



# 2016 SYDNEY BOYS HIGH SCHOOL

## TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

# Mathematics Extension I

### General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided with this paper
- Leave your answers in the simplest exact form, unless otherwise stated
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

### Total marks – 70

#### Section I Pages 3–6

#### 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

#### Section II Pages 8–15

#### 60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

**Examiner:** E.C.

## Section I

10 marks

Attempt Questions 1-10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 Which integral is obtained when the substitution  $u = 1 + 3x$  is applied to  $\int x\sqrt{1+3x} \, dx$ ?

(A)  $\frac{1}{9} \int (u-1)\sqrt{u} \, du$

(B)  $\frac{1}{6} \int (u-1)\sqrt{u} \, du$

(C)  $\frac{1}{3} \int (u-1)\sqrt{u} \, du$

(D)  $\frac{1}{4} \int (u-1)\sqrt{u} \, du$

2 The acceleration of a particle moving along a straight line is given by  $\ddot{x} = -2e^{-x}$ , where  $x$  metres is the displacement from the origin.

If the velocity of the particle is  $v$  m/s, which of the following is a correct statement about  $v^2$ ?

(A)  $v^2 = 2e^{-x} + C$

(B)  $v^2 = 2e^x + C$

(C)  $v^2 = 4e^{-x} + C$

(D)  $v^2 = 4e^x + C$

3 Find  $\frac{d}{dx}(x \cos^{-1} x - \sqrt{1-x^2})$

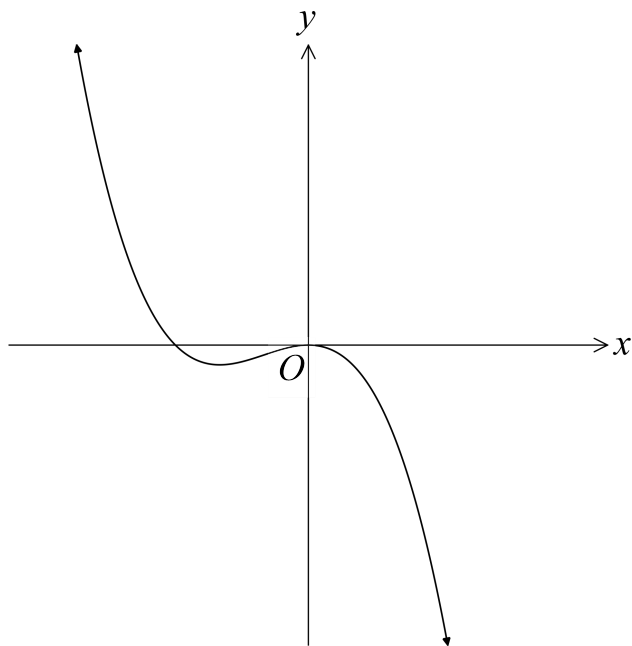
(A)  $\frac{-2}{\sqrt{1-x^2}}$

(B)  $\frac{-1}{\sqrt{1-x^2}}$

(C)  $\cos^{-1} x$

(D)  $\sin^{-1} x$

- 4 What is a possible equation of this function?



- (A)  $f(x) = -x(x-1)(x+1)$
- (B)  $f(x) = -x^2(x+1)$
- (C)  $f(x) = -x^2(x-1)$
- (D)  $f(x) = x^2(x+1)$
- 5 If  $f(x) = 1 + \frac{2}{x-3}$ , which of the following give the equations of the horizontal and vertical asymptotes of  $f^{-1}(x)$ ?
- (A) Vertical asymptote is  $x = 1$  and horizontal asymptote is  $y = 2$
- (B) Vertical asymptote is  $x = 1$  and horizontal asymptote is  $y = 3$
- (C) Vertical asymptote is  $x = 3$  and horizontal asymptote is  $y = 1$
- (D) Vertical asymptote is  $x = 3$  and horizontal asymptote is  $y = 2$

6 The polynomial equation  $x^3 - ax^2 + 8x + (1 - a) = 0$  has roots  $\alpha$ ,  $\beta$  and  $\gamma$ .

Given that  $\alpha + \beta + \gamma < 0$  and  $\alpha\beta\gamma(\alpha + \beta + \gamma) = 20$ , what is the value of  $a$ ?

- (A)  $-4$
- (B)  $4$
- (C)  $-5$
- (D)  $5$

7 If  $t = \tan \frac{\theta}{2}$ , which of the following expressions is equivalent to  $4\sin \theta + 3\cos \theta + 5$ ?

- (A)  $\frac{2(t+2)^2}{1-t^2}$
- (B)  $\frac{(t+4)^2}{1-t^2}$
- (C)  $\frac{2(t+2)^2}{1+t^2}$
- (D)  $\frac{(t+4)^2}{1+t^2}$

8 Which of the following is a correct expression for  $\tan\left(x + \frac{\pi}{4}\right)$ ?

- (A)  $\frac{\cos x + \sin x}{\cos x - \sin x}$
- (B)  $\frac{\cos x + 2\sin x}{\cos x - \sin x}$
- (C)  $\frac{\cos x + \sin x}{\cos^2 x - \sin x}$
- (D)  $\frac{\cos x - \sin x}{\cos x - \sin x}$

- 9** The curve  $y = 2x^{\frac{1}{3}}$  is reflected in the line  $y = x$ .  
What is the equation of the reflected curve?

(A)  $y = \frac{x^3}{16}$

(B)  $y = \frac{x^3}{8}$

(C)  $y = \frac{x^3}{4}$

(D)  $y = \frac{x^3}{2}$

- 10** A particle is moving in simple harmonic motion with displacement  $x$ .  
Its velocity is given by  $v^2 = 9(36 - x^2)$ .  
What is the amplitude,  $A$ , of the motion and the maximum speed of the particle?

- (A)  $A = 3$  and maximum speed  $v = 6$   
(B)  $A = 3$  and maximum speed  $v = 18$   
(C)  $A = 6$  and maximum speed  $v = 18$   
(D)  $A = 6$  and maximum speed  $v = 6$

## Section II

60 marks

Attempt Questions 11-14

Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

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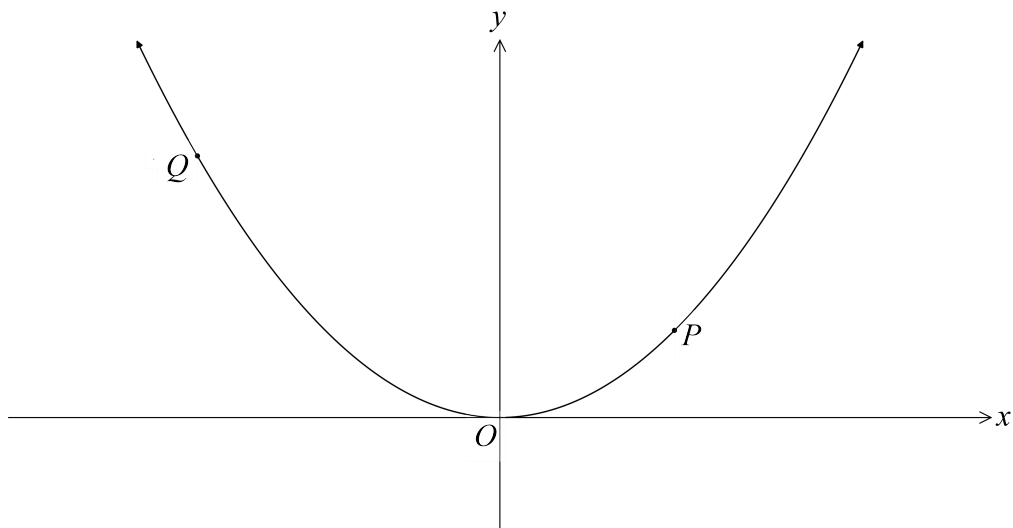
**Question 11** (15 marks) Use a SEPARATE writing booklet.

- (a) Differentiate with respect to  $x$ :

$$y = \tan^{-1}\left(\frac{2}{x}\right) \quad 2$$

- (b) Evaluate  $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-4x^2}}$  2

- (c) The points  $P(2ap, ap^2)$  and  $Q(2aq, aq^2)$  lie on the parabola  $x^2 = 4ay$  2



If the tangents at  $P$  and  $Q$  intersect at  $45^\circ$ , show that  $|1 + pq| = |p - q|$ .

**Question 11 continues on page 9**

Question 11 (continued)

- (d) State the domain and range of the function  $y = 2\cos^{-1} 3x$ . **1**
- (e) The roots of the equation  $x^3 - 3x^2 + 4x + 2 = 0$  are  $\alpha$ ,  $\beta$ , and  $\gamma$ . **2**  
Find the value of  $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ .
- (f) Solve the equation  $4\sin\theta + 3\cos\theta = -5$  for  $0^\circ < \theta < 360^\circ$ . **2**  
Leave your answers correct to the nearest degree.
- (g) (i) Show that the turning points of the curve  $y = \frac{x}{(x+3)(x+4)}$  occur when **2**  
 $x = \pm 2\sqrt{3}$ .
- (ii) Sketch  $y = \frac{x}{(x+3)(x+4)}$  for  $x \geq 0$ . **2**

**End of Question 11**

**Question 12** (15 marks) Use a SEPARATE writing booklet.

- (a) The rate at which perfume evaporates is proportional to the amount of the perfume that has not yet evaporated. That is  $\frac{dN}{dt} = k(P - N)$ , where  $P$  is the initial amount of perfume,  $N$  is the amount that has evaporated at time  $t$  and  $k$  is constant.

- (i) Show that the function  $N = P(1 - e^{-kt})$  satisfies the differential equation

1

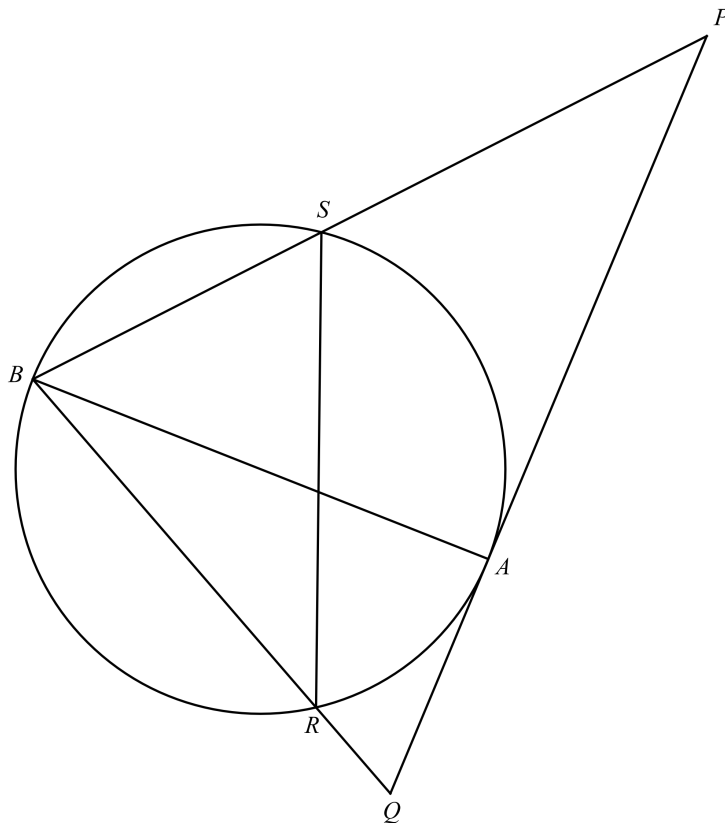
$$\frac{dN}{dt} = k(P - N)$$

- (ii) Show that the time it takes for a quarter of the original amount to evaporate is

2

$$-\frac{(\ln 3 - 2 \ln 2)}{k}$$

- (b) In the diagram below,  $PAQ$  is the tangent to a circle at  $A$ .  $AB$  is a diameter and lines  $PB$  and  $QB$  cut the circle at  $S$  and  $R$  respectively.



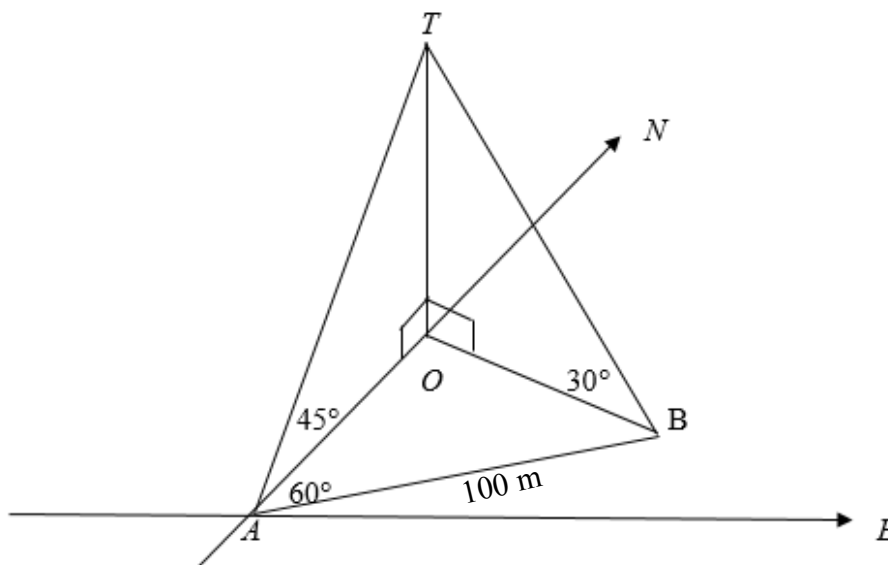
- (i) Copy the diagram to your writing booklet.
- (ii) Prove that  $PQRS$  is a cyclic quadrilateral.

3

**Question 12 continues on page 11**

Question 12 (continued)

- (c) In how many ways would 11 people occupy seats at two circular tables, where one table can accommodate 6 people and the other 5 people? 2
- (d) Consider the function  $f(x) = 3x - x^3$
- (i) Find the largest domain containing the origin for which  $f(x)$  has an inverse function  $f^{-1}(x)$ . 2
- (ii) State the domain of  $f^{-1}(x)$ . 1
- (e) To an observer on a pier  $A$ , the angle of elevation of the top of a cliff  $OT$  due North of the observer is  $45^\circ$ . After the observer travelled 100m by boat from the pier at  $N60^\circ E$  to  $B$ , the angle of elevation of the top of the cliff is  $30^\circ$ .



Find the height of the cliff above the sea level. 2

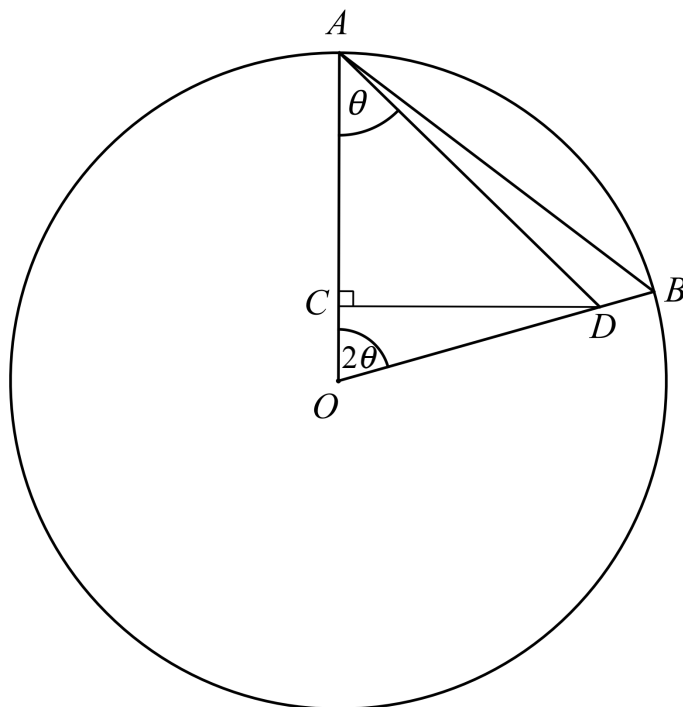
- (f) A particle moves in a straight line with acceleration at any time  $t$  given by  $\ddot{x} = -e^{-2x}$ , where  $x$  metres is the distance measured from a fixed point  $O$ . 2

Initially the particle is at the origin with velocity 1 m/s. Show that  $x = \ln(t+1)$ .

**End of Question 12**

**Question 13** (15 marks) Use a SEPARATE writing booklet.

- (a) The diagram below shows a circle of centre  $O$  and radius 1 m and  $\angle AOD = 2\theta$ .  $D$  is a point on  $OB$  such that  $\angle DAO = \theta$ . Also,  $C$  is a point on  $OA$  such that  $CD \perp OA$ .



Let  $CD = x$ .

- (i) Express  $AC$  in terms of  $x$  and  $\theta$ , and by considering  $\triangle OCD$ , show that

2

$$x = \frac{2 \tan \theta}{3 - \tan^2 \theta}.$$

- (ii) If  $x = \frac{\sqrt{3}}{4}$ , find the value of  $\theta$ , and hence, show that the area of  $\triangle OAB = \frac{\sqrt{3}}{4} \text{ m}^2$ .

2

- (b) Many calculators compute reciprocals by using the approximation  $\frac{1}{a} \doteq x_{n+1}$ ,

2

where  $x_{n+1} = x_n(2 - ax_n)$  for  $n = 1, 2, 3, \dots$

That is if  $x_1$  is an initial approximation to  $\frac{1}{a}$ , then  $x_2 = x_1(2 - ax_1)$  is a better approximation.

This formula makes it possible to use multiplications and subtractions, which can be done quickly, to perform divisions that would be slow to obtain directly.

Apply Newton's method to  $f(x) = \frac{1}{x} - a$ , using  $x_1$  as an initial approximation, to show

$$x_2 = x_1(2 - ax_1)$$

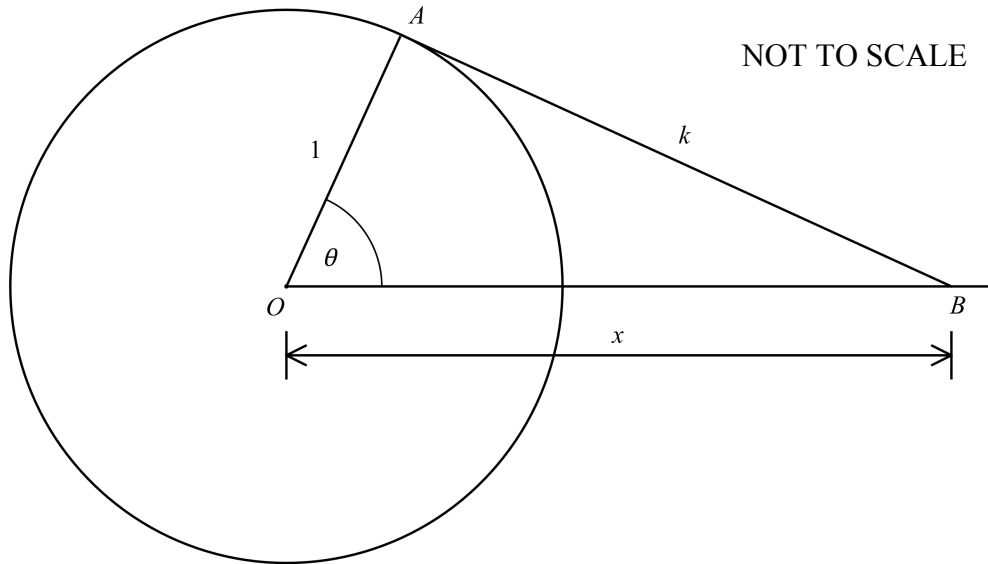
**Question 13 continues on page 13**

Question 13 (continued)

(c) Evaluate  $\lim_{x \rightarrow 0} \frac{\sin 2x}{\tan 3x}$

1

(d) The diagram below shows a circular disc with radius  $OA$ .



The radius of the disc,  $OA$ , is one metre and  $AB$  is a rod of length  $k$  metres ( $k > 1$ ). The end of the rod,  $B$ , is free to slide along a horizontal axis with origin  $O$ . The angle between  $OA$  and  $OB$  is  $\theta$ .

Let  $OB = x$  metres.

(i) Show that  $x = \cos \theta + \sqrt{k^2 - \sin^2 \theta}$ .

3

(ii) Find  $\frac{dx}{d\theta}$  in terms of  $k$  and  $\theta$ .

2

(iii) Given that  $\frac{d\theta}{dt} = 4\pi$  rad/s.

2

Find  $\frac{dx}{dt}$  in terms of  $k$  when  $\theta = \frac{\pi}{6}$ .

(iv) Find  $\theta$ ,  $0 \leq \theta < 2\pi$ , when the velocity of point  $B$  is zero.

1

**End of Question 13**

**Question 14** (15 marks) Use a SEPARATE writing booklet.

- (a) Prove by mathematical induction that  $\sum_{r=1}^n (r+3)2^r = (n+2)2^{n+1} - 4$  **3**  
where  $n$  is a positive integer.

- (b) A particle performs simple harmonic motion on a straight line.  
It has zero speed at the points  $A$  and  $B$  whose distances on the same side from a fixed point  $O$  are  $a$  and  $b$  respectively, where  $b > a$ .

- (i) Find the amplitude of oscillation in terms of  $a$  and  $b$ . **1**

- (ii) The particle has a speed  $V$  when half way between the points  $A$  and  $B$ . **3**  
Show that the period of oscillation is  $\frac{\pi(b-a)}{V}$ .

You may use the following formula:  $v^2 = n^2(c^2 - (x - x_0)^2)$   
(Do **NOT** prove this)

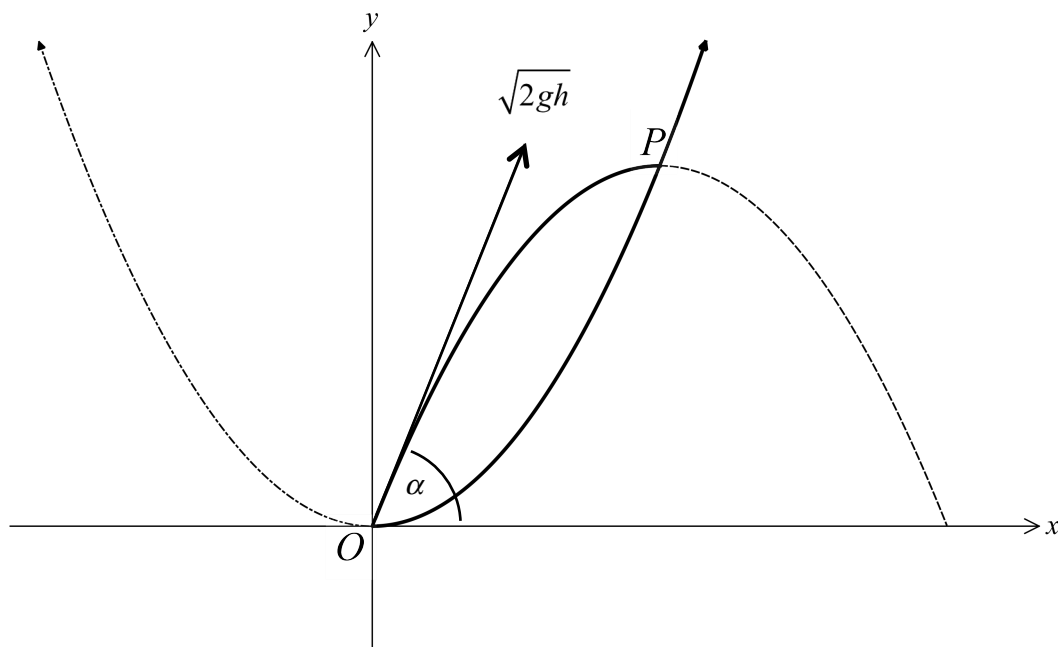
**Question 14 continues on page 15**

Question 14 (continued)

- (c) A vertical section of a valley is in the form of the parabola  $x^2 = 4ay$  where  $a$  is a positive constant.

A gun placed at the origin fires with speed  $\sqrt{2gh}$  at an angle of elevation  $\alpha$  where

$0 < \alpha < \frac{\pi}{2}$  and  $h$  is a positive constant.



The equations of the motion of a projectile fired from the origin with initial velocity  $V \text{ ms}^{-1}$  at angle  $\theta$  to the horizontal are

$$x = Vt \cos \alpha \text{ and } y = Vt \sin \alpha - \frac{1}{2}gt^2 \quad (\text{Do NOT prove this})$$

- (i) If the shell strikes the section of the valley at the point  $P(x, y)$  show that 3

$$x = \frac{4ah}{(a+h)\cot \alpha + a \tan \alpha}$$

- (ii) Let  $f(\theta) = (a+h)\cot \theta + a \tan \theta$  for  $0 < \theta < \frac{\pi}{2}$ . 2

Show that the minimum value of  $f(\theta)$  occurs when  $\tan \theta = \sqrt{\frac{a+h}{a}}$ .

- (iii) Show that the greatest value of  $x$  is given by 3

$$x = 2h\sqrt{\frac{a}{a+h}}$$

**End of paper**



# 2016 SYDNEY BOYS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

## Mathematics Extension I

### Sample Solutions

| Question | Teacher |
|----------|---------|
| Q11      | EC      |
| Q12      | RB      |
| Q13      | AF      |
| Q14      | PB      |

### MC Answers

- |      |      |      |      |       |
|------|------|------|------|-------|
| 1. A | 3. C | 5. B | 7. C | 9. B  |
| 2. C | 4. B | 6. A | 8. A | 10. C |

## Section I

10 marks

Attempt Questions 1-10

Use the multiple-choice answer sheet for Questions 1-10

- 1 Which integral is obtained when the substitution  $u = 1 + 3x$  is applied to  $\int x\sqrt{1+3x} \, dx$ ?

(A)  $\frac{1}{9} \int (u-1)\sqrt{u} \, du$

(B)  $\frac{1}{6} \int (u-1)\sqrt{u} \, du$

(C)  $\frac{1}{3} \int (u-1)\sqrt{u} \, du$

(D)  $\frac{1}{4} \int (u-1)\sqrt{u} \, du$

$$u = 1 + 3x$$

$$\frac{du}{dx} = 3$$

$$du = 3 \, dx$$

$$\frac{du}{3} = dx$$

$$u = 1 + 3x$$

$$\frac{u-1}{3} = x$$

$$\int \left( \frac{u-1}{3} \right) \sqrt{u} \cdot \frac{du}{3}$$

$$\frac{1}{9} \int (u-1)\sqrt{u} \, du$$

- 2 The acceleration of a particle moving along a straight line is given by  $\ddot{x} = -2e^{-x}$ , where  $x$  metres is the displacement from the origin.  
If the velocity of the particle is  $v$  m/s, which of the following is a correct statement about  $v^2$ ?

(A)  $v^2 = 2e^{-x} + C$

(B)  $v^2 = 2e^x + C$

(C)  $v^2 = 4e^{-x} + C$

(D)  $v^2 = 4e^x + C$

$$\frac{d}{dx} \left( \frac{1}{2} v^2 \right) = \ddot{x}$$

$$\int \frac{d}{dx} \left( \frac{1}{2} v^2 \right) dx = \int -2e^{-x} dx$$

$$\frac{1}{2} v^2 = 2e^{-x} + C$$

$$v^2 = 4e^{-x} + C$$

- 3 Find  $\frac{d}{dx} (x \cos^{-1} x - \sqrt{1-x^2})$

(A)  $\frac{-2}{\sqrt{1-x^2}}$

(B)  $\frac{-1}{\sqrt{1-x^2}}$

(C)  $\cos^{-1} x$

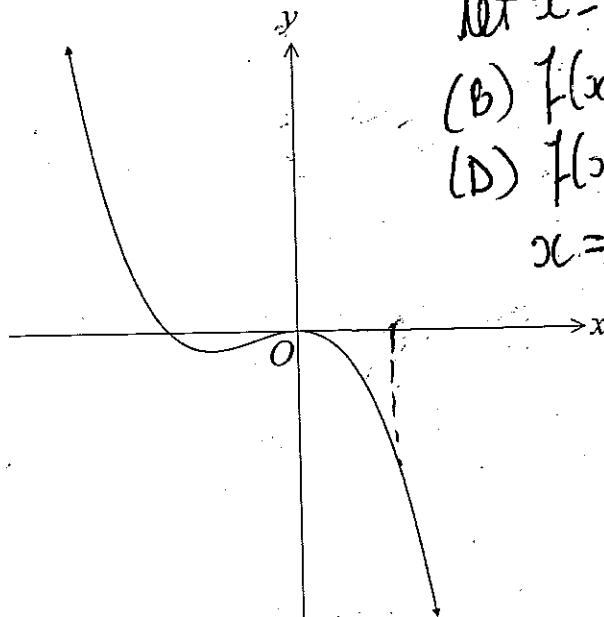
(D)  $\sin^{-1} x$

$$x \times \frac{-1}{\sqrt{1-x^2}} + \cos^{-1} x - \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \times -2x$$

$$\cancel{\frac{-x}{\sqrt{1-x^2}}} + \cos^{-1} x + \cancel{\frac{x}{\sqrt{1-x^2}}}$$

4

What is a possible equation of this function?



let  $x=2$  (say)  
 (b)  $f(x) = -4 \times 3 = -12$   
 (D)  $f(x) = 4 \times 3 = 12$  No!  
 $x=0$  double root  
 $x$  negative single root

(A)  $f(x) = -x(x-1)(x+1)$

(B)  $f(x) = -x^2(x+1)$

(C)  $f(x) = -x^2(x-1)$

(D)  $f(x) = x^2(x+1)$

- 5 If  $f(x) = 1 + \frac{2}{x-3}$ , which of the following give the equations of the horizontal and vertical asymptotes of  $f^{-1}(x)$ ?

- (A) Vertical asymptote is  $x = 1$  and horizontal asymptote is  $y = 2$   
 (B) Vertical asymptote is  $x = 1$  and horizontal asymptote is  $y = 3$   
 (C) Vertical asymptote is  $x = 3$  and horizontal asymptote is  $y = 1$   
 (D) Vertical asymptote is  $x = 3$  and horizontal asymptote is  $y = 2$

the vertical asymptote of  $f(x)$  is  $x = 3$   
 So its inverse, the horizontal asymptote  
 is  $y = 3$ . Hence B.

- 6 The polynomial equation  $x^3 - ax^2 + 8x + (1-a) = 0$  has roots  $\alpha$ ,  $\beta$  and  $\gamma$ .

Given that  $\alpha + \beta + \gamma < 0$  and  $\alpha\beta\gamma(\alpha + \beta + \gamma) = 20$ , what is the value of  $a$ ?

- (A) -4  
(B) 4  
(C) -5  
(D) 5

So

$$* \alpha + \beta + \gamma = -\frac{b}{a} = a$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = 8$$

$$\alpha\beta\gamma = -\frac{d}{a} = a-1$$

$$(a-1)a = 20$$

$$a^2 - a - 20 = 0$$

$$a = 1$$

$$b = -a$$

$$c = 8$$

$$d = 1-a$$

$$(a-5)(a+4) = 0$$

$$a = 5$$

$$a = -4$$

- 7 If  $t = \tan \frac{\theta}{2}$ , which of the following expressions is equivalent to  $4\sin\theta + 3\cos\theta + 5$ ?

~~(A)~~  $\frac{2(t+2)^2}{1-t^2}$

~~(B)~~  $\frac{(t+4)^2}{1-t^2}$

(C)  $\frac{2(t+2)^2}{1+t^2}$

(D)  $\frac{(t+4)^2}{1+t^2}$

$$4 \times \frac{2t}{1+t^2} + 3 \times \frac{(1-t^2)}{1+t^2} + 5$$

$$8t + 3 - 3t^2 + 5 + 5t^2$$

$$\frac{2t^2 + 8t + 8}{1+t^2} = \frac{2(t+2)^2}{1+t^2}$$

- 8 Which of the following is a correct expression for  $\tan\left(x + \frac{\pi}{4}\right)$ ?

(A)  $\frac{\cos x + \sin x}{\cos x - \sin x}$

~~(B)~~  $\frac{\cos x + 2\sin x}{\cos x - \sin x}$

~~(C)~~  $\frac{\cos x + \sin x}{\cos^2 x - \sin x}$

~~(D)~~  $\frac{\cos x - \sin x}{\cos x - \sin x}$

$$\frac{\tan x + 1}{1 - \tan x} = \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}} = \frac{\sin x + \cos x}{\cos x - \sin x}$$

- 9 The curve  $y = 2x^{\frac{1}{3}}$  is reflected in the line  $y = x$ .  
What is the equation of the reflected curve?

- (A)  $y = \frac{x^3}{16}$   
 (B)  $y = \frac{x^3}{8}$   
 (C)  $y = \frac{x^3}{4}$   
 (D)  $y = \frac{x^3}{2}$

$$y = 2x^{\frac{1}{3}}$$

swap  $x$  and  $y$

$$x = 2y^{\frac{1}{3}}$$

$$\frac{x}{2} = y^{\frac{1}{3}}$$

$$\frac{x^3}{8} = y$$

cube

- 10 A particle is moving in simple harmonic motion with displacement  $x$ .  
Its velocity is given by  $v^2 = 9(36 - x^2)$ .  
What is the amplitude,  $A$ , of the motion and the maximum speed of the particle?

- ~~(A)~~  $A = 3$  and maximum speed  $v = 6$   
~~(B)~~  $A = 3$  and maximum speed  $v = 18$   
 (C)  $A = 6$  and maximum speed  $v = 18$   
 (D)  $A = 6$  and maximum speed  $v = 6$

### End of Section I

$$v^2 = 9(36 - x^2)$$

$$= n^2(a^2 - x^2)$$

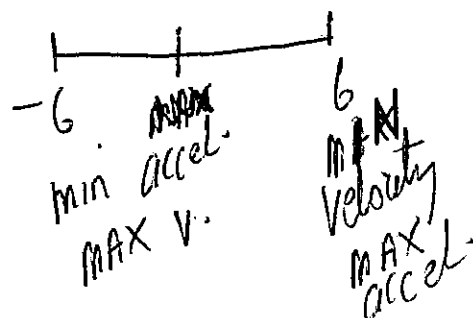
$$n = 3$$

$$a = 6$$

$$v^2 = 9 \times 36$$

$$v = 3 \times 6$$

$$= 18$$



## Question (11)

(a)  $y = \tan^{-1}(2x^{-1})$

$$\frac{dy}{dx} = \frac{-2x^{-2}}{1 + \frac{4}{x^2}} \quad \checkmark$$

$$= \frac{-2}{x^2 + 4} \quad \checkmark$$

Comment:  $x^2 + 4$ .

- Well done
- Straight forward question if you use  $\frac{d}{dx} \tan^{-1}(f(x)) = \frac{f'(x)}{1 + [f(x)]^2}$
- or equivalent merit

(b)  $\int_0^{3/2} \frac{dx}{\sqrt{9 - 4x^2}}$

$$= \frac{1}{2} \int_0^{3/2} \frac{dx}{\sqrt{(\frac{3}{2})^2 - x^2}} \quad \checkmark$$

$$= \frac{1}{2} \left[ \sin^{-1} \frac{2x}{3} \right]_0^{3/2} \quad \checkmark$$

$$= \frac{\pi}{4} \quad \checkmark$$

Comment:

- Deduct (1) if common factor 4 is not taken out.
- Allow CFE and ignore subsequent errors. 1 mark (max) for not taken out.
- Well done for most candidates if

(c)  $\tan 45 = \frac{p-q}{1+pq}$   $\checkmark$   $\begin{matrix} p, q \\ \swarrow \searrow \\ \text{Explained} \end{matrix}$

$$\therefore 1 = \frac{p-q}{1+pq} \rightarrow \left| \frac{a}{b} \right| = \frac{|a|}{|b|} \quad \checkmark$$

Where gradient of  $\tan$  at P is  $p$   
 " " " " at Q is  $q$ .

Comment:

- Acknowledge (show/prove) gradient of  $\tan$  at P and Q is  $p, q$  AND correctly use the (1) Angle between two lines formula.
  - $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$
- Have to show what is  $p, q$ !

$$cd) y = 2\cos^{-1} 3x.$$

$$-1 \leq 3x \leq 1$$

$$-\frac{1}{3} \leq x \leq \frac{1}{3} \quad \checkmark \quad \left(\frac{1}{2}\right)$$

$$\text{Comment } 0 \leq y \leq 2\pi \quad \checkmark \quad \left(\frac{1}{2}\right)$$

|                                                 |
|-------------------------------------------------|
| Well done                                       |
| (1 Mark) From correctly drawn graph             |
| or explicitly specified domain & range as above |

$$(e) x^3 - 3x^2 + 4x + 2 = 0$$

$$\checkmark \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}, \begin{cases} \sum \alpha_i = 3 \\ \sum_{i \neq j} \alpha_i \alpha_j = 4 \end{cases} \checkmark$$

$$= \frac{4}{-2}$$

and correctly evaluated.

Comment 2.

Almost all got this part correct

$$cf) 4 \sin \theta + 3 \cos \theta = -5$$

$$4 \sin \theta + 3 \cos \theta = r \sin(\theta + \alpha) \quad \text{expanding}$$

$$r = \sqrt{4^2 + 3^2} = 5$$

and  $\alpha = 3/4$  in the 1st quad.

$$\alpha = 36^\circ 52'$$

$$5 \sin(\theta + 36^\circ 52') = -5$$

$$\sin(\theta + 36^\circ 52') = -1$$

$$\theta + 36^\circ 52' = 270^\circ, \dots \quad (\text{out of range})$$

$$\text{Comment } \hat{=} 233^\circ 8'$$

|                                                                    |
|--------------------------------------------------------------------|
| Expanding $r \sin(\theta + \alpha)$                                |
| $= 4 \sin \theta + 3 \cos \theta \quad \checkmark$                 |
| Equate $\sin \theta$ & $\cos \theta$                               |
| or 't' substitution $\checkmark$                                   |
| $233^\circ 8' \quad (3^{\text{rd}} \& 4^{\text{th}} \text{ quad})$ |
| correct solution band $\rightarrow$ mark                           |

(g)  $y = \frac{x}{(x+3)(x+4)}$

(i)  $\frac{dy}{dx} = \frac{(x+3)(x+4) - x(2x+7)}{(x+3)^2(x+4)^2}$

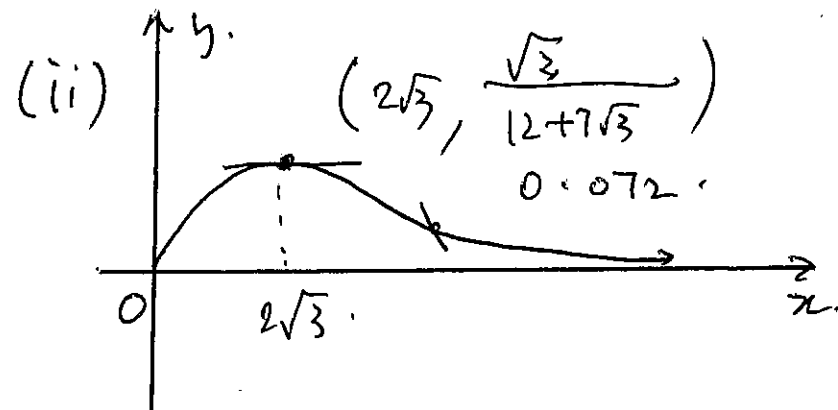
$$= \frac{-x^2 + 7x - 7x + 12}{(x+3)^2(x+4)^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{12 - x^2}{(x+3)^2(x+4)^2} = 0$$

i.e.  $12 - x^2 = 0$

Comment:  $x = \pm\sqrt{12} = \pm 2\sqrt{3}$ .

Well done this part  
Most know how to use  
quotient rule and obtained  
two solutions  $12 - x^2 = 0$ .



- passes through  $(0,0)$  ✓  $\frac{1}{2}$
- max  $(2\sqrt{3}, \frac{\sqrt{3}}{12+7\sqrt{3}})$  ✓  $\frac{1}{2}$   
and test for max.
- Horizontal asymptote  $y=0$   
ad arrow (to infinity...).
- Sign diagram.  
approx markings for pt of inflex ✓  $\frac{1}{2}$

12)

5 unit Trial 11/10  
20/16

$$(a) (i) N = P(1 - e^{-kt})$$

$$N = P - Pe^{-kt}$$

$$\begin{aligned}\frac{dN}{dt} &= -Pe^{-kt} \times -k \\ &= Pke^{-kt} \\ &= k(P - N)\end{aligned}$$

now  $N = P - Pe^{-kt}$   
 $Pe^{-kt} = P - N$

(1)

Generally well answered  
 but some students got  
 very lost in a quite  
 simple proof.

(ii)  $P =$  initial amount

$N =$  amount that has evaporated

Many students assumed:

$$Pe^{-kt} = P - N$$

without stating it.

This made the proof  
 very inconsistent.

$$\frac{1}{4}P = P(1 - e^{-kt})$$

$$\frac{1}{4} = 1 - e^{-kt}$$

$$e^{-kt} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\ln e^{-kt} = \ln\left(\frac{3}{4}\right)$$

$$-kt = \ln 3 - \ln 4$$

$$-kt = \ln 3 - 2\ln 2$$

$$-t = \frac{\ln 3 - 2\ln 2}{k}$$

$$t = -\frac{(\ln 3 - 2\ln 2)}{k}$$

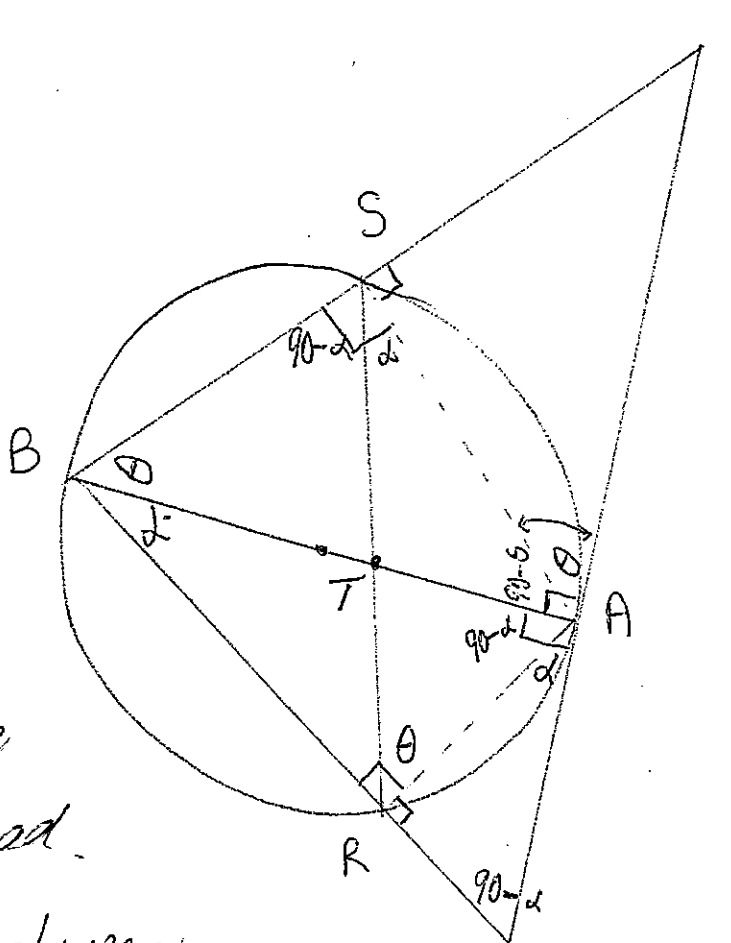
Part (ii) well  
 answered by most!

(2)

(b)

Generally well done. A few people stated that SR is also a diameter. Not so!

A few methods were available to show PQRS is a cyclic quad.



$$\hat{BAP} = 90^\circ \text{ (line between diameter and tangent line)}$$

$$\hat{BRA} = 90^\circ \text{ (angle in a semi circle)}$$

$$\hat{SAP} = \hat{ABS} = \theta \text{ (alternate segment)}$$

$$\text{let } \hat{RBA} = \alpha, \hat{RSA} = \alpha \text{ (angles standing on same arc)}$$

$$\hat{BSA} = 90^\circ \text{ (opposite angle cyclic quad. BSAR, supplementary angles)}$$

$$\hat{ABS} = \hat{ART} = \theta \text{ (angles standing on same arc)}$$

Why is PQRS a cyclic quad?

$$\hat{PQR} = 90 - \alpha, \hat{RSP} = 90 + \alpha$$

$$\text{so } \hat{PQR} + \hat{RSP} = 180^\circ \text{ (opposite angles in cyclic quad add to } 180^\circ)$$

(3)

(C) 11 people.  
2 circular tables

Question  
badly answered  
by a large  
number of  
students.

6 5

$$\frac{11!}{5! \times 6!} \times 5! \times 4!$$

11 C  
Pick the  
people.

6 x 5!  
arrange  
around  
table

x 1 x 4!  
5 people  
left.

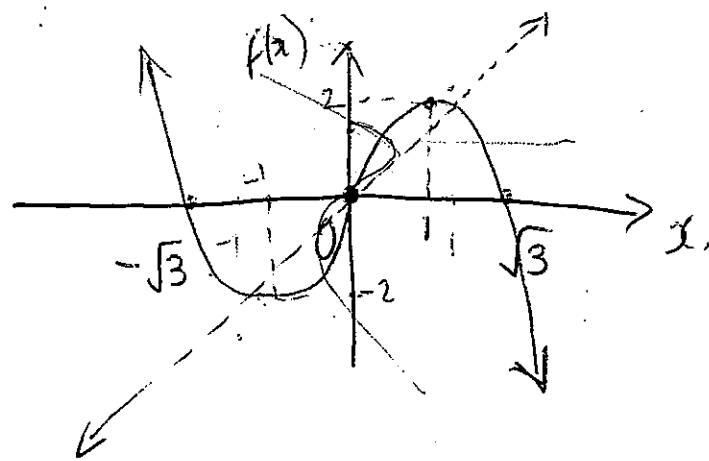
arrange  
around table

$$462 \times 120 \times 24 = 1,330,560$$

(2)

$$\begin{aligned} (d) f(x) &= 3x - x^3 \\ &= x(3 - x^2) \\ &= x(\sqrt{3} - x)(\sqrt{3} + x) \end{aligned}$$

$$\begin{aligned} x &= 3x - x^3 \\ x^3 - 2x &= 0 \\ x(x^2 - 2) &= 0 \\ x(x - \sqrt{2})(x + \sqrt{2}) &= 0 \end{aligned}$$



$$\begin{aligned} \text{let } y &= 3x - x^3 \\ x &= 3y - y^3 \\ x &= y(3 - y^2) \\ &= y(\sqrt{3} - y)(\sqrt{3} + y) \end{aligned}$$

(i)  $-1 \leq x \leq 1$  (2) Badly answered.

(ii)  $-2 \leq x \leq 2$  (1)

$$\begin{aligned} y &= 3 - 3x^2 \\ &= 3(1 - x^2) \\ &= 3(1 - x)(1 + x) \\ x &= 1, -1 \\ y &= 2, -2 \\ y'' &= -6x \end{aligned}$$

Domain of  $f^{-1}(x)$  is the  
range of  $f(x)$  in its specified  
domain.

$$(b) \quad \tan 45^\circ = \frac{OT}{OA}$$

$$OA = OT$$

$$\tan 30^\circ = \frac{OT}{OB}$$

$$\frac{OB}{\sqrt{3}} = OT \Rightarrow OB = \sqrt{3} OT$$

$$\triangle AOB, \quad (AO)^2 + (BO)^2 = 10000$$

$$(OT)^2 + 3(OT)^2 = 10000$$

$$4(OT)^2 = 10000$$

$$(OT)^2 = 2500$$

$$OT = 50 \text{ m}$$

diff is 50 m.

Well answered  
by nearly all  
students.

A bookwork

2.

type question  
+ the diagram

was included  
 $\Rightarrow$  helps a lot!

$$(f) \quad \ddot{x} = -e^{-2x}$$

$$\int \frac{d}{dx} \left( \frac{1}{2} \dot{v}^2 \right) dx = \int -e^{-2x} dx = \boxed{\frac{1}{2}}$$

$$\frac{1}{2} \dot{v}^2 = \frac{1}{2} e^{-2x} + C$$

data  
 $t=0$   
 $x=0$   
 $\dot{v}=1$

$$\frac{1}{2} = \frac{1}{2} e^0 + C$$

$$C = 0 \quad \boxed{\frac{1}{2}}$$

$$\frac{1}{2} \dot{v}^2 = \frac{1}{2} e^{-2x}$$

$$\dot{v}^2 = e^{-2x}$$

$$\dot{v} = (e^{-2x})^{\frac{1}{2}}$$

$$\dot{v} = e^{-x}$$

$$\frac{dx}{dt} = e^{-x} = \frac{1}{e^x}$$

$$\frac{dt}{dx} = e^x$$

$$t = e^x + C_1$$

data  
 $t=0$   
 $x=0$

$$0 = 1 + C_1$$

$$C_1 = -1 \quad \boxed{\frac{1}{2}}$$

$$t = e^x - 1$$

generally well  
 attempted -  
 Using  $\frac{d}{dx} \left( \frac{1}{2} \dot{v}^2 \right)$  was  
 the key -

so  
 $e^x = t+1$   
 $\ln e^x = \ln(t+1)$   
 $x = \ln(t+1) \quad \boxed{\frac{1}{2}}$

total  
 marks 2

$$13) a) i) \tan \theta = \frac{x}{AC}$$

$$AC = \frac{x}{\tan \theta}$$

$$\tan 2\theta = \frac{x}{OC}$$

$$OC = \frac{x}{\tan 2\theta}$$

$$AC + OC = 1$$

$$\frac{x}{\tan \theta} + \frac{x}{\tan 2\theta} = 1$$

$$\frac{x}{\tan \theta} + \frac{x}{\left(\frac{2\tan \theta}{1-\tan^2 \theta}\right)} = 1$$

$$\frac{2x + x - \tan^2 \theta x}{2\tan \theta} = 1$$

$$x(3 - \tan^2 \theta) = 2\tan \theta$$

$$x = \frac{2\tan \theta}{3 - \tan^2 \theta}$$

$$ii) \frac{\sqrt{3}}{4} = \frac{2\tan \theta}{3 - \tan^2 \theta}$$

$$3\sqrt{3} - \sqrt{3}\tan^2 \theta = 8\tan \theta$$

$$\sqrt{3}\tan^2 \theta + 8\tan \theta - 3\sqrt{3} = 0$$

$$\tan \theta = \frac{-(8) \pm \sqrt{(8)^2 - 4(\sqrt{3})(-3\sqrt{3})}}{2(\sqrt{3})}$$

$$= \frac{-8 \pm 10}{2\sqrt{3}}$$

$$= \frac{-4 \pm 5}{\sqrt{3}}$$

$$\tan \theta = \frac{1}{\sqrt{3}} \quad \text{or} \quad -\frac{1}{\sqrt{3}}$$

$$\theta = 30^\circ \quad (\text{since } \theta \text{ is acute})$$

$$2\theta = 60^\circ$$

$$\begin{aligned} A &= \frac{1}{2}(1)(1)\sin 60^\circ \\ &= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{4} \text{ square metres.} \end{aligned}$$

COMMENT:

Part (i) was done reasonably well.

Many students failed to recognise that there was a quadratic in  $\tan \theta$  which could be solved to find  $\theta$ .

$$\begin{aligned} \text{b) } f(x) &= \frac{1}{x} - a \\ f(x) &= x^{-1} - a \\ f'(x) &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 - \frac{\frac{1}{x_1} - a}{-\frac{1}{x_1^2}} \times \frac{-x_1^2}{-x_1^2}$$

$$x_2 = x_1 + x_1 - ax_1^2$$

$$x_2 = 2x_1 - ax_1^2$$

$$x_2 = x_1(2 - ax_1)$$

COMMENT:

A different style of question on first impressions. However, it is just a simple application of Newton's method.

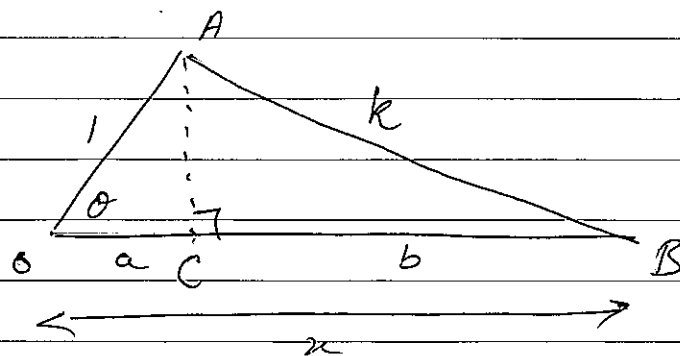
$$c) \lim_{x \rightarrow 0} \frac{\sin 2x}{\tan 3x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times \frac{3x}{\tan 3x} \times \frac{2}{3}$$

$$= 1 \times 1 \times \frac{2}{3}$$

$$= \frac{2}{3}$$

d) i)



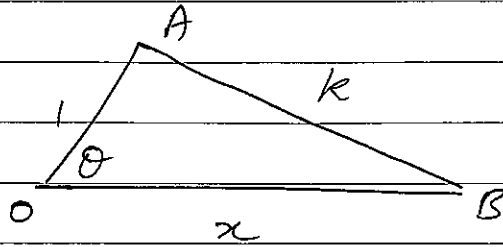
$$\cos \theta = \frac{a}{1}$$

$$\sin \theta = \frac{AC}{1}$$

$$x = a + b$$

$$x = \cos \theta + \sqrt{k^2 - \sin^2 \theta}$$

OR



$$\cos \theta = \frac{1^2 + x^2 - k^2}{2(1)(x)}$$

$$2x \cos \theta = 1 + x^2 - k^2$$

$$x^2 - 2 \cos \theta x + 1 - k^2 = 0$$

$$x^2 - 2 \cos \theta x + \cos^2 \theta + \sin^2 \theta - k^2 = 0$$

$$(x - \cos \theta)^2 = k^2 - \sin^2 \theta$$

$$x - \cos \theta = \pm \sqrt{k^2 - \sin^2 \theta}$$

$$x = \cos \theta \pm \sqrt{k^2 - \sin^2 \theta}$$

$$x = \cos \theta + \sqrt{k^2 - \sin^2 \theta} \quad \text{metres.}$$

since  $x$  is a distance

COMMENT:

Students that assumed AB was a tangent could not get the result.

$$\text{ii) } x = \cos \theta + \sqrt{k^2 - \sin^2 \theta}$$

$$x = \cos \theta + (k^2 - \sin^2 \theta)^{\frac{1}{2}}$$

$$\frac{dx}{d\theta} = -\sin \theta + \frac{1}{2}(k^2 - \sin^2 \theta)^{-\frac{1}{2}}(-2 \sin \theta \cos \theta)$$

$$\frac{dx}{d\theta} = -\sin \theta \left( 1 + \frac{\cos \theta}{\sqrt{k^2 - \sin^2 \theta}} \right) \quad \text{m/rad.}$$

$$\text{iii)} \quad \frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$

$$\frac{dx}{dt} = -\sin\theta \left( 1 + \frac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}} \right) \times 4\pi$$

$$\text{when } \theta = \frac{\pi}{6}$$

$$\frac{dx}{dt} = -\sin\frac{\pi}{6} \left( 1 + \frac{\cos\frac{\pi}{6}}{\sqrt{k^2 - \left(\sin\frac{\pi}{6}\right)^2}} \right) 4\pi$$

$$= -\left(\frac{1}{2}\right) \left( 1 + \frac{\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{k^2 - \left(\frac{1}{2}\right)^2}} \right) 4\pi$$

$$= -2\pi \left( 1 + \frac{\sqrt{3}}{2\sqrt{k^2 - \frac{1}{4}}} \right)$$

$$= -2\pi \left( 1 + \frac{\sqrt{3}}{\sqrt{4k^2 - 1}} \right) \text{ m/s}$$

COMMENT:

This answer could be written a number of different ways.

iv) considering the scenario, the point B will change direction when  $\theta = 0, \pi$ .

From the equation  $\frac{dx}{dt} = 0$  when  $\frac{dx}{d\theta} = 0$

since  $\frac{d\theta}{dt}$  is a constant.

$$-\sin\theta \left( 1 + \frac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}} \right) = 0$$

$$\sin \theta = 0 \quad \text{or} \quad 1 + \frac{\cos \theta}{\sqrt{k^2 - \sin^2 \theta}} = 0$$

$$\underline{\theta = 0, \pi}$$

$$\sqrt{k^2 - \sin^2 \theta} + \cos \theta = 0$$

$$\sqrt{k^2 - \sin^2 \theta} = -\cos \theta$$

$$k^2 - \sin^2 \theta = \cos^2 \theta$$

$$k^2 = \sin^2 \theta + \cos^2 \theta$$

$$k^2 = 1$$

since  $k > 1$ .

no solution.

COMMENT:

Students should not be using the equation which has  $\theta = \frac{\pi}{6}$  substituted.

This should have been an easy mark.

QUESTION 14. (X1)

(a) Claim To prove  $\sum_{r=1}^n (r+3)2^r = (n+2)2^{n+1} - 4.$   
for  $n \in \mathbb{Z}^+$

Step I when  $n=1$

$$\text{LHS} = (1+3)2$$

$$= 8$$

$$\text{RHS} = 3 \times 4 - 4$$

$$= 8$$

$\therefore$  true when  $n=1$ .

Step II Assume  $\sum_{r=1}^k (r+3)2^r = (k+2)2^{k+1} - 4.$

Step III Assuming Step II is true

Prove true for  $n = k+1$ .

[3]

ie.  $\sum_{r=1}^{k+1} (r+3)2^r = (k+3)2^{k+2} - 4.$

now LHS =  $\sum_{r=1}^k (r+3)2^r + (k+4)2^{k+1}$

$$= (k+2)2^{k+1} - 4 + (k+4)2^{k+1}$$

$$= (2k+6)2^{k+1} - 4$$

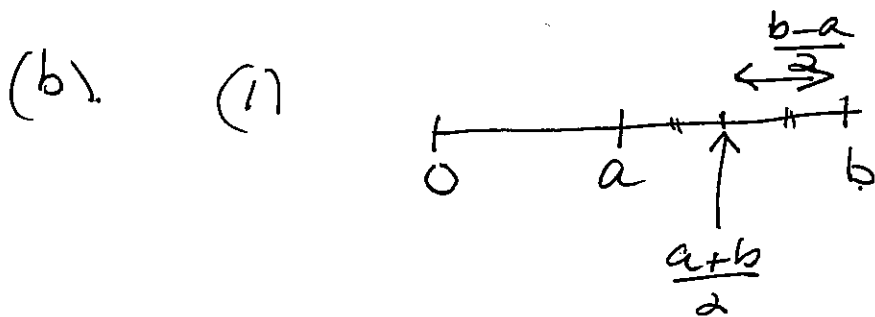
$$= 2(k+3)2^{k+1} - 4$$

$$= (k+3)2^{k+2} - 4$$

$$= \text{RHS}$$

Step IV we conclude that  
by the Principle of Mathematical Induction  
the statement is true for  $n \in \mathbb{Z}^+$

COMMENT: This was a straight forward question on Induction and was well done. Most scored full marks.



[1]

The amplitude is  $\frac{b-a}{2}$ .

COMMENT: The common error was  $\frac{a+b}{2}$  which is the centre of motion.

(ii.) Using  $v^2 = n^2 \left[ \left( \frac{b-a}{2} \right)^2 - \left( x - \left( \frac{a+b}{2} \right) \right)^2 \right]$

we have  $v_{\text{MAX}}^2 = n^2 \left( \frac{b-a}{2} \right)^2$

ie.  $v_{\text{max}} = n \frac{(b-a)}{2}$

$\therefore V = n \frac{(b-a)}{2}$

[3]

$n = \frac{2V}{b-a}$

Hence  $T = \frac{2\pi}{n}$   
 $= \frac{2\pi}{2V/(b-a)}$   
 $= \frac{\pi(b-a)}{V}$

COMMENT: not well done. As is after the case where the answer is provided there was a tendency to continue the answer using a circular argument.

(C) (1) given  $x = Vt \cos \alpha$  &  $y = Vt \sin \alpha - \frac{1}{2}gt^2$ .

$$t = \frac{x}{V \cos \alpha}.$$

$$\therefore y = V \frac{x}{V \cos \alpha} \sin \alpha - \frac{1}{2} \frac{g}{V^2} \frac{x^2}{\cos^2 \alpha}.$$

$$\text{ie. } y = x \tan \alpha - \frac{1}{2} \frac{g x^2}{V^2} \sec^2 \alpha. \quad (A)$$

$$\& V = \sqrt{2gh}. \quad (\text{given})$$

$$\therefore V^2 = 2gh.$$

$\therefore$  (A) becomes

$$y = x \tan \alpha - \frac{1}{2} \frac{g x^2}{2gh} \sec^2 \alpha$$

OR.

$$y = x \tan \alpha - \frac{x^2}{4h} (1 + \tan^2 \alpha) \quad (B)$$

To find P

$$\text{we solve (B) and } y = \frac{x^2}{4a} \quad (C)$$

$$\text{ie. } \frac{x^2}{4a} = x \tan \alpha - \frac{x^2}{4h} (1 + \tan^2 \alpha)$$

$$x^2 \left[ \frac{1 + \tan^2 \alpha}{4h} + \frac{1}{4a} \right] - x \tan \alpha = 0$$

$$x \left[ \left( \frac{1 + \tan^2 \alpha}{4h} + \frac{1}{4a} \right) x - \tan \alpha \right] = 0$$

$$\therefore x = 0 \quad \text{OR} \quad x = \frac{\tan \alpha}{\frac{1 + \tan^2 \alpha}{4h} + \frac{1}{4a}}$$

↑  
x ≠ 0.

$$= \frac{4ah \tan \alpha}{a(1 + \tan^2 \alpha) + h}$$

$$= \frac{4ah \tan \alpha}{a + h + a \tan^2 \alpha}$$

$$= \frac{4ah}{\frac{a+h}{\tan \alpha} + a \tan \alpha}$$

$$= \frac{4ah}{(a+h) \cot \alpha + a \tan \alpha}$$

COMMENT most realised to solve

(B) and (C). Unfortunately not many were able to do so successfully.

[3]

(11). Given  $f(\theta) = (a+h) \cot \theta + a \tan \theta$

$$2a < \theta < \frac{\pi}{2}$$

$$f'(\theta) = (a+h) \times -\operatorname{cosec}^2 \theta + a \sec^2 \theta.$$

For st. point.

$$f'(\theta) = 0$$

$$\therefore a \sec^2 \theta = (a+h) \operatorname{cosec}^2 \theta$$

[2]

$$\frac{a}{\cos^2 \theta} = \frac{(a+h)}{\sin^2 \theta}$$

$$\therefore \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{a+h}{a}$$

$$\tan^2 \theta = \frac{a+h}{a}$$

$$\tan \theta = \pm \sqrt{\frac{a+h}{a}}$$

$$\therefore \tan \theta = \sqrt{\frac{a+h}{a}} \left( \text{as } \tan \theta \neq \text{negative value} \right. \\ \left. \text{since } 0 < \theta < \frac{\pi}{2} \right)$$

Clearly this st. point is a minimum

since  $f(\theta) \rightarrow \infty$  as  $\theta \rightarrow 0$  [ $\cot \theta \rightarrow \infty$ ]

&  $f(\theta) \rightarrow \infty$  as  $\theta \rightarrow \frac{\pi}{2}$  [ $\tan \theta \rightarrow \infty$ ]

COMMENT: Some students failed to justify the positive value for  $\tan \theta$  and lost  $\frac{1}{2}$  mark.

It was possible to show that  $f''(\theta) > 0$

$\therefore$  Min.

(iii) Given  $x = \frac{4ah}{(a+h)\cot\alpha + a\tan\alpha}$  (D-)

the max. value will occur.

when  $(a+h)\cot\alpha + a\tan\alpha$  is least, i.e. when  $\tan\alpha = \sqrt{\frac{a+h}{a}}$ .

∴ in D

$$x_{\text{MAX}} = \frac{4ah}{(a+h)\sqrt{\frac{a}{a+h}} + a\sqrt{\frac{a+h}{a}}}$$

[3]

$$= \frac{4ah}{\sqrt{a(a+h)} + \sqrt{a(a+h)}}$$

$$= \frac{4ah}{2\sqrt{a(a+h)}}$$

$$= 2h\sqrt{\frac{a}{a+h}}$$

as required

COMMENT

~~most~~ students recognized the connection with parts (i) & (ii).

The question proved easy for the majority of ~~most~~ students