SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Tick next to your teacher's name

AF		PSP	
JM		JT	
JJ		MZ	
WB			

2025

YEAR 12
HSC TASK 4
TRIAL HSC

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Only NESA approved calculators may be used
- A reference sheet is provided with this paper
- All answers, unless otherwise stated, should be left in simplified, exact form
- Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations as answers that rely totally on calculator technology may not necessarily receive full marks.

Total Marks: 70

Section I – 10 marks (pages 2 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8 – 16)

- Attempt all Questions in Section II
- Allow about 1 hour and 45 minutes for this section

Examiner:
External Examiner

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

Question 1

A 1 litre tub of ice cream is left out in the open during a lesson. Its temperature T (in ^oC) after t minutes follows Newton's Law of Cooling, as according to

$$T = 18 - 18e^{-2t}.$$

What is its temperature after 2 minutes, to the nearest whole degree?

(A) 18 ^oC

(B) 17 ^oC

(C) 16 ^oC

(D) 15 ^oC

Question 2

What is the angle between the vectors $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$?

(A) $\cos^{-1}(0.06)$

(B) $\cos^{-1}(0.08)$

(C) $\cos^{-1}(0.6)$

(D) $\cos^{-1}(0.8)$

Question 3

What is the remainder when $P(x) = x^3 + 3x^2 - x + 4$ is divided by $(x^2 + 1)$?

(A) $x + 3$

(B) $x - 3$

(C) $2x + 1$

(D) $-2x + 1$

Question 4

Simplify the expression $\dfrac{\sin\theta + \sin\alpha}{\cos\theta + \cos\alpha}$.

(A) $\tan\left(\dfrac{\theta + \alpha}{2}\right)$

(B) $\cot\left(\dfrac{\theta + \alpha}{2}\right)$

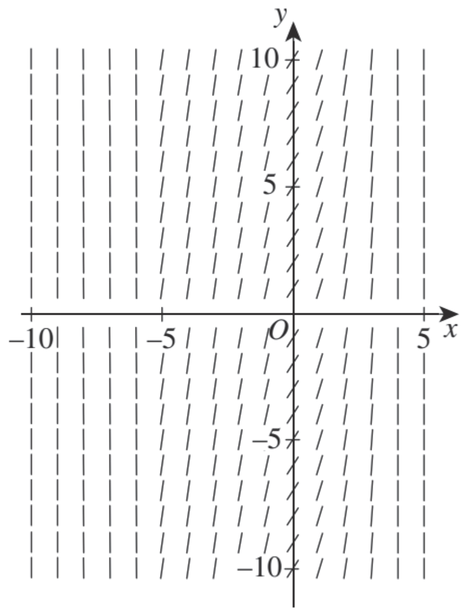
(C) $\tan\left(\dfrac{\theta - \alpha}{2}\right)$

(D) $\cot\left(\dfrac{\theta - \alpha}{2}\right)$

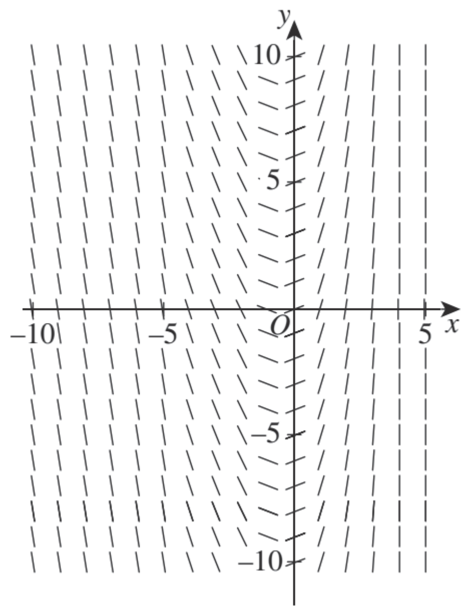
Question 5

Which of the following diagrams best represents the direction field for the differential equation $\dfrac{dy}{dx} = x + e^x$?

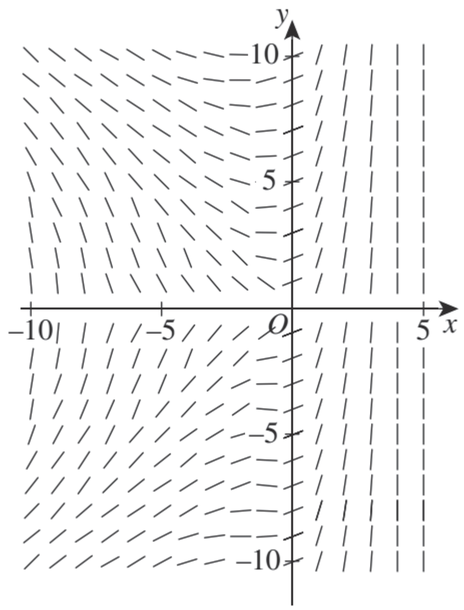
(A)



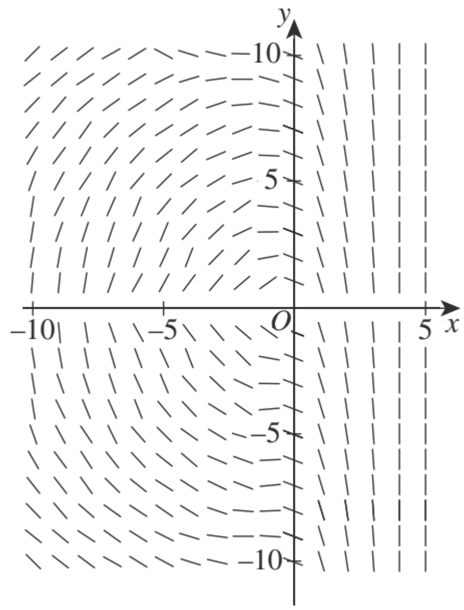
(B)



(C)

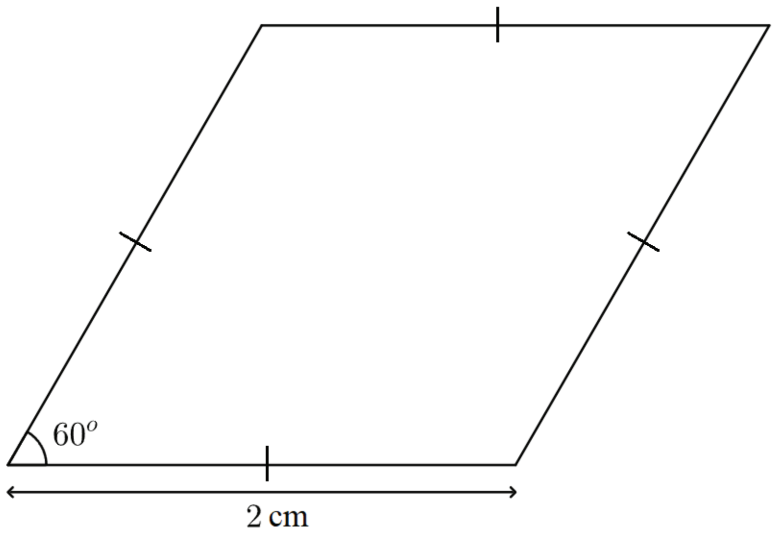


(D)



Question 6

In the rhombus shown, the angle at one of the vertices is 60^o.



What is the minimum number of points that should be randomly placed into the rhombus to guarantee that there is at least two points that are spaced at most 1 cm apart?

(A) 6

(B) 7

(C) 8

(D) 9

Question 7

The function f given by $f(x) = \ln[(x+2)^2]$ has an inverse function $g = f^{-1}$ defined when f is restricted to the domain $x > -2$. What is the value of $g(f(x))$ when $x < -2$?

(A) x

(B) $x - 4$

(C) $4 - x$

(D) $-4 - x$

Question 8

The population, P, of animals in an environment in which there are scarce resources is increasing such that

$$\frac{dP}{dt} = P(100 - P), \text{ where } t \text{ is time.}$$

The initial population is 20 animals.

Which of the following is true?

(A) $P = 100 - 80e^{100t}$

(B) The population is increasing most rapidly when $t = 50$.

(C) The population is increasing most rapidly when $P = 50$.

(D) The maximum population is $P = 50$.

Question 9

What is the coefficient of x in $\left(x - \frac{2}{x^2}\right)^7$?

(A) 84

(B) -84

(C) 280

(D) -280

Question 10

What is the coefficient of x^{293} in the expansion of $(x-1)(x^2-2)(x^3-3)\ldots(x^{24}-24)$?

(A) -7

(B) 13

(C) 20

(D) 28

Section II

Attempt Questions 11 – 14
Allow 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet.
Extra writing booklets are available.
In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE writing booklet.

(a) What values of t satisfy $|1-t|>3$? **1**

(b) Let α, β, γ be the roots to $-2x^3-4x^2-9x+6=0$. Find the value of:

(i) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$; **1**

(ii) $\alpha^2+\beta^2+\gamma^2$. **2**

(c) Solve $\sin\theta+\cos\theta=-1$ for $0\le\theta\le 2\pi$. **2**

(d) Solve $\dfrac{1+x}{x}\ge 2x+1$ for x. **3**

(e) Use the substitution $u=x+1$ to evaluate $\displaystyle\int_0^3 \frac{x-2}{(x+1)\sqrt{x+1}}dx$. **3**

(f) Find the volume generated when the region bounded by the $x-$axis and $y=\sin 2x$ between $x=\dfrac{\pi}{4}$ and $x=\dfrac{\pi}{3}$ is rotated about the $x-$axis to form a solid. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Prove, using mathematical induction, for all $n \in \mathbb{Z}^+$, **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

(b) It is known that $\frac{3}{5}$ of adult Australians take part in some form of organised sport. **2**

A marketing company is undertaking a survey on consumer preferences for a sports clothing manufacturer.

The marketing company requires a sample of sufficient size.

The company needs the probability to be 90% that a member of the sample, who takes part in organised sport, is within 0.025 of $\frac{3}{5}$.

The size of the sample is S.

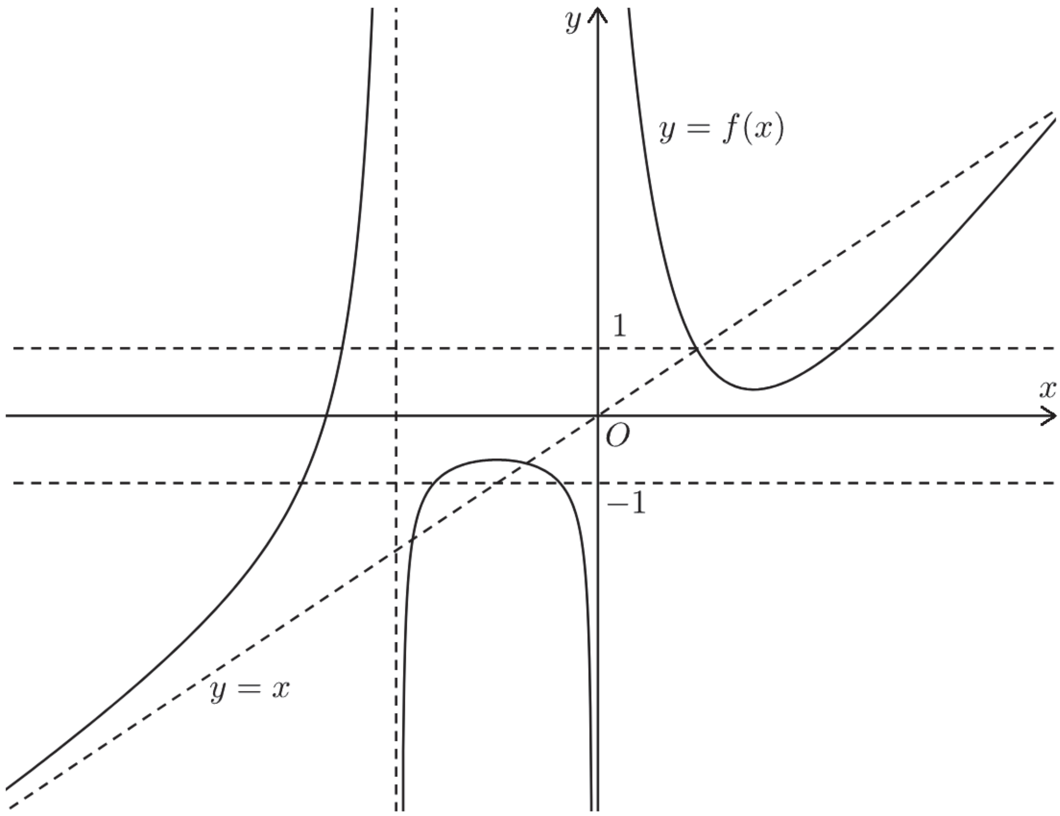
What is the minimum value of S (to the nearest ten people)?

You may assume that the sample size is sufficient for the normal approximation to the binomial distribution to be used (without continuity correction).

You are given that if $Z \sim N(0,1)$ then $P(Z < 1.285) = 0.9$ and $P(Z < 1.645) = 0.95$

Question 12 continues on page 11

(c) The diagram shows the graph of $y = f(x)$. The line $y = x$ is an asymptote.



On the sheet provided, sketch the following graphs, showing all relevant information.

(i) $y = f(|x|)$; **1**

(ii) $y = f(x) - x$; **2**

Question 12 continues on page 12

Question 12 (continued)

(d) A particle moves along a path on the time interval $0 \leq t \leq T$ as according to

$$\underset{\sim}{r}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 4\sin t \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

(i) Prove the particle moves on a circle, stating the centre and radius of this circle. **2**

(ii) The particle completes a total of 3 full revolutions before coming to a stop. **1**

What is the value of T?

(iii) Prove that the acceleration vector is always perpendicular to the velocity vector **1**

of the particle.

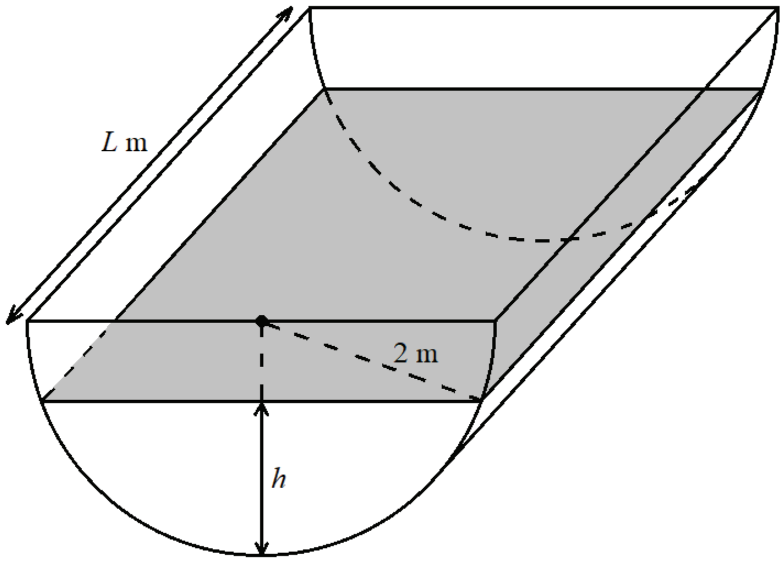
(e) Solve for $f(x)$ if it satisfies the equation **3**

$$\int x^3 [f(x)]^2 dx = f(x) - x^4 + C.$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The cross-section of a L metre long tank is a semi-circle of radius 2 m. The ends are made from the glass and the curved section is made from copper.



The depth of the water is h metres and the volume of water is V m^3 at t minutes.

(i) Show that $V = \left[4\cos^{-1}\left(\dfrac{2-h}{2}\right) + (h-2)\sqrt{h(4-h)}\right] L.$ **3**

(ii) Show that the area of water touching copper at h metres deep is **1**

$$A = 4L\cos^{-1}\left(1 - \frac{h}{2}\right).$$

(iii) The water evaporates proportional to the the rate of change of the area contacting the copper and inversely proportional to the rate of change of height, as given by **3**

$$\frac{dV}{dt} = -k\frac{dA}{dt}\left(\frac{dh}{dt}\right)^{-1}.$$

Prove that the rate of change of the depth of water satisfies

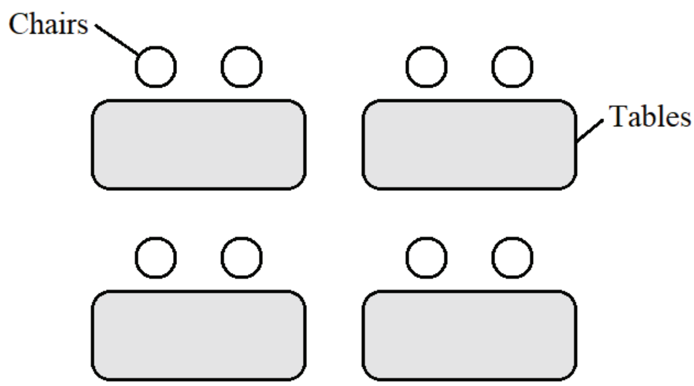
$$\frac{dh}{dt} = \frac{2k}{h^2 - 4h}.$$

(iv) Initially, the tank is completely full. After 3 minutes, the depth halves. How much longer will it take until the tank is completely dry? **2**

Question 13 continues on page 14

Question 13 (continued)

(b) A physics classroom has 4 double desks at the front of the room as shown.



Suppose 5 students, Taf, Jon, Lou, Ale and Bhu wish to sit themselves randomly. Find:

(i) the total number of ways they can sit; **1**

(ii) the probability that Jon and Lou sit on a double desk together, while Taf and Ale sit on another double desk together, given that Bhu sits on his own. **2**

(c) (i) Explain why $\sin\theta = \sin 2\theta\cos\theta - \sin\theta\cos 2\theta$. **1**

(ii) Hence, prove that for $n \in \mathbb{Z}^+$, **2**

$$\sum_{k=0}^{n} \tan(2^k x)\sec(2^{k+1}x) = \tan(2^{n+1}x) - \tan x.$$

End of Question 13

Question 14 (16 marks) Use a SEPARATE writing booklet.

(a) A particular solution to the differential equation **3**

$$\frac{dy}{dx} = \frac{x}{\tan y\ (x^2+1)},$$

where $x \geq 0$ and $-\frac{\pi}{2} < y < 0$, passes through $\left(1, -\frac{\pi}{4}\right)$.

Determine this solution in the form $x = f(y)$.

Leave your answer in simplified form.

(b) (i) Find the unit vector $\hat{\underset{\sim}{u}}$ where $\underset{\sim}{u} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. **1**

(ii) Hence, using vector methods, find the point on the circle $(x-2)^2 + (y-2)^2 = 2$ that is closest to the circle $(x-5)^2 + (y+3)^2 = 6$. **2**

(c) Let S be defined as **4**

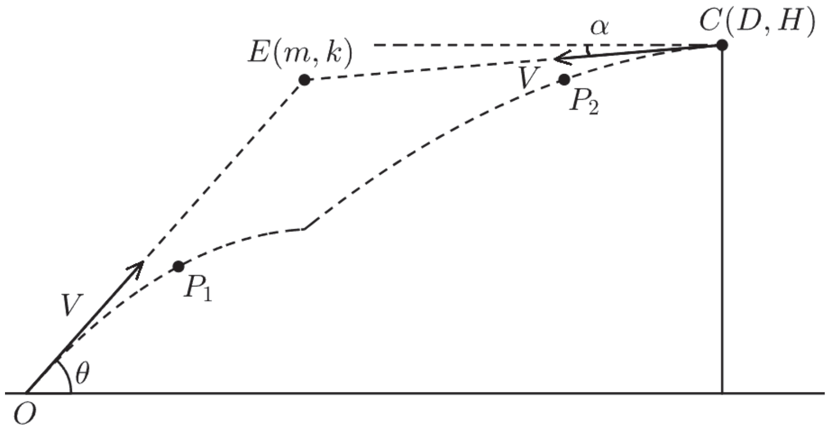
$$S = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000}.$$

By considering $\left(\frac{x}{1+x}\right) S$, or otherwise, find the coefficient of x^{50}.

Question 14 continues on page 16

Question 14 (continued)

(d) Two projectiles, P_1 and P_2 are launched from the origin O and $C(D, H)$ respectively.



Both are shot at speed V m/s towards a chosen point $E(m, k)$ so that the two particles hit each other. Let the angles that the initial velocities of P_1 and P_2 makes with the horizontal be θ and α respectively.

You may assume that the position of a particle launched from the initial position (r, h) with speed U and angle ϕ to the horizontal is given by

$$\underset{\sim}{r}(t) = (Ut\cos\phi + r)\underset{\sim}{i} + \left(Ut\sin\phi - \frac{1}{2}gt^2 + h\right)\underset{\sim}{j} \qquad \text{(Do \textbf{NOT} prove this.)}$$

where g is the acceleration due to gravity.

(i) Prove, using the position vector of each particle, that the point E is equidistant from O and C. **4**

(ii) Hence, prove that the time to impact is $t = \dfrac{\sqrt{m^2 + k^2}}{V}$. **2**

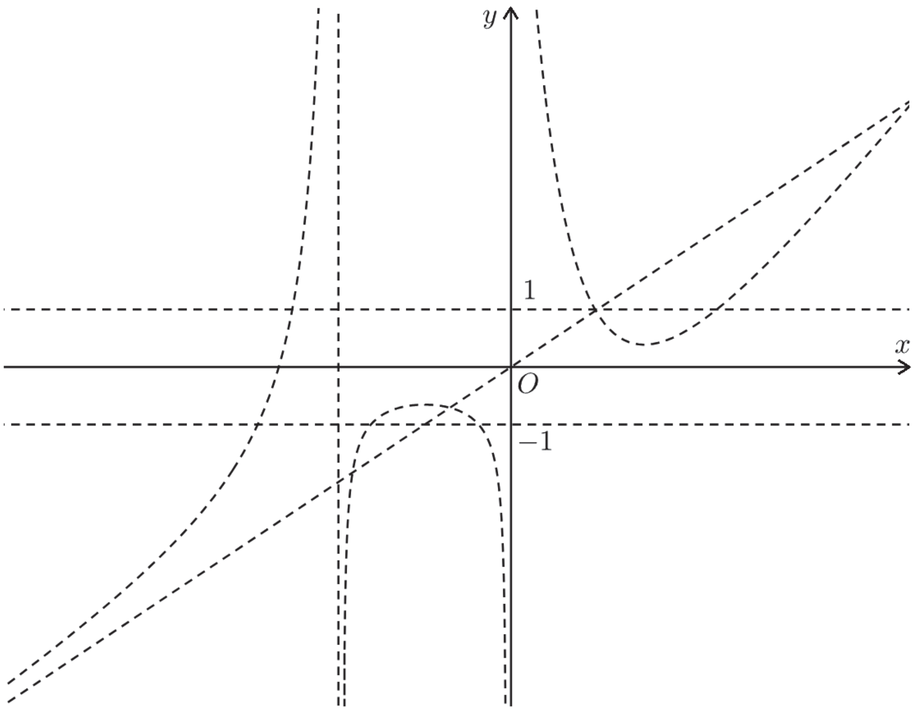
End of Paper

Question 12 (c) **Place inside booklet for Q12 at the end of the exam**

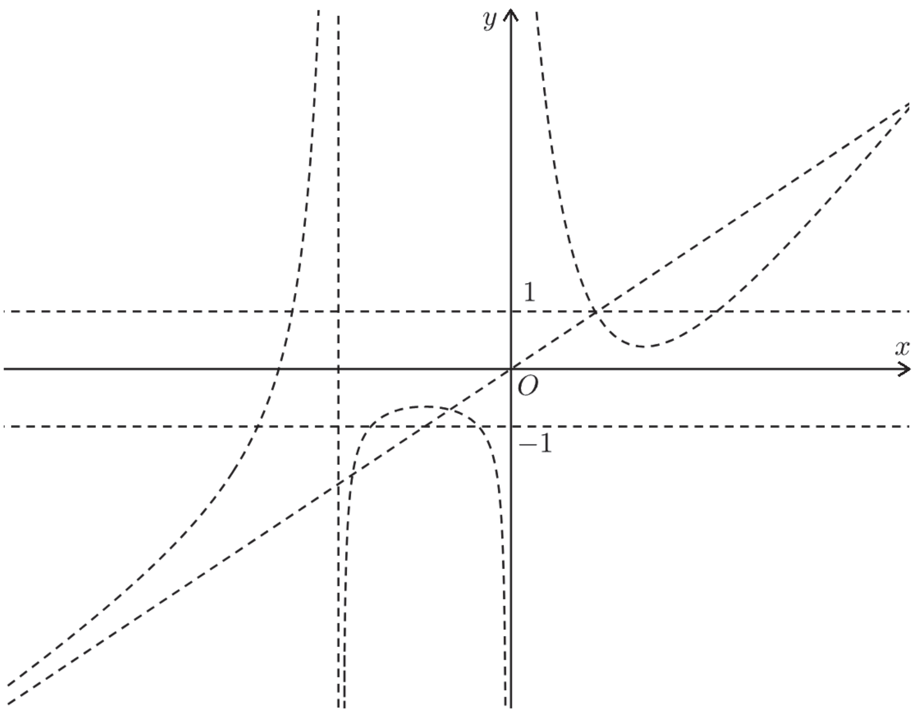
The diagrams show the graph of $y = f(x)$. The line $y = x$ is an asymptote.

Sketch the following graphs, showing all relevant information.

(i) $y = f(|x|)$ **2**



(ii) $y = f(x) - x$ **2**



SYDNEY BOYS HIGH SCHOOL

2025

YEAR 12

HSC TASK 4

Mathematics Extension 1

Sample Solutions

NOTE:

This process of checking your mark is about reading the solutions and the comments. Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown some 'working' does not justify that your solution is worth any marks.

Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

MC Answers

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
A	C	D	A	B	D	D	C	A	B

Question 1

A 1 litre tub of ice cream is left out in the open during a lesson. Its temperature T (in ^oC) after t minutes follows Newton's Law of Cooling, as according to

$$T = 18 - 18e^{-2t}.$$

What is its temperature after 2 minutes, to the nearest whole degree?

(A) $18\ ^oC$

(B) $17\ ^oC$

(C) $16\ ^oC$

(D) $15\ ^oC$

$$T(2) = 18\left(1 - e^{-4}\right) \approx 17.67$$

A	150
B	1
C	2
D	0

Question 2

What is the angle between the vectors $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$?

(A) $\cos^{-1}(0.06)$

(B) $\cos^{-1}(0.08)$

(C) $\cos^{-1}(0.6)$

(D) $\cos^{-1}(0.8)$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \sqrt{5} \times \sqrt{20}\cos\theta$$

$$\therefore\ -2 + 8 = 10\cos\theta \Rightarrow \cos\theta = 0.6$$

A	2
B	1
C	145
D	5

Question 3

What is the remainder when $P(x) = x^3 + 3x^2 - x + 4$ is divided by $(x^2 + 1)$?

(A) $x + 3$

(B) $x - 3$

(C) $2x + 1$

(D) $-2x + 1$

$$x^3 + 3x^2 - x + 4 = x\left(x^2 + 1\right) + 3\left(x^2 + 1\right) - 2x + 1$$

A	2
B	1
C	5
D	145

Question 4

Simplify the expression $\dfrac{\sin\theta + \sin\alpha}{\cos\theta + \cos\alpha}$.

(A) $\tan\left(\dfrac{\theta + \alpha}{2}\right)$

(B) $\cot\left(\dfrac{\theta + \alpha}{2}\right)$

(C) $\tan\left(\dfrac{\theta - \alpha}{2}\right)$

(D) $\cot\left(\dfrac{\theta - \alpha}{2}\right)$

A	110
B	13
C	22
D	7

From the Reference sheet:

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

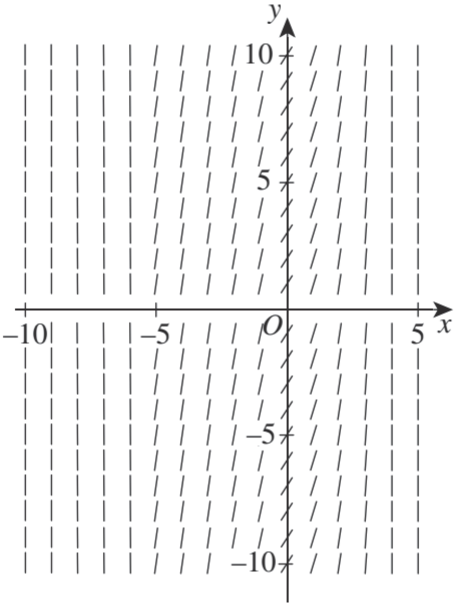
This means the answer must be options A or C i.e. tan( )

Using $\theta = A + B$ & $\alpha = A - B$ then $A = \dfrac{\theta + \alpha}{2}$.

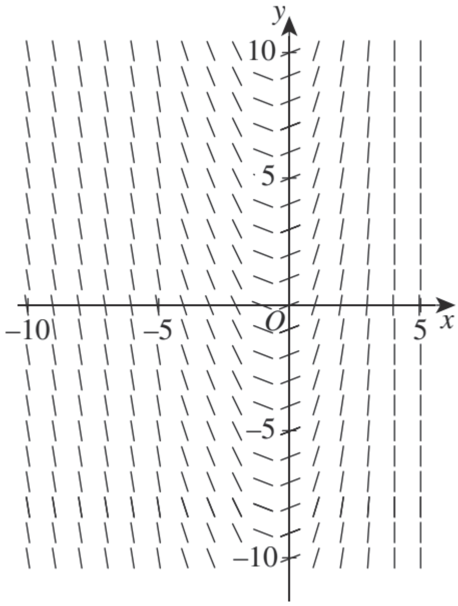
Question 5

Which of the following diagrams best represents the direction field for the differential equation $\dfrac{dy}{dx} = x + e^x$?

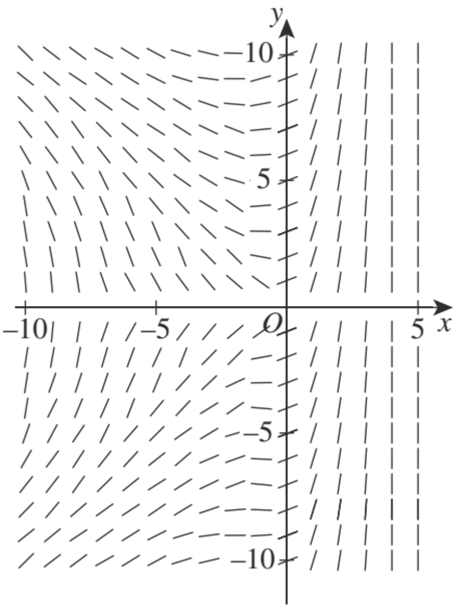
(A)



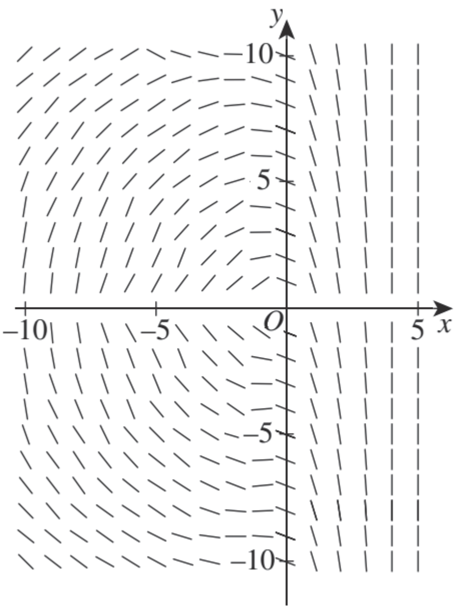
(B)



(C)



(D)



For a particular value of x, y' is independent of y.

$\therefore$ options A or B.

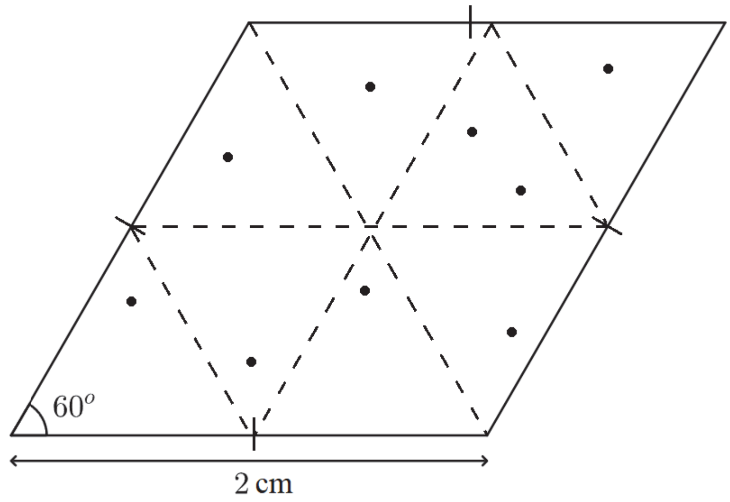
At $x = -1$, $y' < 0$

$\therefore$ option B.

A	17
B	120
C	16
D	0

Question 6

In the rhombus shown, the angle at one of the vertices is 60^o.



What is the minimum number of points that should be randomly placed into the rhombus to guarantee that there is at least two points that are spaced at most 1 cm apart?

(A) 6

(B) 7

(C) 8

(D) 9

Divide the rhombus into 8 equilateral triangles.

In each triangle, two points can be spaced a maximum of 1 cm.

By the Pigeonhole Principle, at least 9 points are required to ensure that there is at least one triangle that contains at least two points.

$\therefore$ option D

A	34
B	25
C	29
D	65

Question 7

The function f given by $f(x) = \ln[(x+2)^2]$ has an inverse function $g = f^{-1}$ defined when f is restricted to the domain $x > -2$. What is the value of $g(f(x))$ when $x < -2$?

(A) x

(B) $x - 4$

(C) $4 - x$

(D) $-4 - x$

Let $h(x) = (x+2)^2$.

Using the symmetry of the parabola in the line $x = -2$:

If $a < -2$, then the x-value, b, equidistant from -2, but with $b > -2$ is $b = -2 + (-2 - a) = -4 - a$

Now $h(a) = h(b)$ by symmetry and so $g(f(b)) = b$

A	25
B	26
C	41
D	60

Question 8

The population, P, of animals in an environment in which there are scarce resources is increasing such that

$$\frac{dP}{dt} = P(100 - P), \text{ where } t \text{ is time.}$$

The initial population is 20 animals.

Which of the following is true?

(A) $P = 100 - 80e^{100t}$

(B) The population is increasing most rapidly when $t = 50$.

(C) The population is increasing most rapidly when $P = 50$.

(D) The maximum population is $P = 50$.

$\dfrac{dP}{dt}$ is a concave down parabola with the maximum turning point at $P = 50$.

A	5
B	15
C	118
D	15

Question 9

What is the coefficient of x in $\left(x - \frac{2}{x^2}\right)^7$?

(A) 84

(B) -84

(C) 280

(D) -280

Using a binomial expansion,

$$\begin{aligned}\left(x - \frac{2}{x^2}\right)^7 &= \sum_{r=0}^{7} \binom{7}{r} x^{7-r} \left(-\frac{2}{x^2}\right)^r \\ &= \sum_{r=0}^{7} \binom{7}{r} (-2)^r x^{7-r} x^{-2r} \\ &= \sum_{r=0}^{7} \binom{7}{r} (-2)^r x^{7-3r}\end{aligned}$$

$\therefore$ For the coefficient of x, $7 - 3r = 1 \Rightarrow r = 2$.

$\therefore$ Coefficient is $\binom{7}{2}(-2)^2 = 84$.

$\therefore$ option A

A	132
B	9
C	5
D	7

Question 10

What is the coefficient of x^{293} in the expansion of $(x-1)(x^2-2)(x^3-3)\ldots(x^{24}-24)$?

(A) -7

(B) 13

(C) 20

(D) 28

$$1+2+\ldots+24=\frac{24\times 25}{2}=300$$

To find the coefficient of x^{293} it is enough to choose the numerical coefficients in the brackets that give x^7.

$7 = 7 + 0$: $(-7) \times 1 = -7$	From $(x^7 - 7)$
$7 = 1 + 6$: $(-1) \times (-6) = 6$	From $(x - 1)(x^6 - 6)$
$7 = 2 + 5$: $(-2) \times (-5) = 10$	From $(x^2 - 2)(x^5 - 5)$
$7 = 3 + 4$: $(-3) \times (-4) = 12$	From $(x^3 - 3)(x^4 - 4)$
$7 = 1 + 2 + 4$: $(-1) \times (-2) \times (-4) = -8$	From $(x - 1)(x^2 - 2)(x^4 - 4)$

$\therefore\ -7 + 6 + 10 + 12 + (-8) = 13$

A	24
B	52
C	47
D	28

End of Section I

Breakdown of MC Marks

Correct	Freq.
10	6
9	19
8	38
7	45
6	25
5	14
4	4
3	1
2	0
1	0
0	0

Question 11 Solutions **14 marks**

(a) What values of t satisfy $|1-t| > 3$? **1**

Before starting anything, note that $|1-t| = |t-1|$. So, we can instead solve $|t-1| > 3$.

Algebraic method:

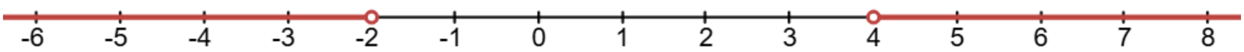
$$|t-1| > 3 \quad \rightarrow \quad \begin{cases} t-1 > 3, & t > 1 \\ t-1 < -3, & t < 1 \end{cases}$$

$\therefore \quad t > 4,\ t < -2$

Number line method:

Recognise that the inequality $|t-1| > 3$ describes the values that are "greater than 3 units from 1".

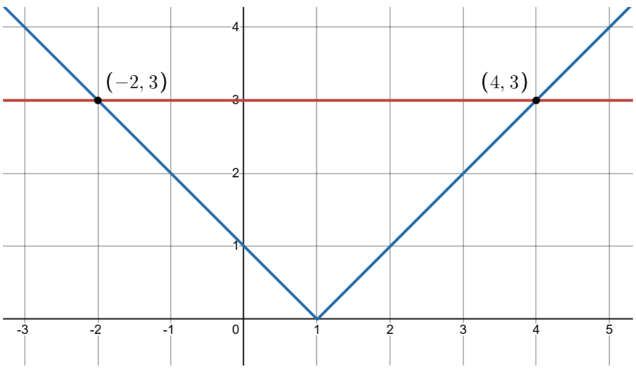
On a number line, we can represent this as:



from which we can observe the solution to be $t > 4, t < -2$.

Graphical method:

Sketching both $y = |t-1|$ and $y = 3$ on a coordinate plane gives the following:



By comparing the y–values of the two graphs, we can see that $|t-1| > 3$ when $t > 4, t < -2$.

1 mark for the correct answer. (no half marks)

Very well done.

(b) Let α, β, γ be the roots to $-2x^3 - 4x^2 - 9x + 6 = 0$. Find the value of:

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$; **1**

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} \\ &= \frac{\frac{-9}{-2}}{-\frac{6}{-2}} \\ &= \frac{3}{2}\end{aligned}$$

0.5 marks for attempting to use sums/products of roots.

0.5 marks for the correct answer.

(ii) $\alpha^2 + \beta^2 + \gamma^2$. **2**

$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) \\ &= \left(-\frac{-4}{-2}\right)^2 - 2\left(\frac{-9}{-2}\right) \\ &= -5\end{aligned}$$

1 mark for expressing the sum of squares in terms of sums/products of roots.

1 mark for the correct answer.

Both parts were done very well.

Commons errors were incorrect substitution of values and incorrect signs of products.

(c) Solve $\sin\theta + \cos\theta = -1$ for $0 \le \theta \le 2\pi$. **2**

$$\sin\theta + \cos\theta = -1$$

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\cos\frac{\pi}{4}\sin\theta + \sin\frac{\pi}{4}\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\sin\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\theta + \frac{\pi}{4} = \frac{5\pi}{4}, \frac{7\pi}{4} \qquad \theta + \frac{\pi}{4} \in \left[\frac{\pi}{4}, \frac{9\pi}{4}\right]$$

$$\theta = \pi, \frac{3\pi}{2}$$

1 mark for correctly expressing the equation in terms of a single trigonometric term.

1 mark for finding all correct values for θ in the given domain.

Generally done ok.

A wide variety of methods were used, many of which introduced extraneous solutions or created null cases.

A common error was forgetting to check which solutions needed to be discarded or forgetting to check the null case for additional solutions.

(c) **Alternate method** (using an auxiliary angle):

Let $\sin\theta+\cos\theta=R\sin(\theta+\alpha)$ where $R>0$ and $\alpha\in[-\pi,\pi]$.

$$\begin{aligned}R\sin(\theta+\alpha)&=R\sin\theta\cos\alpha+R\sin\alpha\cos\theta\\&=\sin\theta+\cos\theta\end{aligned}$$

By equating the coefficients of $\sin\theta$ and $\cos\theta$ we get the equations:

1) $R\cos\alpha=1$
2) $R\sin\alpha=1$

Adding the squares of each equation gives $R^2\left(\cos^2\alpha+\sin^2\alpha\right)=1^2+1^2\to R=\sqrt{2},R>0$.

Dividing equation 2 by equation 1 gives $\tan\alpha=1\to\alpha=\dfrac{\pi}{4}$, since $\cos\alpha,\sin\alpha>0$.

Thus, we can rewrite the equation to be $\sqrt{2}\sin\left(\theta+\dfrac{\pi}{4}\right)=-1$.

$$\begin{aligned}\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)&=-1\\\sin\left(\theta+\frac{\pi}{4}\right)&=-\frac{\sqrt{2}}{2}\\\theta+\frac{\pi}{4}&=\frac{5\pi}{4},\frac{7\pi}{4}\\\theta&=\pi,\frac{3\pi}{2}\end{aligned}$$

Since $\sin\left(\theta+\dfrac{\pi}{4}\right)$ has a period of 2π, there are no other solutions for the given domain.

(c) **Alternate method** (t-method)**:**

Let $\tan\dfrac{\theta}{2}=t$, which allows us to substitute $\sin\theta=\dfrac{2t}{1+t^2}$ and $\cos\theta=\dfrac{1-t^2}{1+t^2}$.

$$\begin{aligned}\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}&=-1\\ 1+2t-t^2&=-1-t^2\\ 2t&=-2\\ t&=-1\end{aligned}$$

Solving $\tan\dfrac{\theta}{2}=-1$ gives the solution $\theta=\dfrac{3\pi}{2}$.

We also need to check the null case $\theta=\pi$ by substituting it into the original equation. This substitution makes the equation true and therefore the final solution is

$$\theta=\pi,\frac{3\pi}{2}.$$

(d) Solve $\dfrac{1+x}{x} \geq 2x+1$ for x. **3**

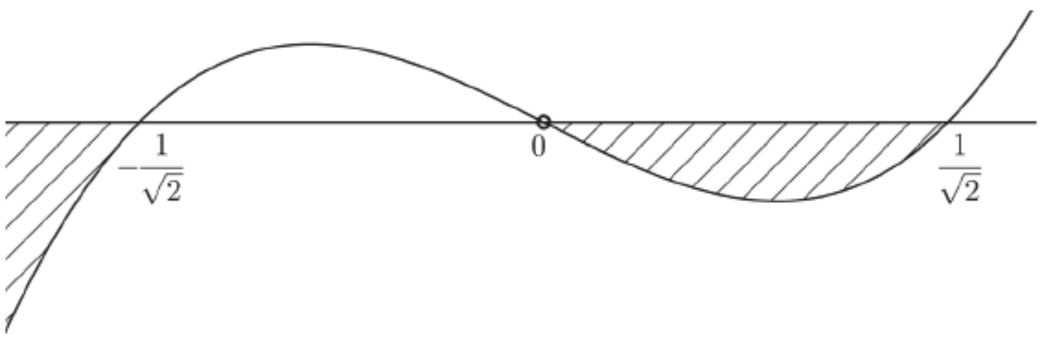
First, we note that $x \neq 0$.

We can then multiply both sides by x^2 and continue solving.

$$\frac{1+x}{x} \geq 2x+1$$
$$x(1+x) \geq x^2(2x+1)$$
$$x+x^2 \geq 2x^3+x^2$$
$$2x^3-x \leq 0$$
$$x(2x^2-1) \leq 0$$
$$x(\sqrt{2}x-1)(\sqrt{2}x+1) \leq 0$$

This gives us boundary values for the inequality at $x=0,\pm\dfrac{1}{\sqrt{2}}$.

Either by testing values or by sketching $y=x(\sqrt{2}x-1)(\sqrt{2}x+1)$



we can determine that the solution is $x \leq -\dfrac{1}{\sqrt{2}}, 0 < x \leq \dfrac{1}{\sqrt{2}}$.

1 mark for determining the correct boundary values for the inequality.

1 mark for determining the appropriate regions of values.

1 mark for recognising that $x \neq 0$.

Generally well done.

Many students forgot to note that $x \neq 0$.

(d) **Alternate method** (critical points and region testing)**:**

First, we note that $x \neq 0$ and is a critical point.

We then consider the corresponding equation, and solve to determine the other critical points:

$$\begin{aligned}
\frac{1+x}{x} &= 2x+1 \\
1+x &= x(2x+1) \\
1+x &= 2x^2+x \\
2x^2-1 &= 0 \\
\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right) &= 0 \\
x &= \pm\frac{1}{\sqrt{2}} \approx \pm 0.707
\end{aligned}$$

Therefore, we have possible solution regions of:

$$\left(-\infty,-\frac{1}{\sqrt{2}}\right),\left(-\frac{1}{\sqrt{2}},0\right),\left(0,\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},\infty\right)$$

Note also that, since the inequality is inclusive, the critical points except $x=0$ are included.

It suffices to check the truth value of the inequality at values $x=\pm 1,\pm\frac{1}{2}$.

By substituting into the original inequality, we find that it holds true at $x=-1$ and $x=\frac{1}{2}$.

Thus, we get the solution: $x \in \left(-\infty,-\frac{1}{\sqrt{2}}\right] \cup \left(0,\frac{1}{\sqrt{2}}\right]$.

(e) Use the substitution $u = x + 1$ to evaluate $\int_0^3 \frac{x-2}{(x+1)\sqrt{x+1}}dx$. **3**

Let $u = x+1$, $du = dx$.

At $x=3, u=4$, and $x=0, u=1$.

$$\int_0^3 \frac{x-2}{(x+1)\sqrt{x+1}}\,dx = \int_1^4 \frac{u-3}{u\sqrt{u}}\,du$$
$$= \int_1^4 \frac{1}{\sqrt{u}} - \frac{3}{u\sqrt{u}}\,du$$
$$= \left[2\sqrt{u} + \frac{6}{\sqrt{u}}\right]_1^4$$
$$= \left(2\sqrt{4} + \frac{6}{\sqrt{4}}\right) - \left(2\sqrt{1} + \frac{6}{\sqrt{1}}\right)$$
$$= -1$$

1 mark for performing the provided substitution correctly.

1 mark for finding the definite integral correctly.

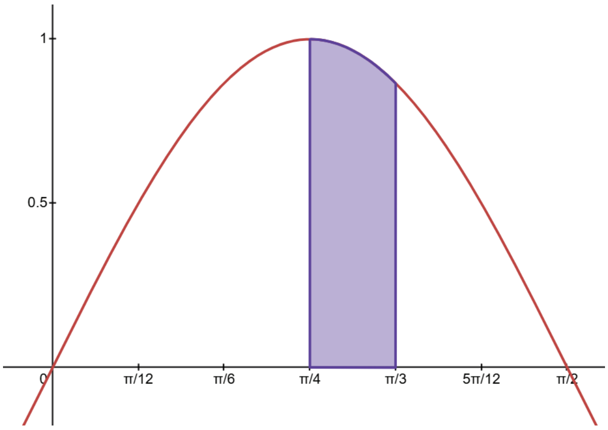
1 mark for the correct final answer, showing at least one line of substitution.

Generally done well.

Common errors occurred during the integration step (particularly on the second term) and surprisingly frequently in the transcription of the bounds.

(f) Find the volume generated when the region bounded by the $x-$axis and $y = \sin 2x$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ is rotated about the $x-$axis to form a solid. **2**

The region bounded is as follows:



By rotating around the x–axis we generate a collection of discs which each have a radius of $\sin 2x$ and a thickness of δx, where the value of x is dependent on where in the region the disc is taken.

By taking the continuous sum of all the discs for the region, or by using the reference sheet, we get the expression for volume:

$$
\begin{aligned}
V &= \pi \int_a^b \left(f(x)\right)^2 \, dx \\
&= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sin 2x)^2 \, dx \\
&= \frac{\pi}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 - \cos 4x \, dx \\
&= \frac{\pi}{2} \left[x - \frac{1}{4} \sin 4x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
&= \frac{\pi}{2} \left[\left(\frac{\pi}{3} - \frac{1}{4} \sin \frac{4\pi}{3} \right) - \left(\frac{\pi}{4} - \frac{1}{4} \sin \frac{4\pi}{4} \right) \right] \\
&= \frac{\pi}{2} \left(\frac{\pi}{12} + \frac{\sqrt{3}}{8} \right) \\
&= \frac{\pi^2}{24} + \frac{\sqrt{3}\pi}{16}
\end{aligned}
$$

1 mark for the correct integral expression for volume.

1 mark for correctly finding the definite integral.

Generally done well.

Most errors occurred during the integration step.

Question 12 Solutions 15 marks

(a) Prove, using mathematical induction, for all $n \in \mathbb{Z}^+$, **3**

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}.$$

Step 1: Prove it's true for $n=1$.

$$\begin{aligned}\text{LHS}&=\frac{1}{(1+1)!}\\&=\frac{1}{2}\\&=1-\frac{1}{2}\\&=1-\frac{1}{(1+1)!}\\&=\text{RHS}\end{aligned}$$

$\therefore$ It's true for $n=1$.

Step 2: Assure it's true for $n=k,\ k\in\mathbb{Z}^+$

i.e. $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\cdots+\dfrac{k}{(k+1)!}=1-\dfrac{1}{(k+1)!}$

Step 3: Prove it's true for $n=k+1,\ k\in\mathbb{Z}^+$

i.e. $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\cdots+\dfrac{k+1}{(k+2)!}=1-\dfrac{1}{(k+2)!}$

$$\begin{aligned}\text{LHS}&=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k+1}{(k+2)!}\\&=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}\\&=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}\qquad(\text{using assumption})\\&=1-\frac{k+2}{(k+2)!}-\frac{k+1}{(k+2)!}\\&=1-\frac{k+2-(k+1)}{(k+2)!}\\&=1-\frac{1}{(k+2)!}\\&=\text{RHS}\end{aligned}$$

$\therefore$ It's true for $n=k+1,\ k\in\mathbb{Z}^+$

Therefore, by mathematical induction, $2^{2n}+3n-1$ is divisible by 3, for all positive integers.

Extremely well done. Most students achieved full marks on this question.

(b) It is known that $\frac{3}{5}$ of adult Australians take part in some form of organised sport. **2**

A marketing company is undertaking a survey on consumer preferences for a sports clothing manufacturer.

The marketing company requires a sample of sufficient size.

The company needs the probability to be 90% that a member of the sample, who takes part in organised sport, is within 0.025 of $\frac{3}{5}$.

The size of the sample is S.

What is the minimum value of S (to the nearest ten people)?

You may assume that the sample size is sufficient for the normal approximation to the binomial distribution to be used (without continuity correction).

You are given that if $Z \sim N(0,1)$ then $P(Z < 1.285) = 0.9$ and $P(Z < 1.645) = 0.95$

Let X denote the number of people who take part in sports, so $X \sim B(S, 0.6)$.

Using normal approximation,

$$\hat{p} \sim N\left(0.6, \frac{0.6\times 0.4}{S}\right)$$

$$\mu = p = 0.6$$

$$\sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.24}{S}}$$

$$P(0.6 - 0.025 < \hat{p} < 0.6 + 0.025) = 0.9$$

$$P\left(\frac{0.575-0.6}{\sqrt{\frac{0.24}{S}}} < Z < \frac{0.625-0.6}{\sqrt{\frac{0.24}{S}}}\right) = 0.9$$

i.e. $$P\left(\frac{-0.025}{\sqrt{\frac{0.24}{S}}} < z < \frac{0.025}{\sqrt{\frac{0.24}{S}}}\right) = 0.9$$

Given $P(Z < 1.645) = 0.95$

$$\begin{aligned} P(0 < Z < 1.645) &= P(Z < 1.645) - P(Z < 0) \\ &= 0.95 - 0.5 \\ &= 0.45 \end{aligned}$$

$\therefore P(-1.645 < Z < 1.645) = 0.9$ (Symmetry)

$$\therefore \frac{-0.025}{\sqrt{\frac{0.24}{S}}} = 1.645$$

$$\therefore S = 0.24 \times \left(\frac{1.645}{0.025}\right)^2 \approx 1040$$

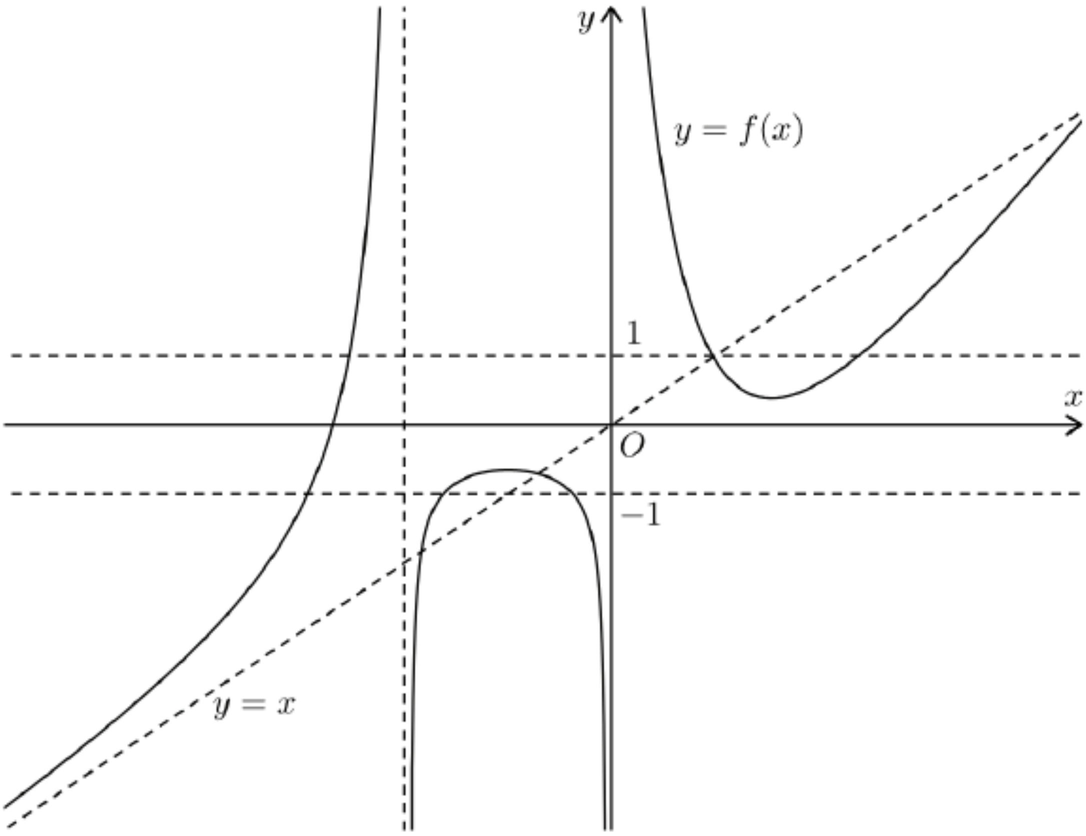
Extremely poorly done. Fewer than 10 students achieved full marks on this question.

A significant number of students did not attempt it.

Common error: used $P(Z < 1.285) = 0.9$

It is strongly recommended that students thoroughly revise this topic of Sample Proportions.

(c) The diagram shows the graph of $y = f(x)$. The line $y = x$ is an asymptote.

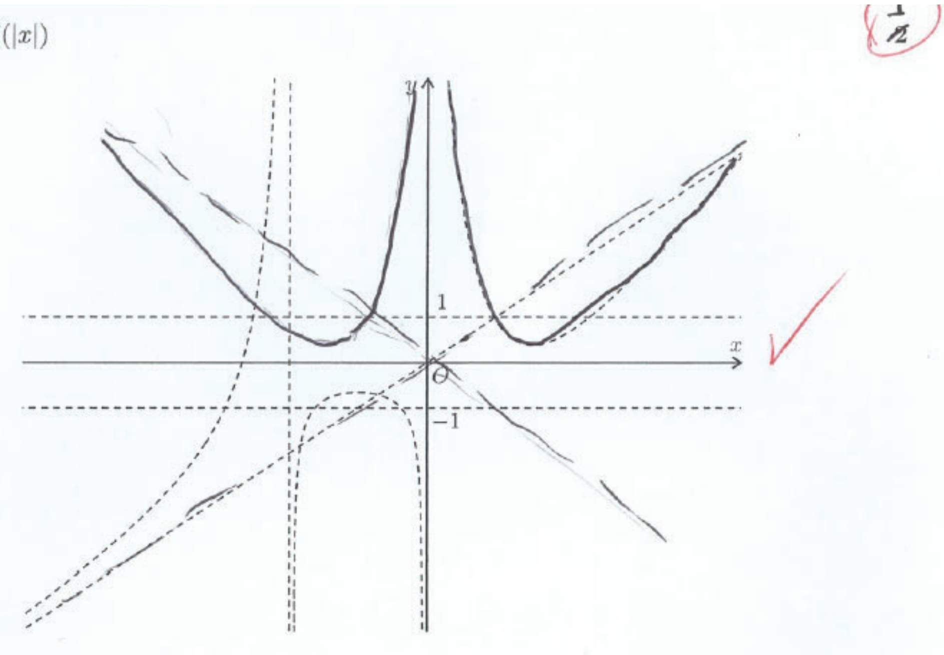


On the sheet provided, sketch the following graphs, showing all relevant information.

(i) $y = f(|x|)$; **1**

(ii) $y = f(x) - x$; **2**

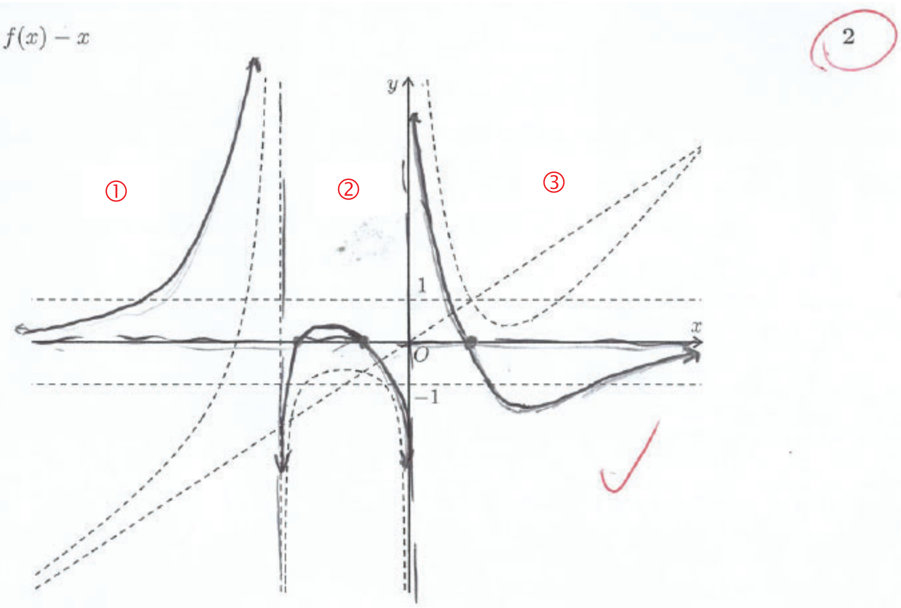
(i) $y = f(|x|)$



Extremely well done. Most students achieved 1 mark.

Marks were deducted if students only sketched the left side of the new curve.

(ii) $y = f(x) - x$



Generally well done

①: *Students need to demonstrate that the x-axis is the asymptote.*

②: *Students need to show that the new curve lies above the original curve.*

③: *Students need to demonstrate an appropriate turning point and show that the new curve approaches the x-axis.*

Three zeroes must be clearly and appropriately shown on the graph: two in the section 2 and one in the section 3.

(d) A particle moves along a path on the time interval $0 \leq t \leq T$ as according to

$$\underset{\sim}{r}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 4\sin t \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

(i) Prove the particle moves on a circle, stating the centre and radius of this circle. **2**

$$\underset{\sim}{r}(t) = \begin{pmatrix} 1+4\sin^2 t \\ 2-4\sin t\cos t \end{pmatrix}$$

$$= \begin{pmatrix} 1+4\times\frac{1}{2}(1-\cos 2t) \\ 2-2\times 2\sin t\cos t \end{pmatrix}$$

$$= \begin{pmatrix} 3-2\cos 2t \\ 2-2\sin 2t \end{pmatrix}$$

Let $x = 3-2\cos 2t$, $y = 2-2\sin 2t$

$$(x-3)^2 + (y-2)^2 = (2\cos 2t)^2 + (2\sin 2t)^2 = 4$$

Therefore, the particle moves on a circle with centre $(3,2)$ and radius 2.

Generally well done.

A few algebraic mistakes were noted, but ECF was accepted.

Multiple methods were accepted; however, the sample answer represents the quickest and easiest approach. Familiarity with double angle content is recommended.

A few students did not complete the question as they did not state the centre and radius of the circle.

(ii) The particle completes a total of 3 full revolutions before coming to a stop. **1**

What is the value of T?

$$\mathrm{T} = \frac{2\pi}{2}\times 3 = 3\pi$$

Generally well done.

Common incorrect answers: $6\pi, \frac{2}{3}\pi, \pi$

Part (d) continues on Page 6

(d) (continued)

(iii) Prove that the acceleration vector is always perpendicular to the velocity vector of the particle. **1**

$$\underset{\sim}{v}(t)\cdot\underset{\sim}{a}(t)=\begin{pmatrix}4\sin 2t\\ -4\cos 2t\end{pmatrix}\cdot\begin{pmatrix}8\cos 2t\\ 8\sin 2t\end{pmatrix}$$

$$=32\sin 2t\cos 2t+\left(-32\cos 2t\sin 2t\right)$$

$$=0$$

$$\therefore \underset{\sim}{a}(t)\perp\underset{\sim}{v}(t)$$

Well done.

Physics content was not accepted here, for example, some students only stated that the direction of acceleration in (uniform) circular motion is towards the centre of the circle.

(e) Solve for $f(x)$ if it satisfies the equation **3**

$$\int x^3[f(x)]^2dx = f(x) - x^4 + C.$$

$$x^3\left[f(x)\right]^2 = f'(x) - 4x^3$$

$$\therefore f'(x) = x^3\left(\left[f(x)\right]^2 + 4\right)$$

Let $y = f(x)$

$$\therefore \frac{dy}{dx} = x^3\left(y^2+4\right) \Rightarrow \frac{dy}{y^2+4} = x^3 dx$$

$$\therefore \int \frac{1}{y^2+4}\,dy = \int x^3\,dx$$

$$\therefore \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = \frac{1}{4}x^4 + D$$

$$\therefore \tan^{-1}\left(\frac{y}{2}\right) = \frac{1}{2}x^4 + K, \text{ where } K = 2D$$

$$\therefore y = 2\tan\left(\frac{1}{4}x^4 + K\right)$$

i.e. $f(x) = 2\tan\left(\frac{1}{4}x^4 + K\right)$

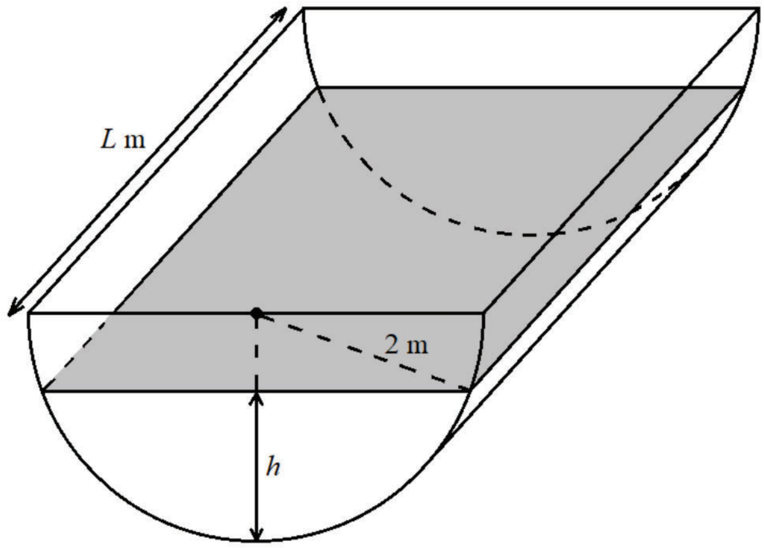
Generally well done.

Some careless mistakes, for example, incorrect copying of the index number. ECF was accepted.

Marks were deducted if students did not include the constant.

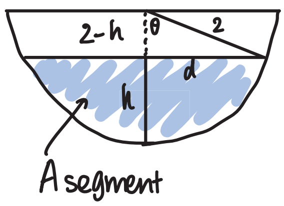
Question 13 **(15 marks)**

(a) The cross-section of a L metre long tank is a semi-circle of radius 2 m. The ends are made from the glass and the curved section is made from copper.



The depth of the water is h metres and the volume of water is V m^3 at t minutes.

(i) Show that $V = \left[4\cos^{-1}\left(\frac{2-h}{2}\right) + (h-2)\sqrt{h(4-h)}\right] L.$ **3**



$$\cos\theta = \frac{2-h}{2}$$

$$\theta = \cos^{-1}\left(\frac{2-h}{2}\right)$$

$$\begin{aligned} d &= \sqrt{2^2 - (2-h)^2} \\ &= \sqrt{4 - (4-4h+h^2)} \\ &= \sqrt{4h - h^2} \\ &= \sqrt{h(4-h)} \end{aligned}$$

$$\begin{aligned} A_{\text{segment}} &= \tfrac{1}{2}(2)^2(2\theta) - d(2-h) \\ &= 4\cos^{-1}\left(\frac{2-h}{2}\right) - (2-h)\sqrt{h(4-h)} \end{aligned}$$

$$\begin{aligned} V &= A_{\text{segment}} \times L \\ &= \left[4\cos^{-1}\left(\frac{2-h}{2}\right) + (h-2)\sqrt{h(4-h)}\right] L \end{aligned}$$

- Many students tried to fudge their way through.
- Common mistake: Assuming the cross-section to be a semicircle – it is a segment
- Drawing a diagram is crucial – you can't expect the marker to know where is 'θ'.

(ii) Show that the area of water touching copper at h metres deep is **1**

$$A = 4L\cos^{-1}\left(1 - \frac{h}{2}\right).$$

$A = r(2\theta)L$

$= 2 \times 2\cos^{-1}\left(\frac{2-h}{2}\right) L$

$= 4L\cos^{-1}\left(\frac{2-h}{2}\right)$

- This is a show question, so simply writing $4 \times \cos^{-1}\left(\frac{2-h}{2}\right) \times L$ is not good enough.
- Students need to explicity show how they got '4' to receive any marks.

(iii) The water evaporates proportional to the the rate of change of the area contacting the copper and inversely proportional to the rate of change of height, as given by **3**

$$\frac{dV}{dt} = -k\frac{dA}{dt}\left(\frac{dh}{dt}\right)^{-1}.$$

Prove that the rate of change of the depth of water satisfies

$$\frac{dh}{dt} = \frac{2k}{h^2 - 4h}.$$

Method 1

$\frac{dV}{dt} = -k\frac{dA}{dt} \times \frac{dt}{dh}$

$= -k\frac{dA}{dh}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{dh}{dV} \times -k\frac{dA}{dh}$ (don't simplify further since we know V and A in terms of h)

$$\frac{dV}{dh} = \left[4 \cdot \frac{-\left(-\frac{1}{2}\right)}{\sqrt{1-\left(\frac{2-h}{2}\right)^2}} + \sqrt{h(4-h)} + (h-2)\cdot \frac{1}{2}(4h-h^2)^{-\frac{1}{2}}(4-2h) \right] L$$

$$= \left[\frac{2\times 2}{\sqrt{4-(2-h)^2}} + \sqrt{h(4-h)} - \frac{(h-2)^2}{\sqrt{(4-h)h}} \right] L$$

$d = \sqrt{h(4-h)} \leftarrow$

$$= \left[\frac{4 + h(4-h) - (h-2)^2}{\sqrt{h(4-h)}} \right] L$$

$$= \left[\frac{\cancel{4} + 4h - h^2 - h^2 + 4h \cancel{-4}}{\sqrt{h(4-h)}} \right] L$$

$$= \frac{2(4h-h^2)}{\sqrt{4h-h^2}} L$$

$$= 2\sqrt{4h-h^2}\, L$$

$$\frac{dA}{dh} = \frac{4L}{\sqrt{4h-h^2}} \quad \text{(previously differentiated)}$$

$$\therefore \frac{dh}{dt} = \frac{dh}{dV} \times -k\,\frac{dA}{dh}$$

$$= \frac{1}{2\sqrt{4h-h^2}\,\cancel{L}} \times -k \times \frac{4\cancel{L}}{\sqrt{4h-h^2}}$$

$$= \frac{-2k}{4h-h^2}$$

$$= \frac{2k}{h^2-4h}$$

- Very poorly done.
- Many students lack the appropriate algebra skills to prove the result.

Method 2

$$\frac{dV}{dt} = -k\frac{dA}{dt} \times \frac{dt}{dh}$$
$$= -k\frac{dA}{dh}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= \frac{dh}{dV} \times -k\frac{dA}{dh}$$
$$= -k\frac{dA}{dV}$$

(but if you did simplify further, you need to find V in terms of $A \rightarrow$ this means eliminate h)

From (ii)

$$A = 4L\cos^{-1}\left(1-\frac{h}{2}\right)$$
$$\frac{A}{4L} = \cos^{-1}\left(1-\frac{h}{2}\right)$$
$$\cos\frac{A}{4L} = 1-\frac{h}{2}$$
$$\frac{h}{2} = 1-\cos\frac{A}{4L}$$
$$h = 2\left(1-\cos\frac{A}{4L}\right)$$
$$= 2-2\cos\frac{A}{4L}$$

$$V = 4L\cos^{-1}\left(\frac{2-h}{2}\right) + L(h-2)\sqrt{h(4-h)}$$
$$= A + L\left(-2\cos\frac{A}{4L}\right)\sqrt{\left(2-2\cos\frac{A}{4L}\right)\left(4-2+2\cos\frac{A}{4L}\right)}$$
$$= A + L\left(-2\cos\frac{A}{4L}\right)\sqrt{\left(2-2\cos\frac{A}{4L}\right)\left(2+2\cos\frac{A}{4L}\right)}$$
$$= A + L\left(-2\cos\frac{A}{4L}\right)\sqrt{4-4\cos^2\frac{A}{4L}}$$
$$= A + L\left(-2\cos\frac{A}{4L}\right)\left(2\sin\frac{A}{4L}\right)$$
$$= A - 2L\sin\left(\frac{A}{2L}\right)$$

$\because \sin\left(\frac{A}{2L}\right) = \sin 2\left(\frac{A}{4L}\right) = 2\sin\frac{A}{4L}\cos\frac{A}{4L}$

$$\frac{dV}{dA} = 1-\cos\left(\frac{A}{2L}\right)$$
$$= 1-\left[2\cos^2\left(\frac{A}{4L}\right)-1\right]$$

$\because \cos\left(\frac{A}{2L}\right) = \cos 2\left(\frac{A}{4L}\right) = 2\cos^2\left(\frac{A}{4L}\right)-1$

$$= 2-2\left(1-\frac{h}{2}\right)^2$$
$$= 2-2+2h-\frac{h^2}{2}$$
$$= 2h-\frac{h^2}{2}$$

$$\therefore \frac{dh}{dt} = -k \cdot \frac{1}{2h-\frac{h^2}{2}}$$
$$= \frac{-2k}{4h-h^2}$$
$$= \frac{2k}{h^2-4h}$$

(iv) Initially, the tank is completely full. After 3 minutes, the depth halves. How much longer will it take until the tank is completely dry? **2**

$$\frac{dh}{dt} = \frac{2k}{h^2 - 4h}$$

$$\int_2^h (h^2 - 4h)\, dh = \int_0^t 2k\, dt$$

$$\left[\frac{h^3}{3} - 2h^2\right]_2^h = \left[2kt\right]_0^t$$

$$\frac{h^3}{3} - 2h^2 - \left(\frac{8}{3} - 8\right) = 2kt$$

$$\frac{h^3}{3} - 2h^2 + \frac{16}{3} = 2kt$$

$t = 3$, $h = \frac{2}{2} = 1$: $\frac{1}{3} - 2 + \frac{16}{3} = 6k$

$$\frac{11}{3} = 6k$$

$$k = \frac{11}{18}$$

$h = 0$: $0 - 0 + \frac{16}{3} = \frac{22}{18}t$

$$t = \frac{48}{11}$$

$$\therefore \frac{48}{11} - 3 = \frac{15}{11} \text{ minutes longer}$$

Common mistakes:

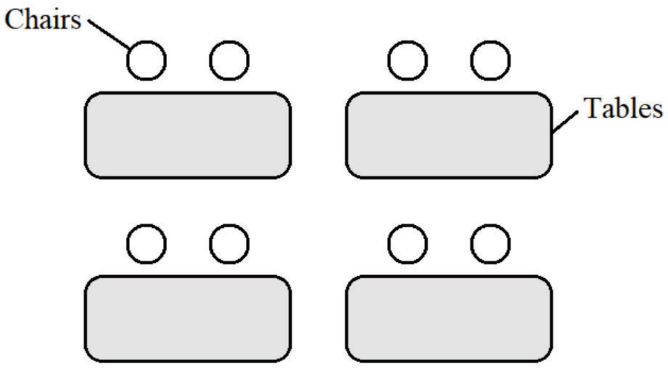
- Separating variables incorrectly.

$$\frac{dh}{h^2 - 4h} = 2k\ dt$$

Note: ME2 students should question why a partial fractions question is in a ME1 paper, and hopefully realise they separated incorrectly.

- Forgetting to subtract 3 to time. Question asked for how much longer.
- Algebraic errors.

(b) A physics classroom has 4 double desks at the front of the room as shown.



Suppose 5 students, Taf, Jon, Lou, Ale and Bhu wish to sit themselves randomly. Find:

(i) the total number of ways they can sit; **1**

$^8P_5 = 6720$

- Order is important.
- Common mistake : $^8C_5$
- No half marks

(ii) the probability that Jon and Lou sit on a double desk together, while Taf and Ale sit on another double desk together, given that Bhu sits on his own. **2**

$$\frac{\overbrace{^4C_1 \times 2!}^{\text{Jon + Lou}} \times \overbrace{^3C_1 \times 2!}^{\text{Taf + Ale}} \times \overset{\text{Bhu}}{4}}{\underset{\text{Bhu}}{^8C_1} \times \underbrace{^6C_4 \times 4!}_{\text{seat remaining 4 people + arrange}}} = \frac{192}{2880} = \frac{1}{15}$$

- Many took the sample space to be their answer from (i). It is different since there are new restrictions in place.
- One mark for getting 192, one mark for getting 2880.
- No half marks or marks for incorrect solution despite getting $\frac{1}{15}$.

(c) (i) Explain why $\sin\theta = \sin 2\theta\cos\theta - \sin\theta\cos 2\theta$. **1**

$$\begin{aligned} LHS &= \sin\theta \\ &= \sin(2\theta-\theta) \\ &= \sin 2\theta\cos\theta - \sin\theta\cos 2\theta \\ &= RHS \end{aligned}$$

- Many students used $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$.
- This is just a simple application of compound angles!

(ii) Hence, prove that for $n \in \mathbb{Z}^+$, **2**

$$\sum_{k=0}^{n} \tan(2^k x)\sec(2^{k+1}x) = \tan(2^{n+1}x) - \tan x.$$

$$\begin{aligned} \tan(2^k x)\sec(2^{k+1}x) &= \frac{\sin(2^k x)}{\cos(2^k x)\cos(2^{k+1}x)} \\ &= \frac{\sin(2.2^k x)\cos(2^k x) - \sin(2^k x)\cos(2.2^k x)}{\cos(2^k x)\cos(2^{k+1}x)} \quad (\text{from (i)}) \\ &= \frac{\sin(2^{k+1}x)}{\cos(2^{k+1}x)} - \frac{\sin(2^k x)}{\cos(2^k x)} \\ &= \tan(2^{k+1}x) - \tan(2^k x) \end{aligned}$$

$$\begin{aligned} LHS &= \sum_{k=0}^{n}\left[\tan(2^{k+1}x) - \tan(2^k x)\right] \\ &= (\cancel{\tan 2x} - \tan x) + (\cancel{\tan 4x} - \cancel{\tan 2x}) + \underbrace{(\cancel{\tan 8x} - \cancel{\tan 4x})}_{\text{some students wrote } (\tan 6x - \tan 3x) \text{ here.}} + \ldots + \left[\tan(2^{n+1}x) - \cancel{\tan(2^n x)}\right] \\ &= \tan(2^{n+1}x) - \tan x \\ &= RHS \end{aligned}$$

- Some students attempted to use mathematical induction, but didn't get anywhere.
- Students need to write out a few terms to show that some cancel out, since the result is already given.

Question 14 (16 marks) Use a SEPARATE writing booklet.

(a) A particular solution to the differential equation **3**

$$\frac{dy}{dx} = \frac{x}{\tan y\,(x^2+1)},$$

where $x \geq 0$ and $-\frac{\pi}{2} < y < 0$, passes through $\left(1, -\frac{\pi}{4}\right)$.

Determine this solution in the form $x = f(y)$.

Leave your answer in simplified form.

$$2\int y' \tan y\, dx = \int \frac{2x}{x^2+1}dx$$
$$-2\ln|\cos y| = \ln\left(x^2+1\right) + C$$
$$\ln\sec^2 y + \ln A = \ln\left(x^2+1\right), -C = \ln A$$
$$x^2 + 1 = A\sec^2 y$$
$$x(1) = -\frac{\pi}{4} \Rightarrow A = 1$$
$$x^2 = \sec^2 y - 1$$
$$x = -\tan y, x \geq 0$$

Note: Some pupils gave the integral of tan y as its derivative. Many incorrectly evaluated the constant of integration. If pupils made this mistake it was impossible to recognise that x squared is equal to the square of the tangent of y. Finally, some pupils did not recognise that for x to be greater than zero, the y function needed a negative sign.

(b) (i) Find the unit vector $\underset{\sim}{\hat{u}}$ where $\underset{\sim}{u} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. **1**

$$\underset{\sim}{\hat{u}} = \frac{7}{\sqrt{3^2 + (-5)^2}} \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{3}{\sqrt{34}} \\ -\dfrac{5}{\sqrt{34}} \end{pmatrix}$$

Note: This was generally well done however, some pupils multiplied the vector by its length. Some pupils gave the result in point form rather than as a vector.

(ii) Hence, using vector methods, find the point on the circle $(x-2)^2 + (y-2)^2 = 2$ that is closest to the circle $(x-5)^2 + (y+3)^2 = 6$. **2**

Radius of first circle is $\sqrt{2}$. The closest point lies in the direction of the vector from $(2,2)$ to $(5,-3)$.

$$\underset{\sim}{p} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \frac{\sqrt{2}}{\sqrt{3^2 + (-5)^2}} \begin{pmatrix} 5-2 \\ -3-2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} \dfrac{3}{\sqrt{17}} \\ \dfrac{-5}{\sqrt{17}} \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{2\sqrt{17}+3}{\sqrt{17}} \\ \dfrac{2\sqrt{17}-5}{\sqrt{17}} \end{pmatrix}$$

$$\therefore \text{ the point is } \left(\frac{2\sqrt{17}+3}{\sqrt{17}}, \frac{2\sqrt{17}-5}{\sqrt{17}} \right)$$

Note: This was poorly done and pupils often gave their response as vector instead of a point.

(c) Let S be defined as **4**

$$S = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000}.$$

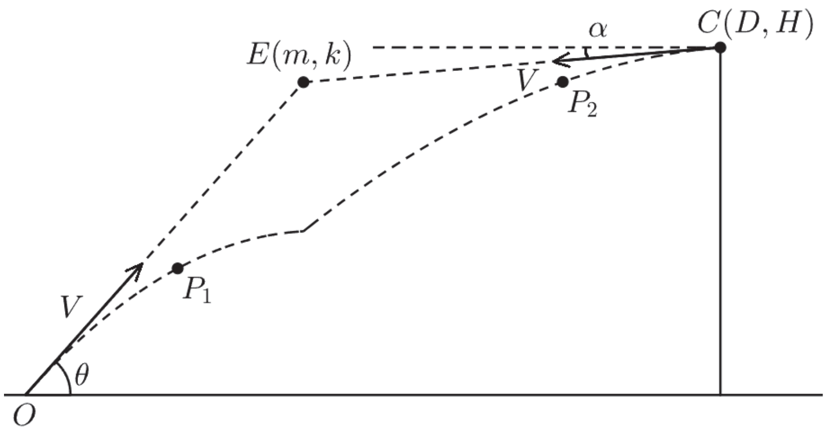
By considering $\left(\dfrac{x}{1+x}\right) S$, or otherwise, find the coefficient of x^{50}.

$$\begin{aligned}
\frac{xS}{1+x} &= x\,(1+x)^{999} + 2x^2\,(1+x)^{998} + \ldots + \frac{1001x^{1001}}{1+x} \\
S - \frac{xS}{1+x} &= (1+x)^{1000} + x\,(1+x)^{999} + \ldots + x^{1000} - \frac{1001x}{1+x} \\
(1+x)\,S\frac{1+x-x}{1+x} &= (1+x)^{1001} + x\,(1+x)^{1000} + \ldots + x^{1000}\,(1+x) - 1001x^{1001} \\
S &= \frac{\left((1+x)^{1002} - x^{1002}\right)}{(1+x-x)} - 1002x^{1001} \\
&= (1+x)^{1002} - x^{1002} - 1002x^{1001}
\end{aligned}$$

$\therefore$ Coefficient of x^{50} is ${}^{1002}\mathrm{C}_{50}$.

Note: Most pupils had an idea of how to start but failed to show what happens with the last few terms. Rarely did pupils go past the first two lines here.

(d) Two projectiles, P_1 and P_2 are launched from the origin O and $C(D, H)$ respectively.



Both are shot at speed V m/s towards a chosen point $E(m, k)$ so that the two particles hit each other. Let the angles that the initial velocities of P_1 and P_2 makes with the horizontal be θ and α respectively.

You may assume that the position of a particle launched from the initial position (r, h) with speed U and angle ϕ to the horizontal is given by

$$\underset{\sim}{r}(t) = (Ut\cos\phi + r)\underset{\sim}{i} + \left(Ut\sin\phi - \frac{1}{2}gt^2 + h\right)\underset{\sim}{j} \qquad \text{(Do \textbf{NOT} prove this.)}$$

where g is the acceleration due to gravity.

(i) Prove, using the position vector of each particle, that the point E is equidistant from O and C. **4**

$$\underline{r_1(t)} = \begin{pmatrix} Vt\cos\theta \\ Vt\sin\theta - \frac{gt^2}{2} \end{pmatrix}; \quad \underline{r_2(t)} = \begin{pmatrix} -Vt\cos\alpha + D \\ -Vt\sin\alpha - \frac{gt^2}{2} + H \end{pmatrix}$$

Let T be the time the particles meet.

$$\Rightarrow VT\cos\theta = D - VT\cos\alpha$$

$$T = \frac{D}{V(\cos\theta + \cos\alpha)}$$

$$\Rightarrow V\sin\theta\frac{D}{V(\cos\theta+\cos\alpha)} - g\frac{T^2}{2} = -V\sin\alpha\frac{D}{V(\cos\theta+\cos\alpha)} - \frac{gT^2}{2} + H$$

$$\Rightarrow \frac{\sin\alpha + \sin\theta}{\cos\alpha + \cos\theta} = \frac{H}{D}$$

(d) (i) (continued)

From the diagram:

$$\sin\alpha = \frac{H-m}{\left|\overrightarrow{EC}\right|}; \sin\theta = \frac{m}{\left|\overrightarrow{OE}\right|}$$

$$\cos\alpha = \frac{D-k}{\left|\overrightarrow{EC}\right|}; \cos\theta = \frac{k}{\left|\overrightarrow{OE}\right|}$$

$$\frac{\dfrac{H-m}{\left|\overrightarrow{EC}\right|}+\dfrac{m}{\left|\overrightarrow{OE}\right|}}{\dfrac{D-k}{\left|\overrightarrow{EC}\right|}+\dfrac{k}{\left|\overrightarrow{OE}\right|}} = \frac{H}{D}$$

$$\frac{\left|\overrightarrow{OE}\right|(H-m)+\left|\overrightarrow{EC}\right|m}{\left|\overrightarrow{OE}\right|(D-h)+\left|\overrightarrow{EC}\right|k} = \frac{H}{D}$$

$$\Rightarrow a\left(\left|\overrightarrow{OE}\right|(H-m)+\left|\overrightarrow{EC}\right|m\right) = bH \quad \text{and}$$

$$a\left(\left|\overrightarrow{OE}\right|(D-h)+\left|\overrightarrow{EC}\right|k\right) = bD$$

Comparing coefficients of m on both sides of either equation $\left|\overrightarrow{EC}\right| - \left|\overrightarrow{OE}\right| = 0.$

$\therefore \left|\overrightarrow{OE}\right| = \left|\overrightarrow{EC}\right|$ or E is equidistant from O & C.

Note: Not one single pupil gave an answer that was as rigorous as this. Barely a single pupil got past half way. A number of students tried explanations using words as their proof which made me laugh. Without mathematics to back them up, words hold little value in a mathematical proof. Many pupils resorted to assume the angles are the same so they could prove collinearity - this is an invalid way to prove it. The same approach was often employed to show *EC* & *OE* were equal in magnitude. Try not to assume that which you are trying to prove in future exams.

(ii) If O, E and C are not collinear, prove that the time to impact is **2**

$$t = \frac{\sqrt{m^2 + k^2}}{V}.$$

Let T be the time the particles meet.

$$\left.\begin{aligned} T &= \frac{D}{V(\cos\theta + \cos\alpha)} \\ \frac{\sin\alpha + \sin\theta}{\cos\alpha + \cos\theta} &= \frac{H}{D} \end{aligned}\right\} \text{From part i.}$$

$$\Rightarrow T = \frac{H}{V(\sin\alpha + \sin\theta)}$$

$\sin\alpha = \dfrac{H - m}{\left|\overrightarrow{EC}\right|}$; $\sin\theta = \dfrac{m}{\left|\overrightarrow{OE}\right|}$ and $\left|\overrightarrow{OE}\right| = \left|\overrightarrow{EC}\right|$, from part i.

$$\left|\overrightarrow{OE}\right| = \sqrt{m^2 + k^2}$$

$$\begin{aligned} \Rightarrow T &= \frac{H}{V} \frac{1}{\dfrac{H - m}{\sqrt{m^2 + k^2}} + \dfrac{m}{\sqrt{m^2 + k^2}}} \\ &= \frac{H}{V} \frac{1}{\dfrac{H}{\sqrt{m^2 + k^2}}} \\ &= \frac{\sqrt{m^2 + k^2}}{V} \end{aligned}$$

Note: This should have been an easy two marks however, since only a handful of pupils successfully did part i) or did the alternative version, this ended up being as difficult as part i). Very few pupils got full marks.

Alternatively:

i) Both are shot at speed V m/s towards a chosen point $E(m, k)$ so that the two particles hit each other. Let the angles that the initial velocities of P_1 and P_2 makes with the horizontal be θ and α respectively.

$\overrightarrow{CE}$ and $\overrightarrow{OE}$ are linear where the magnitudes are Vt after t seconds.
Let T be the time the particles meet.

$$\overrightarrow{CE} = -VT\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}$$
$$\overrightarrow{OE} = VT\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

$$\begin{aligned}\left|\overrightarrow{CE}\right| &= VT\left(\cos^2\alpha + \sin^2\alpha\right)^{\frac{1}{2}} \\ &= VT \\ \left|\overrightarrow{OE}\right| &= VT\left(\cos^2\theta + \sin^2\theta\right)^{\frac{1}{2}} \\ &= VT\end{aligned}$$

ii) From i) $\left|\overrightarrow{OE}\right| = VT$ and $\left|\overrightarrow{OE}\right| = \sqrt{m^2 + k^2}$.

$$\therefore T = \frac{\sqrt{m^2 + k^2}}{V}$$

Those students who wanted to prove a solution exists where O & C are collinear without first proving E is equidistant from O & C should look at the following solution. I want to reiterate that even in this case, E is equidistant from O & C.

(i) The position of particle P_1 is $\underset{\sim}{r}_1(t) = (Vt\cos\theta)\underset{\sim}{i} + \left(Vt\sin\theta - \frac{1}{2}gt^2\right)\underset{\sim}{j}$ (given)

Also, for P_2, $\underset{\sim}{r}_2(t) = (D - Vt\cos\alpha)\underset{\sim}{i} + \left(-Vt\sin\alpha - \frac{1}{2}gt^2 + H\right)\underset{\sim}{j}$ (given)

$\because$ Particles hit each other, $\therefore \underset{\sim}{r}_1(t) = \underset{\sim}{r}_2(t)$

$$\begin{cases} Vt\cos\theta = D - Vt\cos\alpha \\ Vt\sin\theta - \frac{1}{2}gt^2 = -Vt\sin\alpha - \frac{1}{2}gt^2 + H \end{cases} \Rightarrow \begin{cases} Vt(\cos\theta + \cos\alpha) = D \quad (1) \\ Vt(\sin\theta + \sin\alpha) = H \quad (2) \end{cases}$$

From the diagram,

$$\cos\theta = \frac{m}{OE},\ \sin\theta = \frac{k}{OE},\ \cos\alpha = \frac{D-m}{EC},\ \sin\alpha = \frac{H-k}{EC} \quad (3)$$

By using $\frac{(1)}{(2)}$,

$$\therefore \frac{\cos\theta + \cos\alpha}{\sin\theta + \sin\alpha} = \frac{D}{H}$$

$$H(\cos\theta + \cos\alpha) = D(\sin\theta + \sin\alpha)$$

$$H\left(\frac{m}{OE} + \frac{D-m}{EC}\right) = D\left(\frac{k}{OE} + \frac{H-k}{EC}\right) \quad \text{(From (3))}$$

$$\frac{Hm}{OE} + \frac{HD - Hm}{EC} = \frac{Dk}{OE} + \frac{DH - Dk}{EC}$$

$$\frac{Hm - Dk}{OE} = \frac{DH - Dk - (HD - Hm)}{EC}$$

$$\frac{Hm - Dk}{OE} = \frac{Hm - Dk}{EC} \quad (4)$$

Case 1: If $Hm - Dk = 0$, then $\frac{k}{m} = \frac{H}{D}$

$\therefore m_{OE} = m_{OC}$

$\therefore O, E, C$ are collinear.

Case 2: If $Hm - Dk \neq 0$, then $OE = EC$ (from (4))

$\therefore$ E must be equidistant to O and C