SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class: Circle

A B 1 2

P L U S

2023

YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- Unless otherwise stated, all answers should be left in simplified exact form
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 70

Section I – 10 marks (pages 2 – 5)

Attempt Questions 1 – 10
Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 15)

Attempt all Questions in Section II
Allow about 1 hours and 45 minutes for this section

Examiner:
External Examiner

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section.
Use the multiple-choice answer sheet for Questions 1–10.

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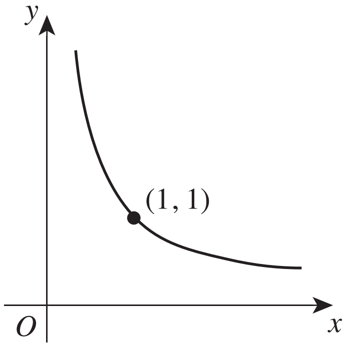
1 Which of the following shows the graph of $x = \sqrt{t-2}$, $y = \dfrac{1}{2-t}$?

A.



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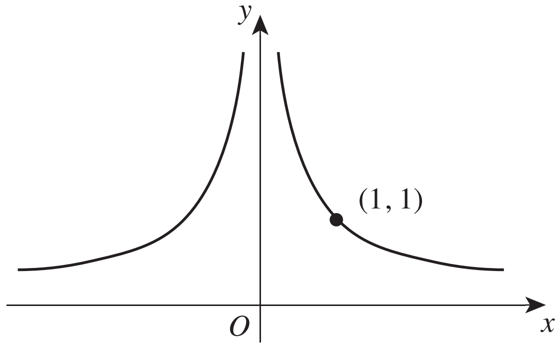

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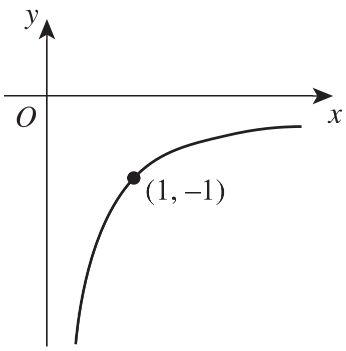

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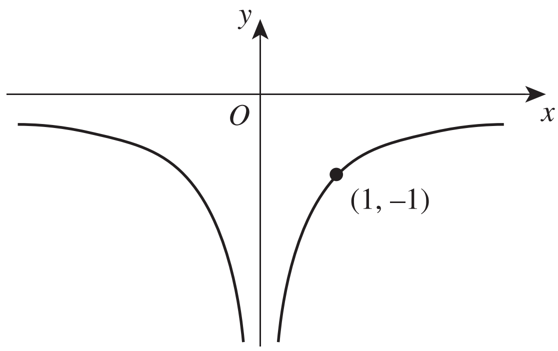


2 The population, N, of wallabies at time t years is given by $N(t) = 155 + Ae^{kt}$, where A and k are positive constants.

Which of the following is the correct differential equation?

A. $\dfrac{dN}{dt} = -k(N - 155)$

B. $\dfrac{dN}{dt} = -k(N + 155)$

C. $\dfrac{dN}{dt} = k(N - 155)$

D. $\dfrac{dN}{dt} = k(N + 155)$

3 A particular monic cubic has three distinct integer roots, each with magnitude less than or equal to 5, and an odd constant term.

How many such cubics are there?

A. 20

B. 40

C. 165

D. 330

4 Which of the following integrals is obtained when the substitution $u = (\ln x)^2$ is applied to

$$\int_e^{e^2} \frac{(\ln x)^3}{x}\, dx \ ?$$

A. $\frac{1}{2}\int_1^4 u\, du$

B. $2\int_1^4 u\, du$

C. $\frac{1}{2}\int_1^4 u^{\frac{3}{2}}\, du$

D. $2\int_1^4 u^{\frac{3}{2}}\, du$

5 Consider $2x^2 + 7x - 15 = 0$

If r and s are the two roots of the equation above and $r > s$, which of the following is the value of $r - s$?

A. $\frac{15}{2}$

B. $\frac{13}{2}$

C. $\frac{11}{2}$

D. $\frac{3}{2}$

6 A one-to one-function $g(x)$ has domain $x \geq 0$ and range $y \geq 0$.
The function $g(x)$ passes through the origin and its inverse is $g^{-1}(x)$.

Given that

$$\int_0^a g^{-1}(x) - x \, dx = \frac{3}{2}a^2$$

what is the value of $\int_0^{g^{-1}(a)} g(x) \, dx$?

A. $ag^{-1}(a) - 2a^2$

B. $ag^{-1}(a) + 2a^2$

C. $ag^{-1}(a) - \frac{3}{2}a^2$

D. $ag^{-1}(a) + \frac{3}{2}a^2$

7 If $\arcsin(1 - x) - 2\arcsin x = \frac{\pi}{2}$, then what is the value of x?

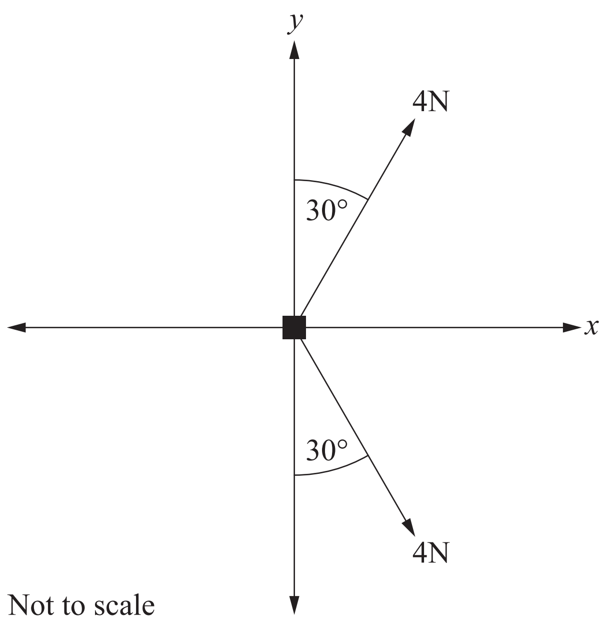
A. $-\frac{1}{2}$

B. 0

C. $\frac{1}{2}$

D. 1

8 Two forces act concurrently on a 2 kg object placed at the origin.



What is the magnitude of the net force acting on the object?

A. 4 N

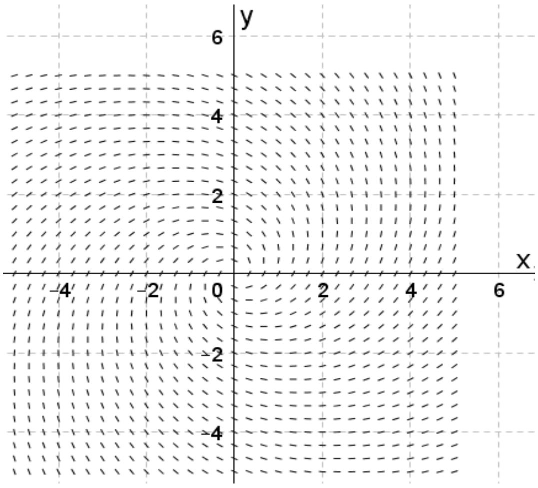
B. $2\sqrt{3}$ N

C. 8 N

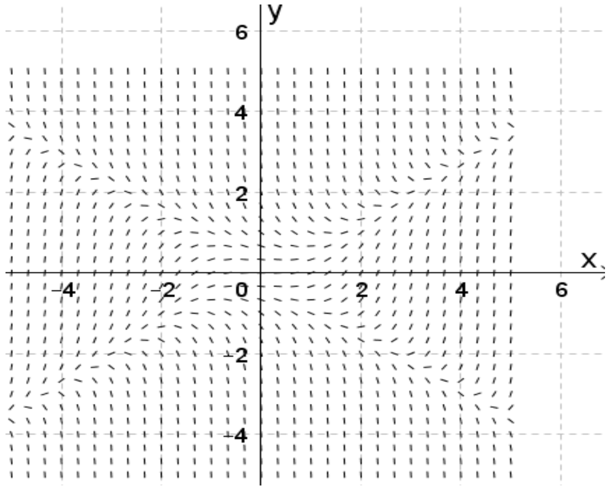
D. $4\sqrt{3}$ N

9 Which of the following direction fields best represents the differential equation $\dfrac{dy}{dx} = \dfrac{y - 2x}{x - 4}$?

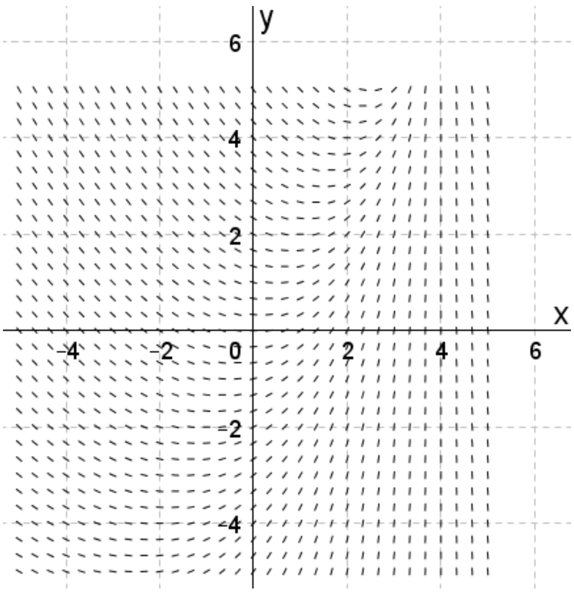
A.



B.



C.



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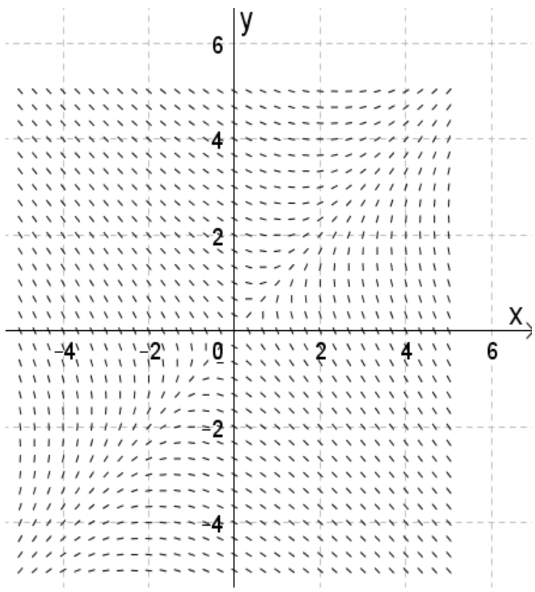


10 If $f(x) = \sin x + \cos x$ and $g(x) = x^2 - 1$, then $g(f(x))$ is a one-to-one function in which of the following intervals?

A. $\left[0, \dfrac{\pi}{2}\right]$

B. $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

C. $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

D. $[0, \pi]$

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

In Questions 11-14, your responses should include ALL relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Find $\displaystyle\int \frac{1}{9+25x^2}\,dx$. **2**

(b) (i) Solve $\dfrac{3x+2}{1-x} < -x+2$. **2**

(ii) Hence, or otherwise, solve $\dfrac{3e^x+2}{1-e^x} < -e^x+2$. **1**

(c) (i) Write down a monic quadratic function that has zeros α and β. **1**

(ii) The polynomial $A(x) = x^4 + px^2 + qx + 36$ has a double zero at $x = 2$, where p and q are real constants. **2**

Factorise $A(x)$, expressing the result as the product of two quadratic functions.

(d) A school principal calls a meeting with two student representatives from each year group from Year 7 to Year 12. **2**
The 12 students and the principal sit at a round table.

In how many ways can the 13 participants sit around the table if at least one pair of students from a year group sits apart?

(e) Show that **2**

$$\frac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A} = 2 - \sin 2A,$$

if $\tan A \neq -1$.

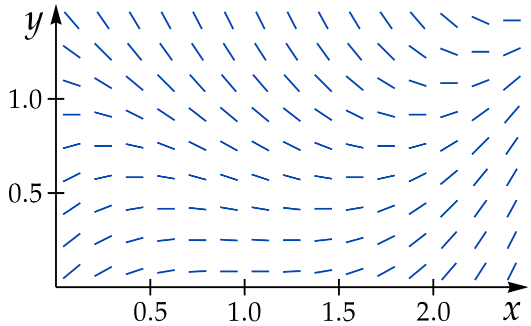
Question 11 continues on page 7

(f) Evaluate $\int_0^{\frac{\pi}{7}} \sin 3x \cos 4x \, dx$ **2**

Leave your answer in exact form.

(g) The diagram below shows part of the slope field for the differential equation **1**

$$\frac{dy}{dx} = (x-1)^2 - y^2.$$



On the Answer sheet provided, sketch the solution curve for $\frac{dy}{dx} = (x-1)^2 - y^2$ where $y(0) = 1$.

Put your Answer Sheet for this question inside your Answer Booklet for Question 11.

End of Question 11

Question 12 (16 marks) Use a SEPARATE writing booklet

(a) Consider the equation $3\sin\theta + k\cos(\theta + 30°) = 7$, where $k \in \mathbb{R}$. **3**

Find the values of k for which this equation has real solutions.

You may assume that $a\sin x + b\cos x = \sqrt{a^2 + b^2}\sin(x + \varepsilon)$, where $\varepsilon \in \mathbb{R}$.

(b) With respect to the origin O, the position vectors of the points A, B, and C are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively.

It is given that $\mathbf{a}$ is a unit vector, $|\mathbf{b}| = 2$, and $\angle AOB = 60°$.

(i) Give a geometrical interpretation of $|\mathbf{a} \cdot \mathbf{c}|$. **1**

(ii) By considering $(2\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} - \mathbf{b})$, or otherwise, find $|2\mathbf{a} - \mathbf{b}|$. **2**

Point C lies on AB such that $AC : CB = 1 : 2$ i.e $\mathbf{c} = \dfrac{\mathbf{b} + 2\mathbf{a}}{3}$.

(iii) By considering the cosine of angles AOC and COB, determine if the line segment OC bisects the angle AOB. **3**

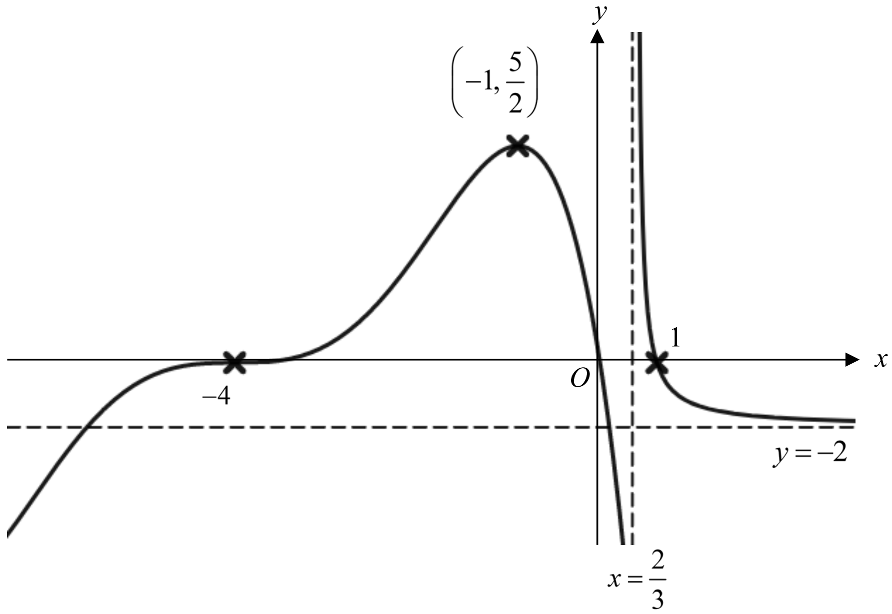
(c) A Mathematics department consists of 12 teachers. Nine of the teachers wear glasses and three of the teachers do not wear glasses. **2**
Five of these teachers go out to dinner together.

In how many ways will there be more teachers who wear glasses than teachers who do not wear glasses in the group who go out for dinner?

Question 12 continues on page 9

Question 12 (continued)

(d) **2**



The diagram shows the curve with equation $y = f(x)$, $x \neq \frac{2}{3}$.

Note: the graph is continuous in its domain.

The curve crosses the axes at $x = -4$, $x = 1$, and the origin.

The curve has asymptotes with equations $x = \frac{2}{3}$ and $y = -2$.

The curve has a stationary point of inflexion at $x = -4$, and a turning point with coordinates $\left(-1, \frac{5}{2}\right)$.

Sketch the curve $y = \frac{1}{f(x)}$, labelling any intercepts, coordinates of stationary points, and the equations of any asymptotes.

(e) The position vector of a particle moving along a curve at time t seconds is given by **3**

$$\underset{\sim}{r}(t) = \frac{t^3}{3}\underset{\sim}{i} + \left(\arcsin(t) + t\sqrt{1-t^2}\right)\underset{\sim}{j},$$

where distances are measured in metres.

The distance d metres that the article travels along the curve in three-quarters of a second is given by

$$d = \int_0^{\frac{3}{4}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Find the exact value of d.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) (i) Factorise $t^3 + 2t^2 + 3t + 2$. **2**

(ii) A curve, D, has parametric equations **2**

$$x = t + \frac{1}{t} + 4, \quad y = t - \frac{1}{t} + 1, \quad t \leq -1$$

A curve E has equation

$$y = x - 2 - \frac{1}{x-3}, \; x \leq 2$$

Show that curves D and E intersect only once and find the coordinates of the point of intersection.

(b) (i) Show that $\dfrac{1}{(n+1)!} - \dfrac{n+1}{(n+2)!} = \dfrac{1}{(n+2)!}$, $n \in \mathbb{Z}$. **1**

(ii) Use mathematical induction to show that, for all integers $n \geq 1$, **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Question 13 continues on page 11

Question 13 (continued)

(c) A curve passes through three points $A(14, 8)$, $B(r, 2)$, and $C(r, 0)$ as shown in Figure 1 below.
The equation of the part joining A and B is $x = \sqrt{4 + 3y^2}$ and the part joining B and C is the vertical line $x = r$.

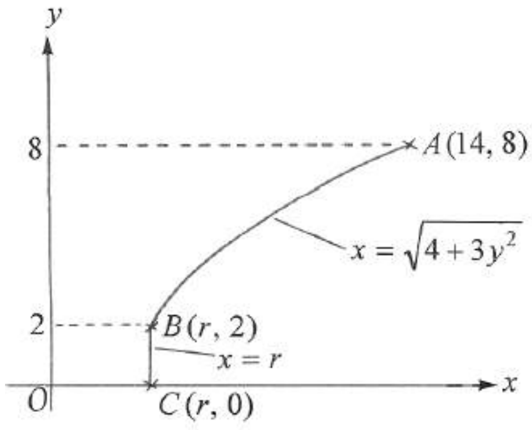


Figure 1

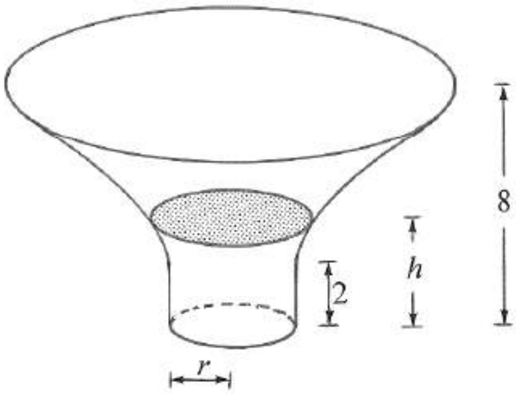


Figure 2

(i) A pot, 8 units in height, is formed by revolving the curve and the line segment OC about the y-axis, where O is the origin. See Figure 2 above. **4**

The pot contains water to a depth of h units, where $h > 2$.
By first finding the value of r, show that the volume of water in the pot is

$$\left(h^3 + 4h + 16\right)\pi \text{ cubic units.}$$

Initially the pot in part (i) contains water to a depth greater than 3 units.
The water is now pumped out at a constant rate of 2π cubic units per second.

(ii) Find the rate of change of the depth of the water in the pot when the depth of the water is 3 units. **2**

(iii) Find the rate of change of the depth of the water in the pot when the depth of the water is 1 unit. **1**

End of Question 13

Question 14 (14 marks) Use a SEPARATE writing booklet

(a) Two students are trying to do mathematical modelling for a recent outbreak of disease in a particular city with a population of 10 000 people.
There are five people who are initially infected by the disease.

Student A proposed modelling the scenario using the following differential equation

$$\frac{dI}{dt} = RI,$$

where I is the number of people in the city who are infected at time t (days) after the initial outbreak and R is a positive constant.

(i) Show that $I = 5e^{Rt}$ satisfies this differential equation. **1**

Student B proposed using a geometric progression for his mathematical model.
His model can be predicted by the equation

$$I_n = 5\left(2^n\right),$$

where the nth term of the geometric progression, I_n, gives the number of people infected with the disease at the beginning of the nth day after the initial outbreak.

It is given that the geometric progression has common ration 2.

(ii) Explain, in the context of the infectious outbreak, the meaning of the common ratio. **1**

(iii) Show that $R = \log_e 2$. **1**

(iv) Explain whether the first or second model is the better model. **1**

(b) A particle P is projected horizontally with speed 15 m/s from the top of a vertical cliff. **4**

At the same instant a particle Q is projected from the bottom of the cliff, with speed 25 m/s at an angle θ above the horizontal.

P and Q move in the same vertical plane. The height of the cliff is 60 m and the ground at the bottom of the cliff is horizontal.

If the particles collide, find the value of θ, and determine whether Q is rising or falling immediately before this collision.

You may assume that the position, $\vec{r}(t)$, of a particle at time t seconds is given by

$$\vec{r}(t) = Ut\cos\theta\,\vec{i} + \left(Ut\sin\theta - 5t^2\right)\vec{j},$$

where the particle is fired at an angle θ to the horizontal, with a speed U m/s from the ground.

Question 14 continues on page 13

Question 14 (continued)

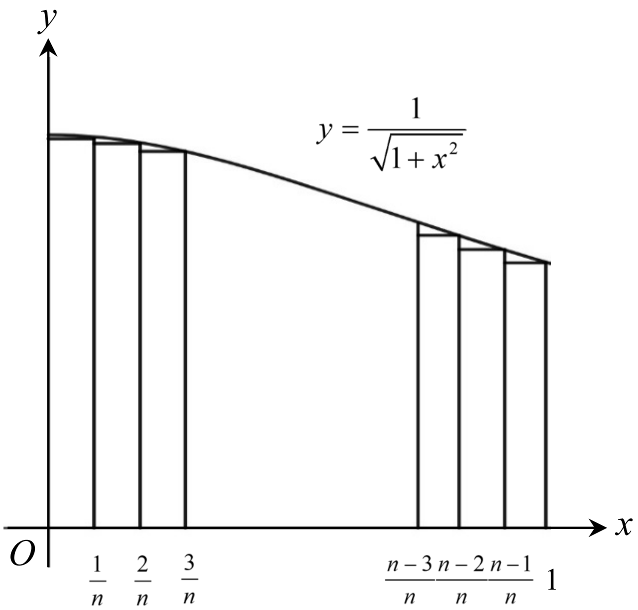
(c) (i) Using the substitution $x = \tan\theta$, find the exact value of **3**

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$$

You may use the fact that $\int \sec x\,dx = \ln\left|\sec x + \tan x\right| + C$. Do NOT prove.

(ii) The graph of $y = \dfrac{1}{\sqrt{1+x^2}}$ for $0 \le x \le 1$, is shown in the diagram below. **2**

Rectangles, each of width $\dfrac{1}{n}$ are drawn under the curve.



Show that the total area A of all n rectangles is given by

$$A = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+4}} + \frac{1}{\sqrt{n^2+9}} + \ldots + \frac{1}{\sqrt{n^2+(n-1)^2}} + \frac{1}{\sqrt{2n^2}}$$

(iii) Find $\lim\limits_{n\to\infty} A$. **1**

End of paper

SYDNEY BOYS HIGH SCHOOL

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2023

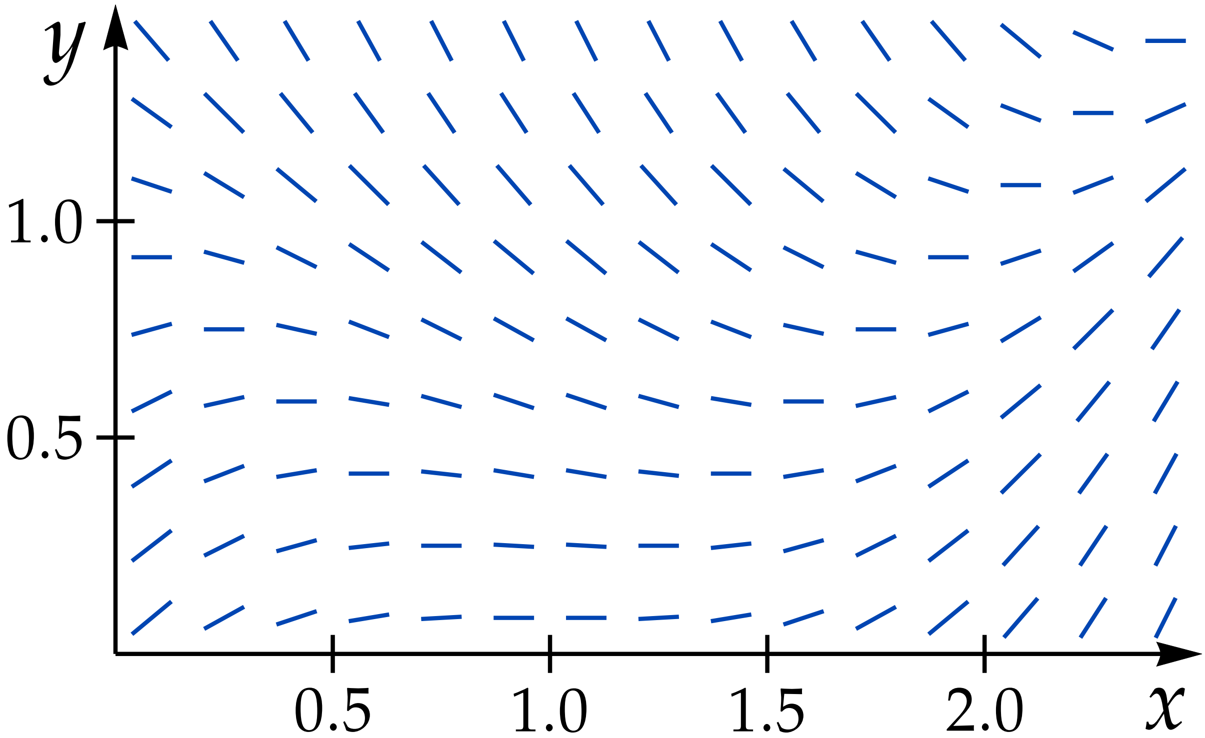
YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 1

Question 11 (g)

(g) The diagram below shows part of the slope field for the differential equation **1**

$$\frac{dy}{dx} = (x - 1)^2 - y^2.$$



On the diagram above, sketch the solution curve for $\frac{dy}{dx} = (x - 1)^2 - y^2$ where $y(0) = 1$.

Put this Answer Sheet inside your Answer Booklet for Question 11.

SYDNEY BOYS HIGH SCHOOL

2023

YEAR 12

TASK 4 – THSC

Mathematics Extension 1

Sample Solutions

NOTE: Some of you may be disappointed with your mark.
This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done. Writing something down does not justify that your working can be linked to a mark.

Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

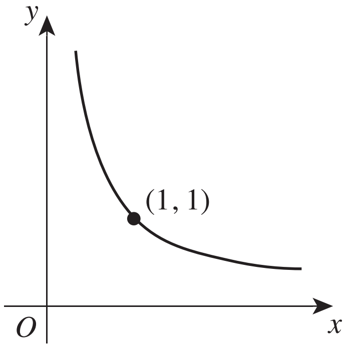
MC Answers

1	C	**6**	A
2	C	**7**	B
3	A	**8**	A
4	A	**9**	C
5	B	**10**	B

Section I Multiple Choice Solutions

1 Which of the following shows the graph of $x = \sqrt{t-2}$, $y = \dfrac{1}{2-t}$?

A.



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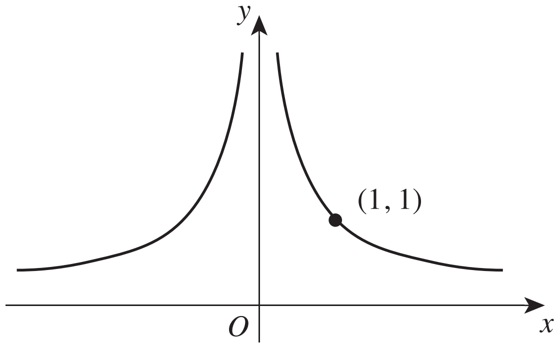
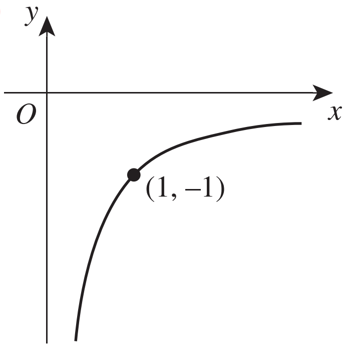


C.



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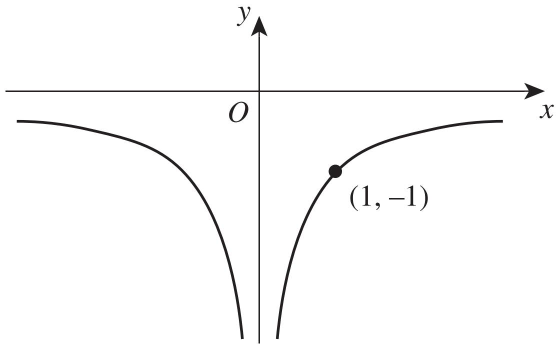


From $x = \sqrt{t-2}$ then $x \geq 0$ and $t \geq 2$. So $y < 0$

2 The population, N, of wallabies at time t years is given by $N(t) = 155 + Ae^{kt}$, where A and k are positive constants.

Which of the following is the correct differential equation?

A. $\dfrac{dN}{dt} = -k(N - 155)$

B. $\dfrac{dN}{dt} = -k(N + 155)$

C. $\dfrac{dN}{dt} = k(N - 155)$

D. $\dfrac{dN}{dt} = k(N + 155)$

A or C are the only possible answers, but A is out as k is given indeterminate about sign.

3 A particular monic cubic has three distinct integer roots, each with magnitude less than or equal to 5, and an odd constant term.

How many such cubics are there?

A. 20

B. 40

C. 165

D. 330

If the roots are α, β, γ then $-5 \le \alpha, \beta, \gamma \le 5$

As the constant term is odd ($= \alpha\beta\gamma$) then there can be no even roots.

$\therefore \alpha, \beta, \gamma \in \{-5, -3, -1, 1, 3, 5\}$.

Given that we need three (and the order is irrelevant) then there are ${}^6C_3 = 20$ ways

4 Which of the following integrals is obtained when the substitution $u = (\ln x)^2$ is applied to

$$\int_e^{e^2} \frac{(\ln x)^3}{x}\, dx \ ?$$

A. $\dfrac{1}{2}\displaystyle\int_1^4 u\, du$

B. $2\displaystyle\int_1^4 u\, du$

C. $\dfrac{1}{2}\displaystyle\int_1^4 u^{\frac{3}{2}}\, du$

D. $2\displaystyle\int_1^4 u^{\frac{3}{2}}\, du$

$$u = (\ln x)^2 \Rightarrow du = 2\ln x \times \frac{1}{u}$$

$$\int_e^{e^2} \frac{(\ln x)^3}{x}\, dx = \frac{1}{2}\int_e^{e^2} (\ln x)^2 \times \frac{2\ln x}{x}\, dx$$

5 Consider $2x^2 + 7x - 15 = 0$

If r and s are the two roots of the equation above and $r > s$, which of the following is the value of $r - s$?

A. $\frac{15}{2}$

B. $\frac{13}{2}$

C. $\frac{11}{2}$

D. $\frac{3}{2}$

$$
\begin{aligned}
(r-s)^2 &= (r+s)^2 - 4rs \\
&= \left(-\tfrac{7}{2}\right)^2 - 4\left(-\tfrac{15}{2}\right) \\
&= \tfrac{49}{4} + 30 \\
&= \frac{169}{4}
\end{aligned}
$$

As $r > s$ then $r - s = \sqrt{\dfrac{169}{4}} = \dfrac{13}{2}$

6 A one-to one-function $g(x)$ has domain $x \geq 0$ and range $y \geq 0$.
The function $g(x)$ passes through the origin and its inverse is $g^{-1}(x)$.

Given that

$$\int_0^a g^{-1}(x) - x \, dx = \frac{3}{2}a^2,$$

what is the value of $\displaystyle\int_0^{g^{-1}(a)} g(x) \, dx$?

A. $ag^{-1}(a) - 2a^2$

B. $ag^{-1}(a) + 2a^2$

C. $ag^{-1}(a) - \dfrac{3}{2}a^2$

D. $ag^{-1}(a) + \dfrac{3}{2}a^2$

$$
\begin{aligned}
\int_0^a g^{-1}(x)\, dx &= \int_0^a x \, dx + \frac{3}{2}a^2 \\
&= \frac{1}{2}a^2 + \frac{3}{2}a^2 = 2a^2
\end{aligned}
$$

Now $\displaystyle\int_0^a g^{-1}(x)\, dx + \int_0^{g^{-1}(a)} g(x)\, dx = ag^{-1}(a)$

7 If $\arcsin(1-x) - 2\arcsin x = \frac{\pi}{2}$, then what is the value of x?

A. $-\frac{1}{2}$

B. 0

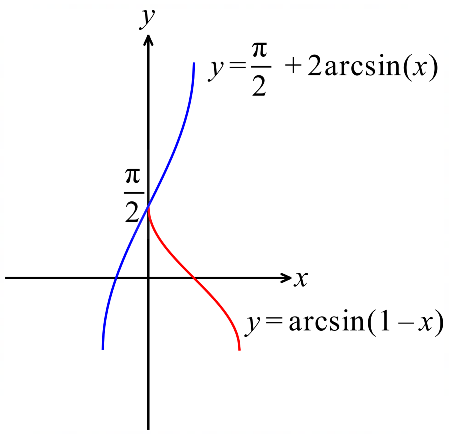
C. $\frac{1}{2}$

D. 1

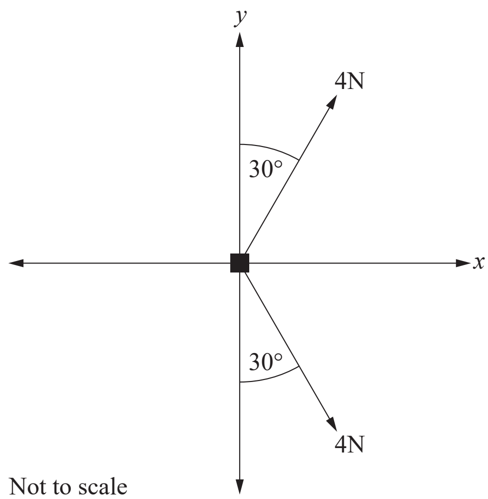
The quickest way to do this is by substitution.

Or $\arcsin(1-x) - 2\arcsin x = \frac{\pi}{2} \Rightarrow \arcsin(1-x) = \frac{\pi}{2} + 2\arcsin x$

Sketching:



8 Two forces act concurrently on a 2 kg object placed at the origin.



What is the magnitude of the net force acting on the object?

A. 4 N

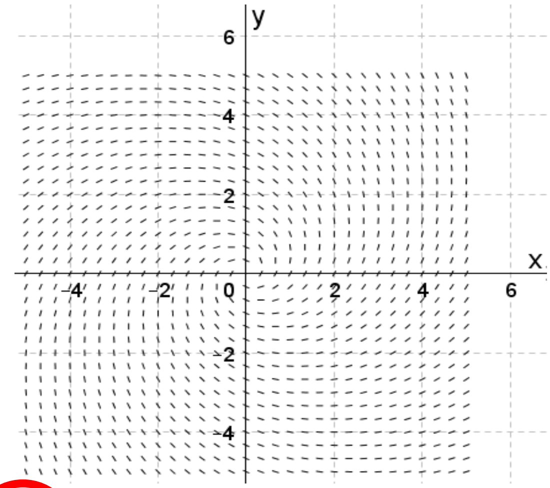
B. $2\sqrt{3}$ N

C. 8 N

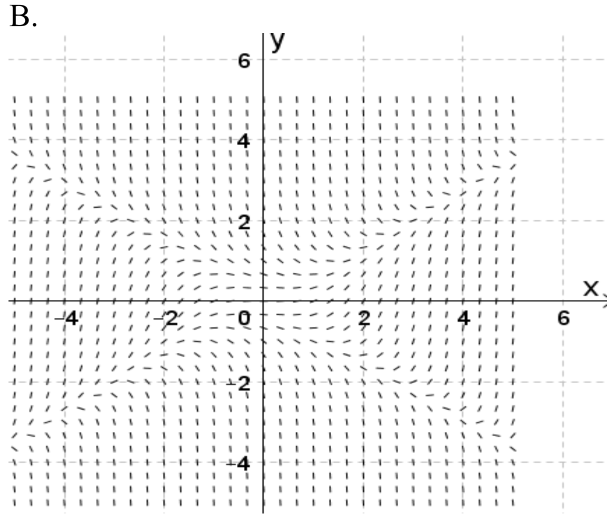
D. $4\sqrt{3}$ N

9 Which of the following direction fields best represents the differential equation $\dfrac{dy}{dx} = \dfrac{y-2x}{x-4}$?

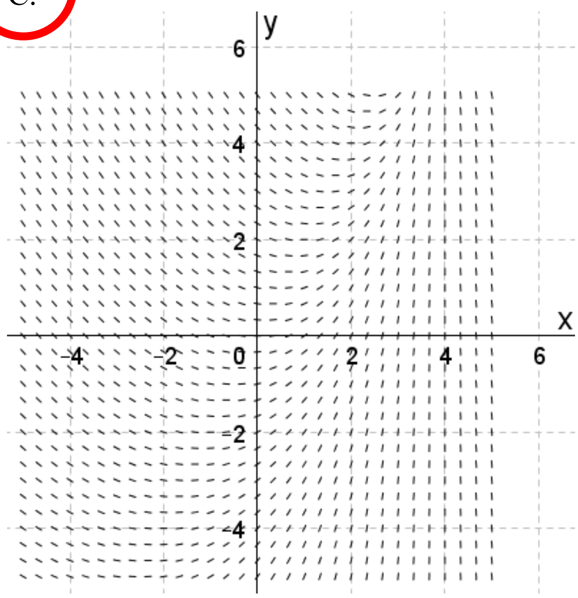
A.



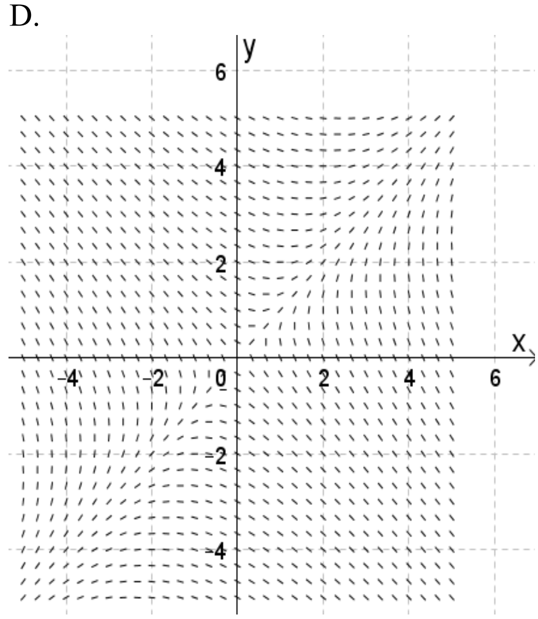
B.



C.



D.



At $x = 4$, the slope lines are vertical

10 If $f(x) = \sin x + \cos x$ and $g(x) = x^2 - 1$, then $g(f(x))$ is a one-to-one function in which of the following intervals?

A. $\left[0, \dfrac{\pi}{2}\right]$

B. $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

C. $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

D. $[0, \pi]$

$$\begin{aligned} g(f(x)) &= (\sin x + \cos x)^2 - 1 \\ &= 2\sin x \cos x \\ &= \sin 2x \end{aligned}$$

$\sin x$ is 1 : 1 for $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, so $\sin 2x$ is 1 : 1 for $x \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$.

<u>2023 HSC ME1 Task 4 – Question 11 Solutions and Comments</u>

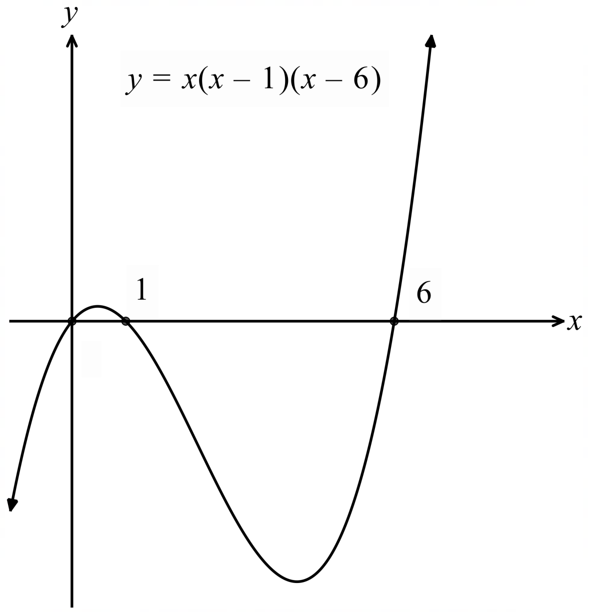
Answers in pencil and/or with white out will **NOT** have their marks changed.

A. Find $\int \frac{1}{9+25x^2}\,dx$. 2

Solution	Comment(s)
From the reference sheet: $$\int \frac{f'(x)}{a^2+\left(f(x)\right)^2}\,dx = \frac{1}{a}\arctan\frac{f(x)}{a}+c$$ Hence: $$\begin{aligned}\int \frac{1}{9+25x^2}\,dx &= \int \frac{1}{3^2+(5x)^2}\,dx \\ &= \frac{5}{5}\int \frac{1}{3^2+(5x)^2}\,dx \\ &= \frac{1}{5}\int \frac{5}{3^2+(5x)^2}\,dx \\ &= \frac{1}{5}\left(\frac{1}{3}\arctan\frac{5x}{3}\right)+c \\ &= \frac{1}{15}\arctan\frac{5x}{3}+c\end{aligned}$$	Common error(s): • Forgetting the factor of $\frac{1}{5}$.

Answers in pencil and/or with white out will **NOT** have their marks changed.

B. I. Solve $\dfrac{3x+2}{1-x} < -x+2$. 2

Solution	Comment(s)
Multiplying both sides by $(1-x)^2$: $(3x+2)(1-x) < (-x+2)(1-x)^2$ Moving $(-x+2)(1-x)^2$ to the LHS and solving by factorising $1-x$ out of expression: $(3x+2)(1-x)-(-x+2)(1-x)^2 < 0$ $(1-x)\big(3x+2-(2-x)(1-x)\big) < 0$ $(1-x)(3x+2-(2-2x-x+x^2)) < 0$ $(1-x)(-x^2+6x) < 0$ $x(x-1)(x-6) < 0$ The graph of $y = x(x-1)(x-6)$ is shown below:  Hence, $x < 0$ or $1 < x < 6$.	Many students lost marks due to various algebraic mistakes as a result of expanding the inequality. Students should attempt to recognise that moving everything to one side and factorising is a more robust method of solving the problem. Common error(s): • Not accounting for the critical point at $x = 1$ when multiplying both sides by $1-x$. • Not factorising all of the negative signs out, leading to incorrect solution sets to the inequality.

II. Hence, or otherwise, solve $\dfrac{3e^x+2}{1-e^x} < -e^x+2$. 1

Solution	Comment(s)
Using Part I: $e^x < 0$ — No solution, as $e^x > 0$. $1 < e^x < 6$ $\ln 1 < \ln(e^x) < \ln 6$ $0 < x < \ln 6$ Hence, $0 < x < \ln 6$.	Students must explicitly address the case of $e^x < 0$ having no solution in order to get the full one mark. Some students annoyingly left their answer in terms of e^x rather than x.

<u>2023 HSC ME1 Task 4 – Question 11 Solutions and Comments</u>

Answers in pencil and/or with white out will **NOT** have their marks changed.

C. I. Write a monic quadratic function that has zeroes α and β. 1

Solution	Comment(s)
$(x-\alpha)(x-\beta) = x^2 - \alpha x - \beta x + \alpha\beta$ $= x^2 - (\alpha+\beta)x + \alpha\beta$	Common error(s): • Writing a function with specific zeroes, such as $(x-1)(x-2)$, rather than a generic one involving α and β. • Writing the function $x^2 + (\alpha+\beta)x + \alpha\beta$, which has zeroes $-\alpha$ and $-\beta$.

II. The polynomial $A(x) = x^4 + px^2 + qx + 36$ has a double zero at $x = 2$, 2
where p and q are real constants.
Factorise $A(x)$, expressing the result as the product of two quadratics.

Solution	Comment(s)
$A(x)$ has a double zero at $x = 2$. Furthermore, $A(x)$ is monic. Hence, using Part I: $A(x) = (x-2)^2(x-\alpha)(x-\beta)$ Consider the sum and product of the roots of $A(x)$: $\alpha + \beta + 2 + 2 = 0$ $\alpha + \beta = -4$ $\alpha\beta \times 2 \times 2 = 36$ $\alpha\beta = 9$ From Part I: $(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$ Hence: $A(x) = (x-2)^2(x^2+4x+9)$ $= (x^2-4x+4)(x^2+4x+9)$	Alternatively, rewriting $A(x)$ as the product of two quadratics: $x^4 + px^2 + qx + 36$ $= (x-2)^2(ax^2+bx+c)$ $= (x^2-4x+4)(ax^2+bx+c)$ By inspection, $a = 1$ and $4c = 36$, which simplifies to $c = 9$. Comparing the coefficients of x^3: $0 = b - 4a$ $= b - 4$ $b = 4$ Hence, $A(x) = (x^2-4x+4)(x^2+4x+9)$. ~ No marks were allocated for finding the values of p and q; It should be clear from the sample solutions that this isn't necessary. Students should appreciate the discrepancy between the number of marks allocated and the additional work required to find p and q to then realise that more efficient methods exist. Common error(s): • Incorrectly assigning $\alpha + \beta + 2 + 2 = -p$ due to forgetting the cubic term in the quartic expression.

<u>2023 HSC ME1 Task 4 – Question 11 Solutions and Comments</u>

Answers in pencil and/or with white out will **NOT** have their marks changed.

D. A school principal calls a meeting with two student representatives from each year group from Year 7 to Year 12. **2**
The 12 students and the principal sit at a round table.
In how many ways can the 13 participants sit around the table if at least one pair of students from a year group sits apart.

Solution	**Comment(s)**
Using the complement, the number of ways in which at least one pair of students sits apart is $A - B$: • A is the number of ways to seat all participants without any restrictions. • B is the number of ways in which all pairs of students sit together. There are $12!$ ways to seat 13 participants at a round table in any order, so $A = 12!$. Next, consider each pair of students as a single unit to be seated at the table. Then, there are $6!$ ways to seat the 7 groups of participants. Furthermore, each pair of students within the unit can be seated in $2!$ ways. This repeats 6 times due to there being 6 pairs, so $B = 6! \times (2!)^6$. Hence: $A - B = 12! - (6! \times (2!)^6)$ $= 12! - (6! \times 2^6)$ $= 478\ 955\ 520$	Common error(s): • Not accounting for the two arrangements of students within each year group pair.

<u>2023 HSC ME1 Task 4 – Question 11 Solutions and Comments</u>

Answers in pencil and/or with white out will **NOT** have their marks changed.

E. Show that for $\tan A \neq -1$: 2

$$\frac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A} = 2 - \sin 2A$$

Solution	**Comment(s)**
LHS $= \dfrac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A}$ $= \dfrac{2(\sin A + \cos A)(\sin^2 A - \sin A\cos A + \cos^2 A)}{\sin A + \cos A}$ $= 2(1 - \sin A\cos A)$ $= 2 - 2\sin A\cos A$ $= 2 - \sin 2A$ $= RHS$	An alternative solution involving the Pythag trig identity $\sin^2 A + \cos^2 A = 1$ and grouping in pairs exists, but the given solution is arguably the most efficient one. Students who attempted to make use of the various forms of $\cos 2A$ could not score any marks, as these answers made no relevant progress towards establishing the proof.

F. Evaluate in exact form: 2

$$\int_0^{\frac{\pi}{7}} \sin 3x \cos 4x \, dx$$

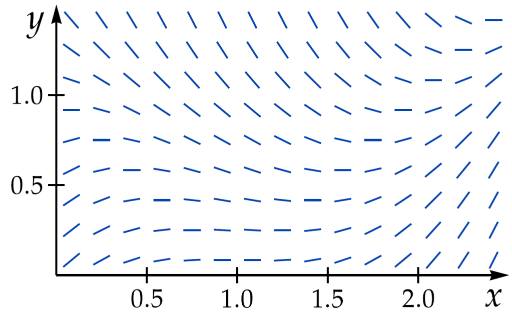
Solution	**Comment(s)**
From the reference sheet: $\sin A\cos B = \dfrac{1}{2}[\sin(A+B) + \sin(A-B)]$ Hence: $\displaystyle\int_0^{\frac{\pi}{7}} \sin 3x\cos 4x\,dx$ $= \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{7}} \sin(3x+4x) + \sin(3x-4x)\,dx$ $= \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{7}} \sin 7x + \sin(-x)\,dx$ $= \dfrac{1}{2}\left[\dfrac{-1}{7}\cos 7x + \cos x\right]_0^{\frac{\pi}{7}}$ $= \dfrac{1}{2}\left(\left(\dfrac{-1}{7}\cos\dfrac{7\pi}{7} + \cos\dfrac{\pi}{7}\right) - \left(\dfrac{-1}{7}\cos 0 + \cos 0\right)\right)$ $= \dfrac{1}{2}\left(\dfrac{1}{7} + \cos\dfrac{\pi}{7} + \dfrac{1}{7} - 1\right)$ $= \dfrac{1}{2}\left(\cos\dfrac{\pi}{7} - \dfrac{5}{7}\right)$ $= \dfrac{1}{2}\cos\dfrac{\pi}{7} - \dfrac{5}{14}$	Students who could not recognise the product to sum form could not score any marks; These students should remember to consult the reference sheet. Common error(s): • Various errors in evaluating the integral.

2023 HSC ME1 Task 4 – Question 11 Solutions and Comments

Answers in pencil and/or with white out will **NOT** have their marks changed.

G. The diagram below shows part of the slope field for the differential equation: 1

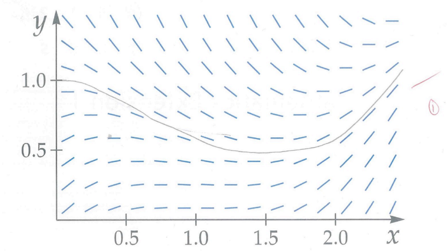
$$\frac{dy}{dx} = (x-1)^2 - y^2$$

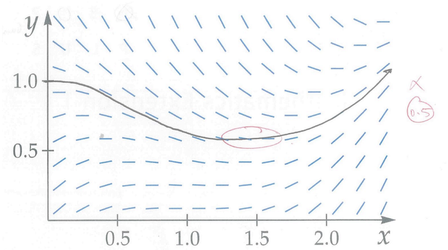


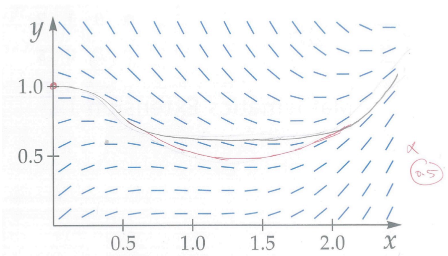
On the Answer Sheet provided, sketch the solution curve for $\dfrac{dy}{dx} = (x-1)^2 - y^2$,

where $y(0) = 1$.

Put your Answer Sheet for this question inside your Answer Booklet for Question 11.

<table>
<tr><th>Solution</th><th>Comment(s)</th></tr>
<tr><td>
Sample student response that scored the full one mark:


Some sample student responses that were penalised are shown below:




</td><td>
Students were penalised if there were any clear discrepancies between their solution curve and the tangent lines within the vicinity of the sketch.

Students were also penalised if it was difficult to determine whether their solution curve crossed over any of the tangent lines in the slope field.

See sample student responses on the left for reference.

This ambiguity can be avoided if students clearly sketch their solution curve between the tangent lines rather than on them.

As such, any student appeal regarding the precision and/or clarity of their solution curve will not be considered.

Common error(s):
• Not recognising $y(0) = 1$ to mean that the solution curve passes through the point $(0, 1)$.
</td></tr>
</table>

Notes and Solutions For THSC Ext1 - Q12, 2023.

Notes on (a):

There really wasn't much wiggle room on this question. The solution of successful pupils was nearly the same as what is given here. That said, if you made a mistake near the start, it was still possible to earn the rest of the marks.

Missing a k on the next line or failing to copy an index to the next line hurt quite a lot of pupils. Many did not put their final answer as a union of intervals and simply gave specific valves for k. Some did not apply the cosine of the sum two angles correctly - they should have double checked with the reference sheet. Some pupils went to the trouble of working out the value of the auxiliary angle - but it was not used in finding the solution.

Mostly though, pupils got to a point where they did not know how to proceed.

Notes on (b):

Part i - An equation is not an interpretation. An explanation in words was required, no exceptions.
Having said that, this was a tough question for me. There are rarely any reasons to consider what the dot product of two vectors means geometrically if it is not zero, let alone it's magnitude. The ONLY reason this works is because one of the vectors is a unit vector.

Part ii - Pupils rarely (I'm not exaggerating) used proper vector notation in their working. As a result, many treated the vectors like scalar pro-numerals. This should have been an easy two marks for most - it wasn't. If pupils used the dot product in their solution, they then had to work out the dot product of vectors a and b, but made mistakes doing so. Some got the correct result for the square of the absolute value, then failed to give the absolute value.

Part iii - Where to begin?... This is not a difficult question. The varied solutions given do not require any stretching of one's imagination.
Again vectors were treated as though they were scalars and the ratio given was assumed by many as the actual lengths. Many times pupils used trigonometric ratios on triangles they had not shown to be right angled. Assuming right-angles, with no reason as to why, occurred too often and the justification would not have been too hard. It simplified the question and I insisted on reasons for going this route.
It seems clear that most pupils find it difficult to do dot products when at least one of the vectors is a linear combination of two or more vectors. Algebra on vectors was rarely done well.

Notes on (c):

This was probably the question that rewarded the most amount of pupils with full marks.

Notes on (d): A reasonable approximation of the curves in the solution (which you can see is NOT computer generated) was required. Apart from getting these correct (which a lot of pupils were able to do), things that let them down were failing to identify critical values from the original graph (local max becomes local min) and the asymptote at x equal to two thirds becomes at least a limit point of zero at this same x value. I'm being wary here - careful not to start an argument in staff rooms. My interpretation is obvious from the diagram.

Another issue was that students were not identifying the asymptotes on the diagram. They were doing it in legends off to the side. Technically only the diagram should be marked. It was annoying to have to go back to the diagram and remark. I did this begrudgingly. I strongly doubt that HSC markers would allow students to get away with this.

Notes on (e): This question had even less wiggle room than the first. If pupils were not able to find the derivative of the y function it was extremely difficult to earn any more marks. If they happen to do that part correctly, more often than not, they were not able to simplify the sum of the square of the derivatives.

From the supplied solution, one can see that this simplification was a square. Pupils were then required to realise that the square root of a square is an absolute value and this needed to be expressed in the integral somehow (that is, before they end up getting a negative result and realising that they need to swap the sign).

A big problem was that students who ended up with lots of ugly algebra failed to see that this was because they had made a mistake earlier. If pupils had just gone back to check the derivative of the y function they may have been able to earn full marks.

(a) Consider the equation $3\sin\theta + k\cos(\theta + 30°) = 7$, where $k \in \mathbb{R}$. **3**

Find the values of k for which this equation has real solutions.

You may assume that $a\sin x + b\cos x = \sqrt{a^2 + b^2}\sin(x + \varepsilon)$, where $\varepsilon \in \mathbb{R}$.

$$3\sin\theta + k\cos\theta \times \frac{\sqrt{3}}{2} - \frac{k}{2}\sin\theta = 7$$

$$(6-k)\sin\theta + \sqrt{3}k\cos\theta = 14$$

$$\sqrt{(6-k)^2 + 3k^2}\sin(\theta + \alpha) = 14, \text{ where } \sin\alpha = \frac{\sqrt{3}k}{\sqrt{(6-k)^2 + 3k^2}}, \cos\alpha = \frac{6-k}{\sqrt{(6-k)^2 + 3k^2}}.$$

$$sin\,(\theta + \alpha) = \frac{14}{\sqrt{(6-k)^2 + 3k^2}}$$

This has real solutions when the denominator is greater than the numerator on the right hand side of the equation.

$$\sqrt{(6-k^2) + 3k^2} \geq 14$$

$$4k^2 - 12k + 36 \geq 196$$

$$k^2 - 3k - 40 \geq 0$$

$$(k-8)(k+5) \geq 0$$

k satisfies either: $k \leq -5, k \geq 8$

(b) With respect to the origin O, the position vectors of the points A, B, and C are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively.

It is given that $\mathbf{a}$ is a unit vector, $|\mathbf{b}| = 2$, and $\angle AOB = 60°$.

(i) Give a geometrical interpretation of $|\mathbf{a} \cdot \mathbf{c}|$. **1**

(ii) By considering $(2\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} - \mathbf{b})$, or otherwise, find $|2\mathbf{a} - \mathbf{b}|$. **2**

Point C lies on AB such that $AC : CB = 1 : 2$ i.e $\mathbf{c} = \dfrac{\mathbf{b} + 2\mathbf{a}}{3}$.

(iii) By considering the cosine of angles AOC and COB, determine if the line segment OC bisects the angle AOB. **3**

(i)

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}| \cos\theta$$

$$|\mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}|\,|\mathbf{y}|\,|\cos\theta|$$

Then $|\mathbf{a} \cdot \mathbf{c}| = |\mathbf{a}|\,|\mathbf{c}|\,|\cos(\angle AOC)|$

$$|\mathbf{a} \cdot \mathbf{c}| = |\mathbf{c}|\,|\cos(\angle AOC)|\,,\, |\mathbf{a}| = 1.$$

This is <u>the magnitude of the projection vector of **c** on **a**.</u>

The answer must be in words!

Merely for confirmation: $|proj_{\mathbf{a}}\mathbf{c}| = |\mathbf{c}||\cos(\angle AOC)| \times \dfrac{|\hat{\mathbf{a}}|}{|\mathbf{a}|}$

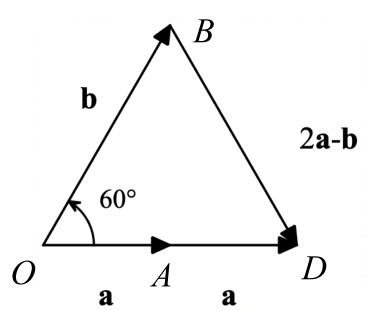
$$= |\mathbf{c}| \times \frac{|\mathbf{a} \cdot \mathbf{c}|}{|\mathbf{c}|\,|\mathbf{a}|^2} \times |\hat{\mathbf{a}}|\,,\, \cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$$

$$= |\mathbf{a} \cdot \mathbf{c}|\,,\, |\mathbf{a}| = 1$$

(ii)

$$\begin{aligned} |2\mathbf{a} - \mathbf{b}| &= \sqrt{(2\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} - \mathbf{b})} \\ &= \sqrt{4|\mathbf{a}|^2 - 4(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2} \\ &= \sqrt{4 - 4|\mathbf{a}||\mathbf{b}| \cos\left(\frac{\pi}{3}\right) + 4} \\ &= \sqrt{8 - 4 \times 2 \times \frac{1}{2}} \\ &= 2 \end{aligned}$$

OR:



Let 2**a** be the vector that represents a point D. Then $\overline{OB} = \overline{OD}$.

$\overrightarrow{BD} = 2\mathbf{a} - \mathbf{b}$.

Also, the angles opposite the intervals OB and OD are equal, since these are equal sides on an Isosceles triangle. Therefore, these base angles must also be 60 degrees. That can only mean OBD is an equilateral triangle.

Hence $|2\mathbf{a} - \mathbf{b}| = 2$.

(iii)

$$\begin{aligned}
\cos(\angle AOC) &= \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}|} \\
&= \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{c}|} \\
&= \frac{\frac{1}{3}\mathbf{a}\cdot(2\mathbf{a}+\mathbf{b})}{\frac{1}{3}\sqrt{(2\mathbf{a}+\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})}} \\
&= \frac{2\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}}{\sqrt{4\mathbf{a}\cdot\mathbf{a}+4\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}}} \\
&= \frac{2+|\mathbf{a}||\mathbf{b}|\cos\left(\frac{\pi}{3}\right)}{\sqrt{8+4|\mathbf{a}|\,|\mathbf{b}|\cos\left(\frac{\pi}{3}\right)}} \\
&= \frac{3}{2\sqrt{3}} \\
&= \frac{\sqrt{3}}{2}
\end{aligned}$$

$$\therefore \angle AOC = \frac{\pi}{6}$$

$$\cos\angle AOB = \cos\left(\angle AOC + \angle BOC\right)$$

$$\angle AOB = \angle AOC + \angle BOC$$

$$\frac{\pi}{3} = \frac{\pi}{6} + \angle BOC$$

$$\frac{\pi}{6} = \angle BOC$$

$$\cos\left(\angle AOC\right) = \cos\left(\angle BOC\right)$$

Therefore $\overrightarrow{OC}$ bisects the angle made by $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Or consider:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$$

$$2\mathbf{a}\cdot\mathbf{b} = 2\mathbf{a}\cdot\mathbf{b}$$

$$2\mathbf{a}\cdot\mathbf{b}+4 = 2\mathbf{a}\cdot\mathbf{b}+4$$

$$2\mathbf{a}\cdot\mathbf{b}+4\mathbf{a}\cdot\mathbf{a} = 2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}$$

$$2\mathbf{a}\cdot(2\mathbf{a}+\mathbf{b}) = \mathbf{b}\cdot(2\mathbf{a}+\mathbf{b})$$

$$2\mathbf{a}\cdot\left(\frac{2\mathbf{a}+\mathbf{b}}{3}\right) = \mathbf{b}\cdot\left(\frac{2\mathbf{a}+\mathbf{b}}{3}\right)$$

$$2\mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c}$$

$$\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{c}|} = \frac{\mathbf{b}\cdot\mathbf{c}}{2|\mathbf{c}|}$$

$$\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|}$$

$$\cos\left(\angle AOC\right) = \cos\left(\angle BOC\right)$$

Therefore **c** bisects the angle made by **a** and **b**.

OR: $\cos(\angle AOC) = \dfrac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}|}$ and $\cos(\angle BOC) = \dfrac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}||\mathbf{c}|}$

$$\begin{aligned}
\cos\angle BOC &= \frac{\mathbf{b}\cdot\mathbf{c}}{2|\mathbf{c}|}\\
&= \frac{\mathbf{b}\cdot\left(\dfrac{2\mathbf{a}+\mathbf{b}}{3}\right)}{2|\mathbf{c}|}\\
&= \frac{2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}}{6|\mathbf{c}|}\\
&= \frac{2\mathbf{a}\cdot\mathbf{b}+4\mathbf{a}\cdot\mathbf{a}}{6|\mathbf{c}|}\\
&= \frac{\mathbf{a}\cdot\mathbf{b}+2\mathbf{a}\cdot\mathbf{a}}{3|\mathbf{c}|}\\
&= \frac{\mathbf{a}\cdot\left(\dfrac{\mathbf{b}+2\mathbf{a}}{3}\right)}{|\mathbf{c}|}\\
&= \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{c}|}\\
&= \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}|}\\
&= \cos\angle AOC
\end{aligned}$$

Hence $\cos(\angle AOC) = \cos(\angle BOC)$.

Therefore $\overrightarrow{OC}$ bisects the angle made by $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

OR: Show $\cos(\angle AOC) = \cos(\angle BOC)$.

Then $\cos(\angle AOC) - \cos(\angle BOC) = 0$.

$$\begin{aligned}
\text{LHS} &= \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} - \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|}\\
&= \frac{2\mathbf{a}\cdot\mathbf{c}}{2\,|\mathbf{c}|} - \frac{\mathbf{b}\cdot\mathbf{c}}{2|\mathbf{c}|}\\
&= \frac{(2\mathbf{a}-\mathbf{b})\cdot\mathbf{c}}{2|\mathbf{c}|}\\
&= \frac{(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})}{6\,|\mathbf{c}|}\\
&= \frac{4|\mathbf{a}|^2-|\mathbf{b}|^2}{6|\mathbf{c}|}\\
&= \frac{4-4}{6|\mathbf{c}|}\\
&= 0\\
&= RHS
\end{aligned}$$

Therefore $\overrightarrow{OC}$ bisects the angle made by $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

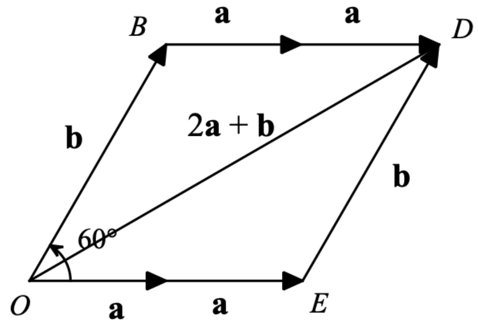
My favourite solution!

Finally: Let $2\mathbf{a} + \mathbf{b}$ be the positional vector of point D. $\mathbf{c}$ is parallel to $\mathbf{2a} + \mathbf{b}$.

Let $2\mathbf{a}$ be the positional vector of point E.

The quadrilateral $OBDE$ is a rhombus. Diagonals of a rhombus bisect its angles. $2\mathbf{a} + \mathbf{b}$ bisects angle BOE and hence the angle BOA.

Therefore $\overrightarrow{OC}$ bisects the angle made by $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



(c) A Mathematics department consists of 12 teachers. Nine of the teachers wear glasses and three of the teachers do not wear glasses. **2**
Five of these teachers go out to dinner together.

In how many ways will there be more teachers who wear glasses than teachers who do not wear glasses in the group who go out for dinner?

As long as there are 3 or more who wear glasses, there will be more at dinner who wear glasses.

Hence, we subtract the number of ways the group can be chosen with 3 teachers who wear glasses from the total number of ways any group of 5 can be formed (without restrictions).

$$^{9}C_5 - {}^{9}C_2\,{}^{3}C_3 = 756$$

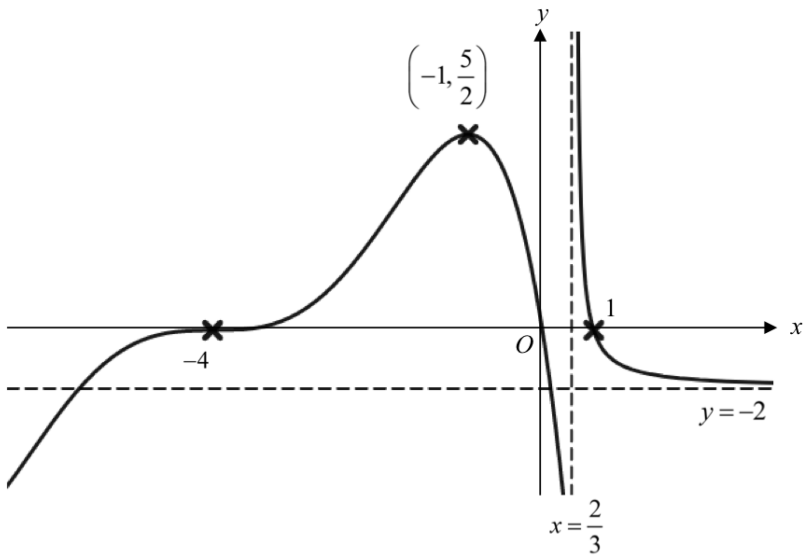
Alternate method: Add the three cases below

.

Case 1: All teachers wear glasses.
Cast 2: 4 teachers who wear glasses.
Case 3: 3 teachers who wear glasses.

$$^{9}C_3\,{}^{3}C_2 + {}^{9}C_4\,{}^{3}C_1 + {}^{9}C_5 = 756$$

(d) **2**



The diagram shows the curve with equation $y = f(x)$, $x \neq \frac{2}{3}$.

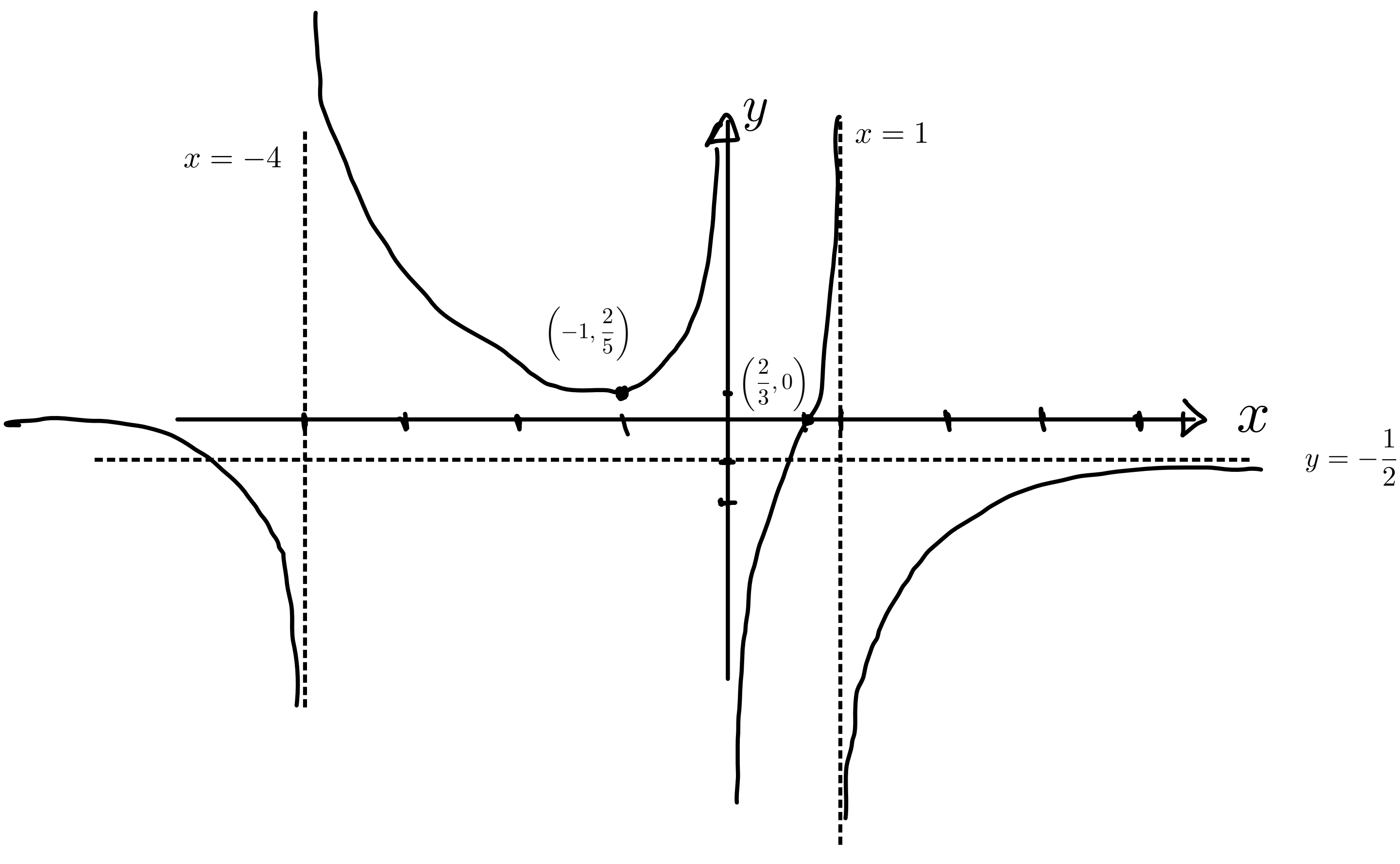
Note: the graph is continuous in its domain.

The curve crosses the axes at $x = -4$, $x = 1$, and the origin.

The curve has asymptotes with equations $x = \frac{2}{3}$ and $y = -2$.

The curve has a stationary point of inflexion at $x = -4$, and a turning point with coordinates $\left(-1, \frac{5}{2}\right)$.

Sketch the curve $y = \dfrac{1}{f(x)}$, labelling any intercepts, coordinates of stationary points, and the equations of any asymptotes.



(e) The position vector of a particle moving along a curve at time t seconds is given by **3**

$$\underset{\sim}{r}(t) = \frac{t^3}{3}\underset{\sim}{i} + \left(\arcsin(t) + t\sqrt{1-t^2}\right)\underset{\sim}{j},$$

where distances are measured in metres.

The distance d metres that the article travels along the curve in three-quarters of a second is given by

$$d = \int_0^{\frac{3}{4}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Find the exact value of d.

$$x(t) = \frac{t^3}{3} \Rightarrow \frac{dx}{dt} = t^2$$

$$y(t) = \arcsin(t) + t\sqrt{1-t^2} \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}} + t \times \frac{1}{2}(-2t)\frac{1}{\sqrt{1-t^2}} + \sqrt{1-t^2}$$

$$= 2\sqrt{1-t^2}$$

$$\begin{aligned}
d &= \int_0^{\frac{3}{4}} \sqrt{(t^2)^2 + \left(2\sqrt{1-t^2}\right)^2}\, dt \\
&= \int_0^{\frac{3}{4}} \sqrt{t^4 + 4(1-t^2)}\, dt \\
&= \int_0^{\frac{3}{4}} \sqrt{t^4 + 4 - 4t^2}\, dt \\
&= \int_0^{\frac{3}{4}} \sqrt{(t^2-2)^2}\, dt \\
&= \int_0^{\frac{3}{4}} |t^2 - 2|\, dt \quad * \qquad\qquad * \; 2 - t^2 > 0, t \in \left[0, \frac{3}{4}\right] \\
&= 2t - \frac{t^3}{3}\Big|_0^{\frac{3}{4}} \\
&= \frac{3}{2} - \frac{\frac{27}{64}}{3} - \left(2 \times 0 - \frac{0^3}{3}\right) \\
&= \frac{96}{64} - \frac{9}{64} \\
&= \frac{87}{64}
\end{aligned}$$

Question 13

(a) (i) $t^3 + 2t^2 + 3t + 2$

$= t^3 + 2t^2 + t + 2t + 2$

$= t(t^2 + 2t + 1) + 2(t+1)$

$= t(t+1)^2 + 2(t+1)$

$= (t+1)[t(t+1)+2]$

$= (t+1)(t^2 + t + 2)$

Marking Scale	Marker's Comment
2 marks: Correct answer with appropiate method 1 mark: Working towards solution	* Most common method provided by candidates was to use factor theorem and polynomial division. * Some candidates were not able to factorise this polynomial correctly. Considering factorisation were taught in Stage 4 and 5, this is concerning for those doing Ext 1 Mathematics.

a) ii) $x = t + \frac{1}{t} + 4$ ① $t \leq -1$

$y = t - \frac{1}{t} + 1$ ② $t \leq -1$

$y = x - 2 - \frac{1}{x-3}$ ③ $x \leq 2$

Sub ① and ② into ③ to find point of intersection

$t - \frac{1}{t} + 1 = (t + \frac{1}{t} + 4) - 2 - \frac{1}{(t + \frac{1}{t} + 4) - 3}$

$t - \frac{1}{t} + 1 = t + \frac{1}{t} + 2 - \frac{1}{t + \frac{1}{t} + 1}$

$-\frac{2}{t} - 1 = -\frac{1}{\frac{t^2}{t} + \frac{1}{t} + \frac{t}{t}}$

$\frac{-2-t}{t} = -\frac{t}{t^2+1+t}$

$(-2-t)(t^2+1+t) = -t^2$

$-2t^2 - 2 - 2t - t^3 - t - t^2 = -t^2$

$t^3 + 2t^2 + 3t + 2 = 0$

From (i)

$t + 1 = 0$ or $t^2 + t + 2 = 0$

$\Delta = b^2 - 4ac = 1 - 4 \times 1 \times 2$

$= -7 < 0$

$\therefore$ $t + 1 = 0$,

$t = -1$

$\therefore$ Only intersect at one point

$x = (-1) + \frac{1}{(-1)} + 4 = -1 - 1 + 4 = 2$

$$y = (-1) - \left(\frac{1}{-1}\right) + 1$$
$$= -1 + 1 + 1$$
$$= 1$$

$\therefore$ Point of intersection $= (2, 1)$

Marking Scale	Marker's Comments
2 marks : Correct Solution 1 mark : Substituting curve D into curve E with some progress towards correct answer.	* Candidates whom tried to make curve D into cartesian form and did not substitute in curve E were not given any marks as it did not progress towards correct answer. * Candidates whom were able to get the result $t^3 + 2t^2 + 3t + 2 = 0$ should also explain why $t^2 + t + 2 = 0$ is not valid as discriminant is negative. * Some candidates did not answer the 2nd part of the question i.e. find the coordinates of point of intersection.

b) i) LHS: $\dfrac{1}{(n+1)!} - \dfrac{n+1}{(n+2)!}$

$= \dfrac{n+2}{(n+2)(n+1)!} - \dfrac{(n+1)}{(n+2)!}$

$= \dfrac{n+2-n-1}{(n+2)!}$

$= \dfrac{1}{(n+2)!}$

$=$ RHS.

Marking Scale	Marker's comments
1 mark = Correct solution	* Majority of candidates did well in this question.

(ii) Test for $n=1$

$LHS = \frac{1}{2!}$

$= \frac{1}{2}$

$RHS: 1 - \frac{1}{(1+1)!}$

$= 1 - \frac{1}{2!}$

$= 1 - \frac{1}{2}$

$= 1/2$

$\therefore LHS = RHS$

$\therefore$ True for $n=1$

Assume the statement is true for $n=k$, $k \in Z^+$

i.e. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$

Hence to prove the statement is true for $n=k+1$

RTP: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$

$LHS = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$ (From assumption)

$= 1 - \left(\frac{1}{(k+1)!} - \frac{k+1}{(k+2)!}\right)$

$= 1 - \frac{1}{(k+2)!} = RHS$

From (i)

$\therefore$ Hence by the principle of Mathematical Induction, the statement is proven true for $n \in \mathbb{Z}^+$

Marking Scale	Marker's comments
3 marks: for full proof clearly showing true for $n=1$ and for $n=k+1$ when assumed true for $n=k$	* Done well by majority of candidates.
2 marks: for proving true for $n=1$ and for progress towards proof for $n=k+1$	* Candidates whom used (i) should show step by step how they used it. i.e $1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}$ $=1-\left(\frac{1}{(k+1)!}-\frac{k+1}{(k+2)!}\right)$ $=1-\frac{k}{(k+2)!}$ (From (i))
1 mark: For proving for $n=1$ correctly and no further progress.	* Candidates should always mention when they used the assumption In proving $n=k+1$ i.e. by assumpti

c) i) At point $B(r, 2)$, sub into $x = \sqrt{4+3y^2}$

$$r = \sqrt{4+3(2)^2}$$
$$= \sqrt{4+12}$$
$$= \sqrt{16}$$
$$= 4$$

$$V = \text{Volume of cylinder} + \pi \int_2^h x^2 \, dy$$

$$= \pi r^2 h + \pi \int_2^h \left(\sqrt{4+3y^2}\right)^2 dy$$

$$= \pi \times 4^2 \times 2 + \pi \int_2^h 4 + 3y^2 \, dy$$

$$= 32\pi + \pi \left[4y + y^3\right]_2^h$$

$$= 32\pi + \pi \left[(4h + h^3) - (8 + 8)\right]$$

$$= 32\pi + 4h\pi + h^3\pi - 16\pi$$

$$= (h^3 + 4h + 16)\pi \text{ units}^3$$

Marking Scale:	Marker's comments
4 marks: Correct Solution	* Mostly done well by candidates.
3 marks: Finding the value of r, and significant progress towards the correct result.	* Some candidates did fudge their answer to fit the result which they were penalised for.
2 marks: Finding the value of r, and some progress towards the correct result.	

1 mark: Find the value of r.

(ii) $V = (h^3 + 4h + 16)\pi$ (From (i))

$$\frac{dV}{dh} = (3h^2 + 4)\pi$$

At $h = 3$

$$\frac{dV}{dh} = (3(3)^2 + 4)\pi$$

$$= 31\pi$$

$$\frac{dh}{dV} = \frac{1}{31\pi}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{31\pi} \times -2\pi$$

$= -\frac{2}{31}$ units / seconds

OR

$= \frac{2}{31}$ units / second decreasing.

Marking Scale	Marker's Comments
→ 2 marks : Correct answer with correct answer	→ Candidates were penalised half a mark if they did not have negative for $\frac{dV}{dt}$ or state it is decreasing with positive result for $\frac{dh}{dt}$
→ 1 mark : Working towards correct solution.	

→ Some candidates made arithmetic errors which they need to take more care.

(ii) When $h=1$, the water remained is in the cylindrical part only.

$\therefore \; V = \pi r^2 h$

$\frac{dV}{dh} = \pi \times r^2$

$= \pi \times 4^2 = 16\pi$

$\therefore \; \frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}}$

$= \frac{-2\pi}{16\pi}$

$= -\frac{1}{8}$ units / second

or $\frac{1}{8}$ units / second decrease.

Marking Scale:	Marker's comments
1 mark: Correct answer.	* Many candidates did poorly in this question as they didnt recognise $h=1$ is only in the cylindrical part which means the $\frac{dV}{dh}$ will be different calculation to part (i). * If candidates were penalised half mark in part (i) for not recognising $\frac{dV}{dt}$ is negative, no further penalt was applied in (ii).

Question 14 Solutions

(a) (i) Given DE: $\frac{dI}{dt} = RI$

Possible solution: $I = 5e^{Rt}$

$$\frac{dI}{dt} = R.5e^{Rt}$$
$$= RI$$

$\therefore$ This is a solution.

Alternative method: Integration by variables separable.

[Well answered by most candidates. Many integrated.]

(ii) It means that the number of people infected will double every day.

[Nearly every candidate received the mark for this part.]

(iii) Using model B, $I_0 = 5$, and $I_1 = 10$.

Using model A: For $n = 1$

$$10 = 5e^{R\times 1}$$
$$2 = e^R$$
$$\therefore\ R = \ln 2$$

[Most did well here, but some failed to connect the two models.]

(iv) Model A is better because it may be used for fractional days.

OR

Model B is better because the number of people must be integral.

OR

Any plausible argument.

[Very few failed to give a reasonable answer.]

(b) $\overrightarrow{r_P}(t) = 15t\hat{\imath} + (60 - 5t^2)\hat{\jmath}$

$\overrightarrow{r_Q}(t) = 25\cos\theta\, t\hat{\imath} + (25\sin\theta\, t - 5t^2)\hat{\jmath}$

At the point of collision, horizontal (and vertical) components are equal.

Hence $15t = 25t\cos\theta$ $\qquad$ $60 - 5t^2 = 25t\sin\theta - 5t^2$

$\theta = \cos^{-1}\frac{3}{5} = 53°\,8'$ $\qquad$ $t\sin\theta = \frac{60}{25}$

$\therefore t = 3$ sec

We seek to find vertical velocity at this time:

$$\dot{y} = 25\sin\theta - 10t = -10$$

Hence the particle is falling.

[Many were able to find the angle, but only a small percentage could prove "falling".]

(c) $I = \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$ Substitute $x = \tan\theta$ when $x = 0, \theta = 0$

$dx = \sec^2\theta\, d\theta$ $x = 1, \theta = \pi/4$

$$I = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 x}} \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta} \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec\theta\, \theta\, d\theta \qquad (*)$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} d\theta$$

$$= [\ln(\sec\theta + \tan\theta)]_0^{\frac{\pi}{4}}$$

$$= \ln(1 + \sqrt{2})$$

[About 50% achieved this result, though some got only as far as (*).]

(ii) Area of first rectangle $A_1 = \frac{1}{n} \times \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{n^2+1}}$

second rectgangle $A_2 = \frac{1}{n} \times \frac{1}{\sqrt{1+\frac{4}{n^2}}} = \frac{1}{\sqrt{n^2+4}}$

k-th rectangle $A_k = \frac{1}{n} \times \frac{1}{\sqrt{1+\frac{k^2}{n^2}}} = \frac{1}{\sqrt{n^2+k^2}}$

Hence the result is the sum of all n such areas.

[Most had no trouble with this. Several attempted to use the trapezoidal rule, and failed.]

(iii) $\lim_{n\to\infty} A = \int_0^1 \frac{1}{\sqrt{1+x^2}} dx = \ln(1 + \sqrt{2})$ from (i).

[Most who attempted this received the mark, some by "ecf".]