1

(C) $-\frac{1}{3}$

(D) 0

9.

$$\int \frac{1}{\sqrt{25-4x^2}} \cdot dx =$$

(A) $\frac{1}{4} \sin^{-1} \left(\frac{5x}{2} \right) + C$

(B) $\frac{1}{4} \sin^{-1} \left(\frac{2x}{5} \right) + C$

(C) $\frac{1}{2} \sin^{-1} \left(\frac{5x}{2} \right) + C$

(D) $\frac{1}{2} \sin^{-1} \left(\frac{2x}{5} \right) + C$

10. The coefficient of x^{2n} in the binomial expansion of $(1+x)^{4n}$ is

(A) $\frac{4n!}{2n!2n!}$

(B) $\frac{(4n)!}{2(n!)^2}$

(C) $\frac{(4n)!}{(2n)!}$

(D) None of the above

End of Section A

Section II

60 marks

Attempt Questions 11 – 14

Allow about 1 hour and 45 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW Writing Booklet

- (a) Determine the acute angle, between the line $x - 3y + 2 = 0$ and the line BC where B is $(-1, -1)$ and C is $(1, 3)$. 2

- (b) Evaluate 1

$$\lim_{x \rightarrow 0} \frac{3x}{2 \sin 4x}$$

- (c) Solve for x , 2

$$\frac{(x - 2)}{(x - 1)(x - 3)} \geq 0$$

- (d) Write down a general solution to the equation $\cos 2x = -\frac{1}{2}$. Leave your answer in terms of π . 2

- (e) 2
- (i) Express $12 \cos x - 5 \sin x$ in the form $A \cos(x + \alpha)$ where A is positive and $0^\circ \leq \alpha \leq 180^\circ$, correct α to the nearest minute.

- (ii) Hence find the maximum value of $12 \cos x - 5 \sin x$ and the smallest positive value of x for which this maximum occurs. 2

- (f) Calculate the number of different arrangements which can be made using all the letters of the word BANANA. 1

Question 11 continues on page 7

(g)

(i) Differentiate $\cot x$ with respect to x . **1**

(ii) Hence differentiate $x \cot x$ with respect to x . **1**

(iii) Hence find **1**

$$\int x \operatorname{cosec}^2 x \cdot dx$$

End of Question 11

Question 12 (15 Marks) Start a NEW Writing Booklet

- (a) Express $\sin 2\theta$ and $\cos 2\theta$ in terms of $t = \tan \theta$ to show that 2

$$\frac{1 + \sin 2\theta - \cos 2\theta}{1 + \sin 2\theta + \cos 2\theta} = \tan \theta$$

- (b) In the expansion of $(1 + 2x)^n(1 - x)^2$, the coefficient of x^2 is 9. Find the coefficient of x in the expansion. 3

- (c) If the roots of $x^3 - 6x^2 + 3x + k = 0$ are consecutive terms of an arithmetic series, find k . 2

- (d) Evaluate 2

$$\int_0^{\frac{3}{4}} x\sqrt{1-x} \, dx$$

using the substitution $u = 1 - x$, express your answer in simplest exact form.

- (e) (i) Prove by the Principle of Mathematical Induction that 3

$$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n = (n - 1) \times 2^{n+1} + 2$$

for all positive integers n .

- (ii) Using the result of (i), simplify 2

$$\sum_{r=1}^n (r + 1) \times 2^r$$

- (f) Brian is to celebrate his 16th birthday by having a dinner with 11 other family members. At this dinner, Brian will sit at the head of a non-circular table. In how many ways can everyone be seated? 1

End of Question 12

Question 13 (15 Marks) Start a NEW Writing Booklet

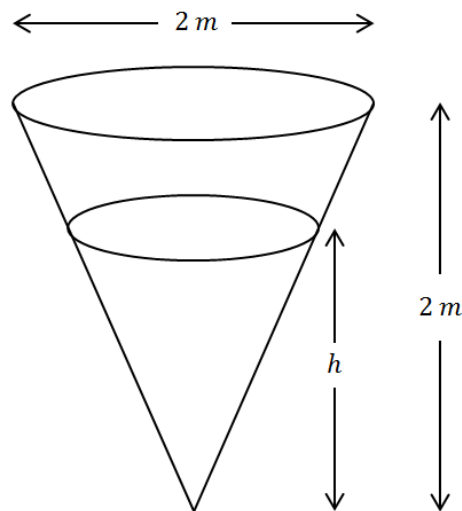
(a) A particle moves up and down so that its vertical displacement, x from a point O , is given by $x = 10 + 8 \sin 2t + 6 \cos 2t$ where x is in metres and t is in seconds.

(i) Show that the particle moves in Simple Harmonic motion. 1

(ii) What is the period of the motion? 1

(iii) What is the amplitude? 1

(b) A container in the shape of a right cone with both height and diameter 2 m is being filled with water at a rate of $\pi\text{ m}^3/\text{min}$.



(i) Show that 2

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

(ii) Find the rate of change of height h of the water when the container is $\frac{1}{8}th$ full by volume. 2

Question 13 continues on page 10

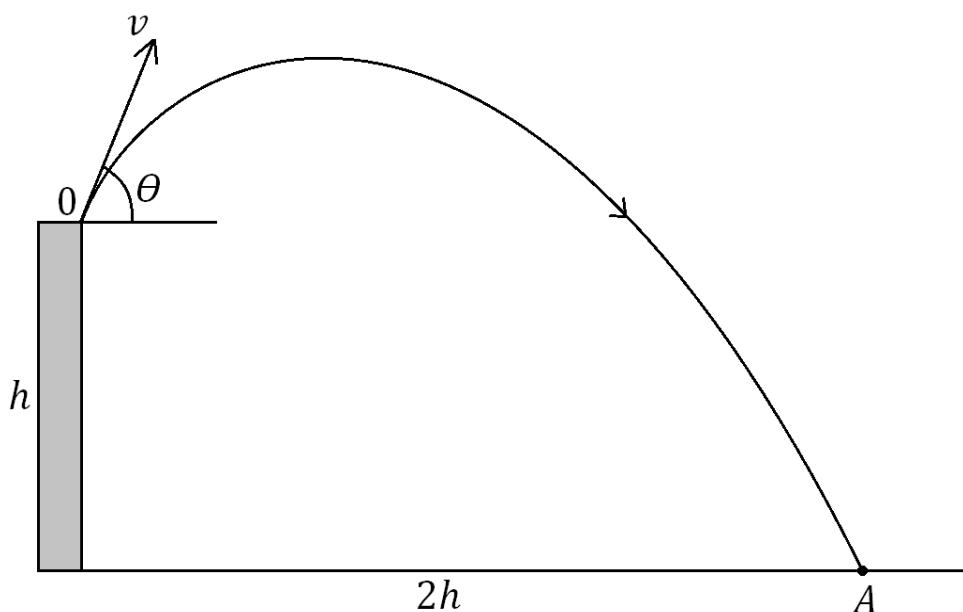
- (c) The rate of change in the number of members of the Sydney Boys High School Old Boys Mathematical Society, M , is given by

$$\frac{dM}{dt} = k(M - 50)$$

The number of members of this society at the start of 1995 was 70.

- (i) Show that $M = 20e^{kt} + 50$ satisfies the differential equation above. 1
- (ii) In 2000, the number of members was 150. Find the number of members in 2005. 1
- (iii) There is a year that this society will eventually become a “ghost society” with no members. Do you agree? Give reasons. 1

(d)



A projectile is fired with speed $\sqrt{\frac{4gh}{3}}$ at an angle θ to the horizontal from the top of a cliff of height h and the projectile strikes the ground a horizontal distance $2h$ from the base of the cliff.

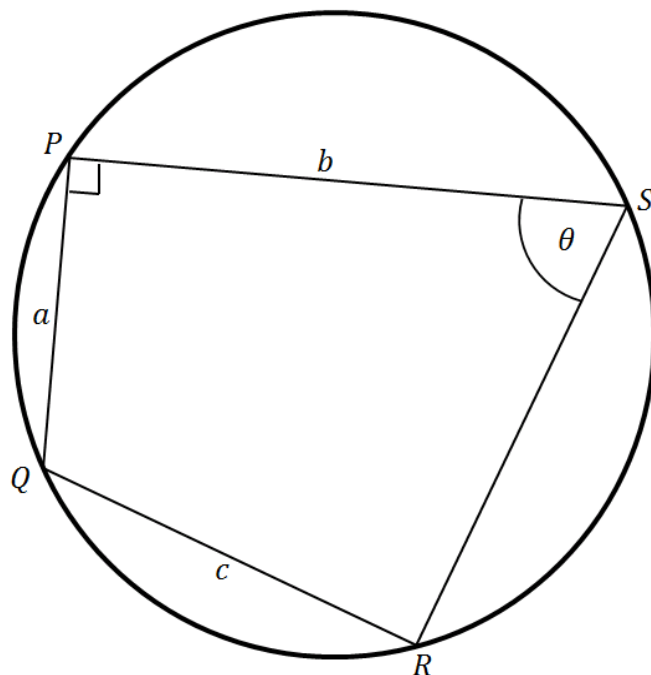
You may assume $y = Vt \sin \theta - \frac{1}{2}gt^2$ and $x = Vt \cos \theta$.

- (i) Show that $y = x \tan \theta - \frac{gx^2}{2V^2}(1 + \tan^2 \theta)$. 1
- (ii) Find the 2 possible values of $\tan \theta$. 2

Question 13 continues on page 11

- (e) In the diagram below, $PQRS$ is a cyclic quadrilateral, $\angle QPS = 90^\circ$ and $\angle PSR = \theta$, $PQ = a$, $PS = b$ and $QR = c$.

2



Show that $(a^2 + b^2) \sin^2 \theta = a^2 + c^2 + 2ac \cos \theta$.

End of Question 13

Question 14 (15 Marks) Start a NEW Writing Booklet

(a) At an election, 30% of the voters favoured party A. If 5 voters were selected at random, what is the probability (as a decimal) that

(i) exactly 3 favoured party A. 1

(ii) at most 2 favoured party A. 1

(b)

(i) Show that 3

$$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

(ii) If x satisfies the equation $\tan 3x = \cot 2x$, show that x also satisfies the equation $5 \tan^4 x - 10 \tan^2 x + 1 = 0$. 2

(iii) Using the result of (ii), deduce that 4

$$\tan \frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$$

(c) In the expansion of $(1 + x)^n$, let S_1 be the terms containing the coefficients ${}^nC_0, {}^nC_2, {}^nC_4, \dots$
whilst S_2 be the terms containing the coefficients ${}^nC_1, {}^nC_3, {}^nC_5, \dots$
Prove that,

(i) $4S_1S_2 = (1 + x)^{2n} - (1 - x)^{2n}$ 2

(ii) $(S_1)^2 - (S_2)^2 = (1 - x^2)^n$ 2

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$



Student Number: _____

SOLUTIONS

Mathematics Extension 1 Trial HSC 2014

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9
A ☐ B ☒ C ☐ D ☐

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ☒ B ☒ C ☐ D ☐

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

A ☒ B ☒ C ☐ D ☐
correct

Section I: Multiple choice answer sheet.

Completely colour the cell representing your answer. Use black pen.

1. (A) (B) (C) ☒
2. (A) (B) ☒ (D)
3. (A) (B) ☒ (D)
4. ☒ (B) (C) (D)
5. (A) ☒ (C) (D)
6. (A) (B) ☒ (D)
7. (A) ☒ (C) (D)
8. (A) ☒ (C) (D)
9. (A) (B) (C) ☒
10. (A) (B) (C) ☒

Question 11

a. $B(-1, -1)$ $C(1, 3)$

$$m_1 = \frac{3+1}{1+1}$$

$$= 2$$

$$x - 3y + 2 = 0$$

$$3y = x + 2$$

$$y = \frac{1}{3}x + \frac{2}{3}$$

$$\therefore m_2 = \frac{1}{3}$$

$$\tan \alpha = \frac{2 - \frac{1}{3}}{1 + 2 \cdot \frac{1}{3}}$$

$$\alpha = 45^\circ$$

b. $\lim_{x \rightarrow 0} \frac{3x}{2 \sin 4x}$

$$= \lim_{x \rightarrow 0} \frac{4 \cdot 3x}{4 \cdot 2 \sin 4x}$$

$$= \lim_{x \rightarrow 0} \frac{3}{8} \cdot \frac{4x}{\sin 4x}$$

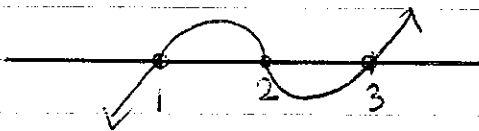
$$= 3/8$$

c. $\frac{x-2}{(x-1)(x-3)} > 0$

$$x \neq 1, 3$$

consider

$$y = (x-2)(x-1)(x-3)$$



$$\therefore 1 < x \leq 2, x > 3$$

d. $\cos 2\alpha = -1/2$

$$\cos \alpha = 1/2$$

$$\alpha = \pi/3$$

$\sqrt{s}$	A
$\sqrt{t}$	C

$$2\alpha = \frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi$$

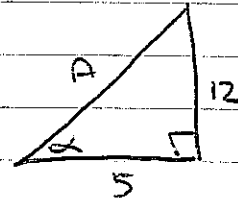
$$\alpha = \frac{\pi}{3} + k\pi, \frac{2\pi}{3} + k\pi$$

e. $12 \cos \alpha - 5 \sin \alpha \equiv A \cos(\alpha + \phi)$

$$\equiv A \cos \alpha \cos \phi - A \sin \alpha \sin \phi$$

$$12 = A \cos \phi, \quad 5 = A \sin \phi$$

$$\cos \phi = \frac{12}{A}, \quad \sin \phi = \frac{5}{A}$$



$$\therefore A = 13$$

$$\tan \phi = 5/12$$

$$\phi = 22^\circ 37'$$

ii. $12 \cos \alpha - 5 \sin \alpha \equiv 13 \cos(\alpha + 22^\circ 37')$

$$\text{Max value} = 13$$

$$\text{occurs when } \cos(\alpha + 22^\circ 37') = 1$$

$$\cos(\alpha + 22^\circ 37') = 1$$

$$\alpha + 22^\circ 37' = 360$$

$$\alpha = 337^\circ 23'$$

$$-1/2 \text{ for neg angle}$$

$$(5.89)$$

$$f. \text{ Banana} = \frac{6!}{2!3!} = 60 \quad (1)$$

$$g. \text{ i) } \frac{d \cot x}{dx} = \frac{1}{\tan x} dx \quad \begin{array}{l} u = 1 \quad u' = 0 \\ v = \tan x \quad v' = \sec^2 x \end{array}$$

$$= \frac{\cancel{1} 0 - \sec^2 x}{\tan^2 x}$$

$$= \frac{-1}{\cos^2 x} \times \frac{\cos^2 x}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x \quad (1)$$

$$\text{ii. } \frac{d(x \cot x)}{dx} = x(-\operatorname{cosec}^2 x) + \cot x$$

$$= \cot x - x \operatorname{cosec}^2 x \quad (1)$$

$$\text{iii } \int x \operatorname{cosec}^2 x \, dx = - \int -x \operatorname{cosec}^2 x \, dx$$

$$= - \int (\cot x - x \operatorname{cosec}^2 x - \cot x) \, dx$$

$$= - \int (\cot x - x \operatorname{cosec}^2 x - \frac{\cos x}{\sin x}) \, dx$$

$$= - [x \cot x - \ln(\sin x)] + c$$

$$= \ln(\sin x) - x \cot x + c \quad (1)$$

12) a) $t = \tan \theta$

$$\text{LHS} = \frac{1 + \sinh 2\theta - \cosh 2\theta}{1 + \sinh 2\theta + \cosh 2\theta}$$

$$= \frac{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{1+t^2}{1+t^2}$$

$$= \frac{\cancel{1+t^2} + 2t - \cancel{1+t^2}}{\cancel{1+t^2} + 2t + \cancel{1-t^2}}$$

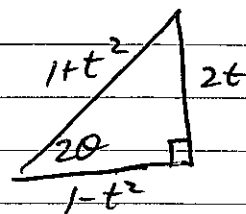
$$= \frac{2t^2 + 2t}{2t + 2}$$

$$= \frac{\cancel{2t}(\cancel{t+1})}{\cancel{2}(\cancel{t+1})}$$

$$= t$$

$$= \tan \theta$$

$$= \text{RHS}$$



b) $(1+2x)^n(1-x)^2 \equiv \left({}^nC_0 + {}^nC_1(2x) + {}^nC_2(2x)^2 + \dots \right) (1-2x+x^2)$

coefficient of x^2 is 9

$${}^nC_0 + {}^nC_1(2)(-2) + {}^nC_2(2)^2 = 9$$

$$1 - 4n + \frac{4n(n-1)}{2} = 9$$

$$1 - 4n + 2n^2 - 2n = 9$$

$$2n^2 - 6n - 8 = 0$$

$$n^2 - 3n - 4 = 0 \quad \begin{array}{l} \times \quad -4 \\ + \quad -3 \\ \hline \end{array}$$

$$(n-4)(n+1) = 0$$

$$\textcircled{n=4}, -1$$

coefficient of x is

$${}^4C_0(-2) + {}^4C_1(2) = -2 + 8 \\ = 6$$

c) let roots be $\alpha - \beta, \alpha, \alpha + \beta$

$$(\alpha - \beta) + \alpha + (\alpha + \beta) = -\frac{b}{a}$$

$$3\alpha = -\frac{(-6)}{1}$$

$$\alpha = 2$$

Since α is a root $p(2) = 0$

$$(2)^3 - 6(2)^2 + 3(2) + k = 0$$

$$-10 + k = 0$$

$$k = 10$$

$$d) \int_0^{3/4} x \sqrt{1-x} dx$$

$$u = 1-x$$

$$\frac{du}{dx} = -1$$

$$dx = -du$$

limit change when $x=0, u=1$
 $x=3/4, u=1/4$

$$= \int_1^{1/4} (1-u) \sqrt{u} \cdot -du$$

$$= \int_{1/4}^1 (u^{1/2} - u^{3/2}) du$$

$$= \left[\frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right]_{1/4}^1$$

$$= \frac{2}{3} (1)^{3/2} - \frac{2}{5} (1)^{5/2} - \left(\frac{2}{3} \left(\frac{1}{4}\right)^{3/2} - \frac{2}{5} \left(\frac{1}{4}\right)^{5/2} \right)$$

$$= \frac{47}{240}$$

e) i) Prove $1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \dots + n \times 2^n = (n-1) \times 2^{n+1} + 2$

Prove true for $n=1$

$$\begin{aligned} \text{LHS} &= 1 \times 2^1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= (1-1) \times 2^{1+1} + 2 \\ &= 2 \end{aligned}$$

$$\text{LHS} = \text{RHS}$$

$\therefore$ true for $n=1$

Assume true for $n=k$, where $k \in \mathbb{N}$

$$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \dots + k \times 2^k = (k-1) \times 2^{k+1} + 2$$

Prove true for $n=k+1$

$$\text{ie } 1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \dots + k \times 2^k + (k+1) \times 2^{k+1} = k \times 2^{k+2} + 2$$

$$\text{LHS} = 1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \dots + k \times 2^k + (k+1) \times 2^{k+1}$$

$$= (k-1) \times 2^{k+1} + 2 + (k+1) \times 2^{k+1}$$

$$= 2^{k+1} (\cancel{k-1} + \cancel{k+1}) + 2$$

$$= 2^{k+1} (2k) + 2$$

$$= k \times 2^{k+2} + 2$$

$$= \text{RHS}$$

$\therefore$ true for $n=k+1$

$\therefore$ true by induction for positive integers n .

$$\text{ii) } \sum_{r=1}^n (r+1) 2^r$$

$$= 2 \times 2^1 + 3 \times 2^2 + 4 \times 2^3 + \dots + (n+1) \times 2^n$$

$$= \frac{1}{2} (2 \times 2^2 + 3 \times 2^3 + 4 \times 2^4 + \dots + (n+1) \times 2^{n+1})$$

$$= \frac{1}{2} (1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \dots + (n+1) \times 2^{n+1}) - 1$$

$$= \frac{1}{2} (n \times 2^{n+2} + 2) - 1$$

$$= n \times 2^{n+1} + 1 - 1$$

$$= n \times 2^{n+1}$$

OR

$$\sum_{r=1}^n (r+1)2^r = \sum_{r=1}^n r \times 2^r + \sum_{r=1}^n 2^r$$

from (i) geometric series

$$S_n = a \frac{(r^n - 1)}{r - 1}$$

$$S_n = 2 \frac{(2^n - 1)}{2 - 1}$$

$$= 2^{n+1} - 2$$

$$\sum_{r=1}^n r \times 2^r = (n-1)2^{n+1} + 2$$

$$\therefore \sum_{r=1}^n (r+1)2^r = (n-1)2^{n+1} + 2 + 2^{n+1} - 2$$

$$= 2^{n+1} (n-1+1)$$

$$= n \times 2^{n+1}$$

f) $1 \times 11! = 39916800$

Question 13

(a) $x = 10 + 8 \sin 2t + 6 \cos 2t$

(i) $\dot{x} = 16 \cos 2t - 12 \sin 2t$
 $\ddot{x} = -32 \sin 2t - 24 \cos 2t$
 $\ddot{x} = -4(8 \sin 2t + 6 \cos 2t)$
 $\therefore \ddot{x} = -4(x - 10)$

Now let $X = x - 10$

$\therefore \ddot{X} = -4X$ and thus the motion is SHM.

(ii) Clearly $n = 2$.

$$\therefore T = \frac{2\pi}{2}$$
$$= \pi$$

(iii) Amplitude
 $a = \sqrt{8^2 + 6^2}$
 $= 10$

(b) (i) $\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt}$
 $V = \frac{1}{3} \pi \left(\frac{h}{2} \right)^2 h$
 $= \frac{1}{12} \pi h^3$
 $\frac{dV}{dh} = \frac{1}{4} \pi h^2$
 $\therefore \frac{dV}{dt} = \frac{\pi h^2}{4} \frac{dh}{dt}$

(ii) Maximum Volume

$$V_{\max} = \frac{1}{12} \pi (2)^3$$

One eighth full means

$$V = \frac{8\pi}{12} \div 8$$
$$= \frac{\pi}{12}$$

Thus

$$\frac{\pi}{12} = \frac{1}{12} \pi h^3$$

$$h = 1\text{m}$$

We seek $\frac{dh}{dt}$ when $h = 1$.

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \frac{dV}{dt} \\ &= \frac{4}{\pi h^2} \pi \\ &= 4 \text{ m/min}\end{aligned}$$

(c) $\frac{dM}{dt} = k(M - 50)$

When $t = 0$, $M = 70$.

(i) Consider

$$M = 20e^{kt} + 50$$

$$\begin{aligned}\frac{dM}{dt} &= 20ke^{kt} \\ &= k(20e^{kt}) \\ &= k(M - 50) \\ \therefore \text{ Satisfies D.E.}\end{aligned}$$

(ii) When $t = 5$, $M = 150$.

$$\begin{aligned}150 &= 20e^{5k} + 50 \\ 100 &= 20e^{5k} \\ 5 &= e^{5k} \\ k &= \frac{\ln 5}{5} \\ &\approx 0.3219\end{aligned}$$

Thus when $t = 10$

$$\begin{aligned}M &= 20e^{10k} + 50 \\ &= 550\end{aligned}$$

(iii) No, as $k > 0$ membership always increases.

(d) Given $y = Vt \sin \theta - \frac{1}{2}gt^2$ and $x = Vt \cos \theta$

(i) $t = \frac{x}{V \cos \theta}$, substitute to obtain

$$y = \frac{x \sin \theta}{\cos \theta} - \frac{1}{2} g \left(\frac{x^2}{V^2 \cos^2 \theta} \right)$$

$$y = x \tan \theta - \frac{gx^2}{2V^2} (1 + \tan^2 \theta) \text{ as required.}$$

(ii) The point A is $(2h, -h)$. Substitute:

$$-h = 2h \tan \theta - \frac{g(2h)^2}{2V^2} (1 + \tan^2 \theta)$$

$$\text{But } V^2 = \frac{4gh}{3}$$

$$-h = 2h \tan \theta - \frac{3h}{2} (1 + \tan^2 \theta)$$

$$\text{Thus } 3 \tan^2 \theta - 4 \tan \theta + 1 = 0$$

$$\text{So } \tan \theta = 1 \text{ or } \frac{1}{3}$$

(e) Required to prove $(a^2 + b^2) \sin^2 \theta = a^2 + c^2 + 2ac \cos \theta$

Join PR , QS . QS is the diameter, so

$\angle QPR = 90^\circ$ (angle in a semicircle).

In $\triangle PQS$ $QS^2 = a^2 + b^2$ (Pythagoras's Thm)

$$\sin \angle PQS = \frac{b}{QS}$$

$$\therefore QS = \frac{b}{\sin \angle PQS}$$

In quad $PQRS$ $\angle PQR = 180^\circ - \theta$ (opposite angles of cyclic quadrilateral)

$\angle PRS = \angle PQS$ (angles in same segment)

In $\triangle PQR$

$$PR^2 = a^2 + c^2 - 2ac \cos(180^\circ - \theta)$$

$$= a^2 + c^2 + 2ac \cos \theta$$

In $\triangle PRS$

$$\frac{PR}{\sin \theta} = \frac{b}{\sin \angle PRS}$$

$$= \frac{b}{\sin \angle PQS}$$

$$= QS \quad \text{from above}$$

$$\therefore PR = QS \sin \theta$$

$$PR^2 = QS^2 \sin^2 \theta$$

$$= a^2 + c^2 + 2ac \cos \theta$$

$$\text{Thus } (a^2 + b^2) \sin^2 \theta = a^2 + c^2 + 2ac \cos \theta \quad \text{QED}$$

QUESTION 14 (X1)

$$(a) \quad (i) \quad P(x=3) = \binom{5}{3} (0.3)^3 (0.7)^2$$
$$\quad \quad \quad \boxed{\div 0.1323}$$

$$(ii) \quad P(x=0) + P(x=1) + P(x=2)$$
$$= \binom{5}{0} (0.3)^0 (0.7)^5 + \binom{5}{1} (0.3)^1 (0.7)^4 + \binom{5}{2} (0.3)^2 (0.7)^3$$
$$\quad \quad \quad \boxed{\div 0.83692}$$

$$(b) \quad (i) \quad \text{Let } t = \tan x \quad \therefore \tan 3x = \tan(2x+x)$$
$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \cdot \tan x}$$
$$= \frac{\frac{2t}{1-t^2} + t}{1 - \frac{2t}{1-t^2} \cdot t}$$
$$= \frac{2t + t(1-t^2)}{1-t^2 - 2t^2}$$
$$= \frac{3t - t^3}{1 - 3t^2}$$
$$= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

$$(ii) \quad \text{Let } \tan 3x = \cot 2x.$$
$$\text{ie. } \tan 3x = \tan\left(\frac{\pi}{2} - 2x\right)$$

$$3x = \frac{\pi}{2} - 2x$$

$$5x = \frac{\pi}{2}$$

$$x = \frac{\pi}{10}$$

$$\text{OR, } \tan 3x = \cot 2x$$

$$\Rightarrow \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = \frac{1}{\tan 2x}$$

(11) (contd)

$$\text{ie. } \frac{3t - t^3}{1 - 3t^2} = \frac{1 - t^2}{2t}$$

$$2t(3t - t^3) = (1 - 3t^2)(1 - t^2)$$

$$6t^2 - 2t^4 = 1 - t^2 - 3t^2 + 3t^4$$

$$5t^4 - 10t^2 + 1 = 0. \quad (A)$$

(11) From $x = \frac{\pi}{10}$ is also a solution to (A)

$$\text{ie. } t^2 = \frac{10 \pm \sqrt{100 - 20}}{10}$$

$$= \frac{10 \pm 4\sqrt{5}}{10}$$

$$= \frac{5 \pm 2\sqrt{5}}{5}$$

$$\text{ie. } t = \pm \sqrt{\frac{5 \pm 2\sqrt{5}}{5}}$$

$$\text{ie. } \tan \frac{\pi}{10} = \pm \sqrt{\frac{5 \pm 2\sqrt{5}}{5}}$$

$$\text{Clearly } \tan \frac{\pi}{10} > 0$$

$$\therefore \tan \frac{\pi}{10} = \sqrt{\frac{5 \pm 2\sqrt{5}}{5}}$$

$$\begin{aligned} \text{now } \tan \frac{\pi}{5} &= \frac{2 \tan \frac{\pi}{10}}{1 - \tan^2 \frac{\pi}{10}} \\ &= \frac{2 \sqrt{\frac{5 \pm 2\sqrt{5}}{5}}}{1 - \frac{5 \pm 2\sqrt{5}}{5}} \end{aligned}$$

(iii) CONTD.

$$= \frac{2 \sqrt{\frac{5 \pm 2\sqrt{5}}{5}}}{\frac{2\sqrt{5}}{5}}$$

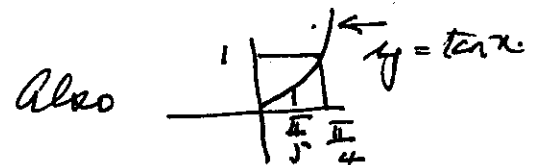
$$= \frac{\pm \sqrt{\frac{5 \pm 2\sqrt{5}}{5}}}{\frac{\sqrt{5}}{5}}$$

$$= \pm \sqrt{5 \pm 2\sqrt{5}}$$

$$= \sqrt{5 \pm 2\sqrt{5}}$$

(clearly $\tan \frac{\pi}{5} > 0$)

$$\therefore \boxed{\tan \frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}}$$



$$\tan \frac{\pi}{5} < \tan \frac{\pi}{4} = 1$$

$$\therefore \tan \frac{\pi}{5} \neq \sqrt{5 + 2\sqrt{5}}$$

which is > 1 .

(c) Now $S_1 = \binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \dots$

$$\& S_2 = \binom{n}{1}x + \binom{n}{3}x^3 + \binom{n}{5}x^5 + \dots$$

$$\therefore S_1 + S_2 = (1+x)^n$$

$$\& S_1 - S_2 = (1-x)^n$$

(i) RHS = $(1+x)^{2n} - (1-x)^{2n}$

$$= (S_1 + S_2)^2 - (S_1 - S_2)^2$$

$$= S_1^2 + 2S_1S_2 + S_2^2 - (S_1^2 - 2S_1S_2 + S_2^2)$$

$$= 4S_1S_2$$

$$= \text{LHS}$$

$$\begin{aligned}
 \text{(ii)} \quad LHS &= S_1^2 - S_2^2 \\
 &= (S_1 + S_2)(S_1 - S_2) \\
 &= (1+x)^n (1-x)^n \\
 &= (1-x^2)^n \\
 &= RHS.
 \end{aligned}$$