SYDNEY GIRLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE, 1993

3/4 UNIT MATHEMATICS

TIME ALLOWED: Two hours (including reading time)

DIRECTIONS TO CANDIDATES:

* All questions may be attempted
* All questions are of equal value
* All necessary working should be shown in every question
  Marks may not be awarded for careless or badly arranged work
* Standard integrals are printed on a separate paper
* Calculators may be used
* Start a new page for each question

N.B. This is a TRIAL PAPER ONLY and does not necessarily reflect the content or format of the final Higher School Certificate Examination Paper for this subject

Question 1:

a) Solve $x - \frac{1}{x} < 0$

b) If $x = \sqrt{3} + \sqrt{2}$ find the value of $x + \frac{1}{x}$, hence or otherwise find the value of $\left(x^2 + \frac{1}{x^2}\right)$

c) The point P (6,9) divides the interval AB in the ratio -3:2. Find the point B given that A is (1,4)

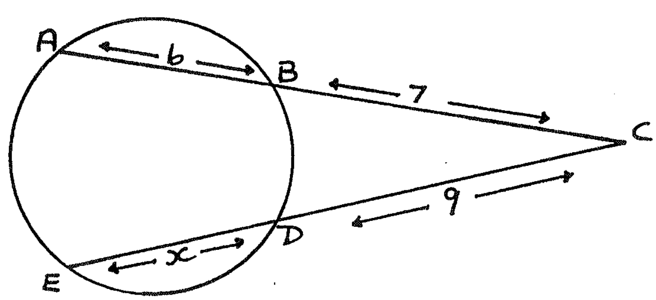
d) $\int \frac{-x}{\sqrt{1 - x^2}}\,dx$

e) $\int \frac{3}{16 + x^2}\,dx$

Question 2:

a) Solve $\sin x + \sin 2x = 0 \quad 0° \le x \le 360°$

b)



Find x, give reasons

c) The positive square root of 37 is approximately 6
Use one application of Newton's Method to find a better approximation, correct to 2 decimal places

d) A function is defined by the rules

$$f(x) = \begin{cases} \sin^{-1} x & \text{if } -1 \le x < 0 \\ \cos^{-1} x & \text{if } 0 \le x \le 1 \end{cases}$$

i) Sketch the graph of the function for $-1 \le x \le 1$

ii) For this function evaluate exactly $f(\tfrac{1}{2}) + 2f(0) - f(-\tfrac{1}{2})$

Question 3:

a) A certain particle moves along the x axis in accordance with the law

$$t = 2x^2 - 4x + 3$$

where x is measured in cms and t in seconds
Initially the particle is 1.5cms to the right of 0 and moving away from 0

i) Prove that the velocity, v cm/sec is given by

$$v = \frac{1}{4x - 4}$$

ii) Find an expression for the acceleration a cm/sec² in terms of x

iii) Find the velocity and acceleration of the particle when

a) $x = 2$cms
b) $t = 9$secs

iv) Describe in words the motion of the particle

b) Prove by induction that

$$3^n > 1 + 2n \text{ for } n > 1$$

Question 4:

The points P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$

a) Derive the equation of the normal to the parabola at P

b) Find the coordinates of the point N the intersection of the normals at P and Q

c) Find the equation of the chord PQ and determine the condition necessary for PQ to be a focal chord

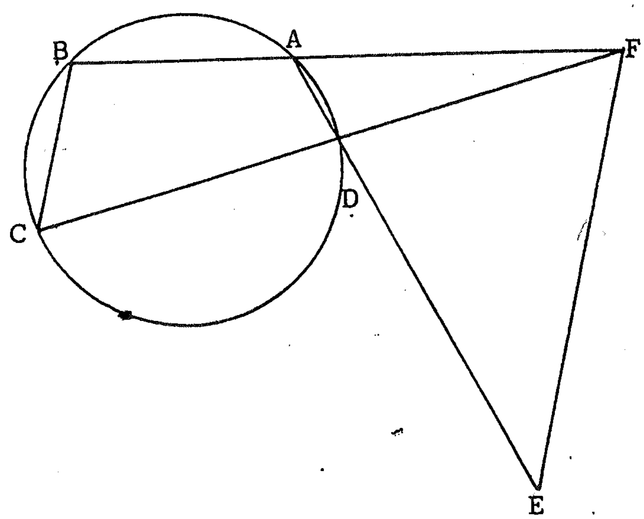
d) If PQ is a focal chord and N is the intersection of the normals at P and Q, find the equation of the locus of N. Describe the locus

Question 5:

a) Prove that

$$\frac{1 - \cos\theta}{1 + \cos\theta} = \tan^2\frac{\theta}{2}$$

b)



BC || EF

ABCD is a cyclic quadrilateral
Prove that

$$EF^2 = EA \times ED$$

c) The product of two of the roots of the equation

$$24x^3 + 14x^2 - 11x - 6 = 0$$

is equal to the other root. Find the values of all three roots of the equation

Question 6:

a) Find the value of

$$\int_0^{\frac{\pi}{2}} \frac{\sec^2(\sin x)\,dx}{\sec x}$$

using the substitution

$u = \sin x$

b) Express $7\cos\alpha - \sin\alpha$ in the form $R\cos(\alpha + \beta)$ where $R > o$ and $0° < \beta < 360°$. Hence solve the equation

$7\cos\alpha - \sin\alpha = 5$

c) Assume that the rate at which a body cools in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This rate can be expressed by the differential equation

$$\frac{dT}{dt} = -k(T - A)$$

where t is the time in minutes and k is a constant

i) Show that $T = A + Ce^{-kt}$ is a solution to the differential equation (C is a constant)

ii) The body of a murder victim is discovered at 2am when its temperature is 35°C. One hour later its temperature has fallen to 34°C. If the room temperature remains constant at 21°C, find the value of k

iii) Calculate the time the murder was committed. (Normal body temperature is approximately 37°C.)

Question 7:

a) Find the volume of the solid of revolution obtained by revolving the area bounded by $y = 1 + \sin x$ and the x-axis between $x = 0$ and $x = \pi$ about the x axis

b) A particle undergoes simple harmonic motion about the origin 0. Its displacement x cm from 0 at time t seconds is given by

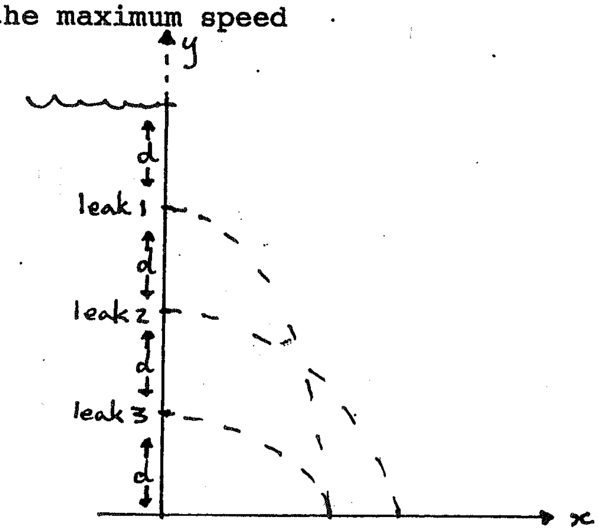
$$x = 5\cos\left(4t + \frac{\pi}{2}\right)$$

i) Find the acceleration in terms of t and hence or otherwise as a function of displacement

ii) Write down the amplitude of the motion

iii) Find the maximum speed

c)



A dam wall is 4d units high. Small leaks occur at 3 evenly spaced points 3d, 2d and 1d units down from the water level

i) Using axes as shown, show that the parametric equations for the water from leak 1 are:

$$x = Vt \qquad\qquad y = 3d - \tfrac{1}{2}gt^2$$

ii) Find the parametric equations for leak 2 and leak 3

iii) Show that the jets of water from leak 1 and leak 3 reach the ground level at the same point given that the velocity of water issuing from a leak is $V^2 = kh$ where h is the height of water above the leak and k is a constant

iv) Find the velocity of impact of the water from leak 2 in terms of g, k and d

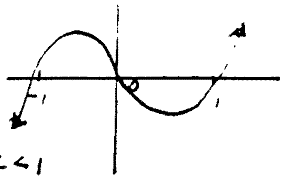
v) A leak occurs at a distance ℓ units above ground. Show that it hits the ground a maximum distance from the dam if $\ell = 2d$

...tion 1

$\ldots -\frac{1}{x} < 0$

$\ldots - x < 0$

$\ldots -1)(x+1) < 0$



$\ldots -1 \text{ or } 0 < x < 1$

$\frac{\ldots}{\ldots+\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{\sqrt{3}-\sqrt{2}}{1}$

$\ldots\sqrt{3}+\sqrt{2} + \frac{1}{\sqrt{3}+\sqrt{2}} = \sqrt{3}+\sqrt{2}+\sqrt{3}-\sqrt{2}$

$= \underline{2\sqrt{3}}$

$\ldots + \frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2 - 2$

$= (2\sqrt{3})^2 - 2$

$= 12 - 2$

$= 10$

$AP : PB = -3 : 2$

$\ldots \frac{k_1 x_2 + k_2 x_1}{k_1 + k_2}$ $\quad y = \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2}$

$= \frac{-3x_2 + 2\times 1}{-1}$ $\quad 9 = \frac{-3y + 2\times 4}{-1}$

$= 3x - 2$ $\quad = 3y - 8$

$= \frac{8}{3}$ $\quad y = \frac{17}{3}$

$\therefore B\left(\frac{8}{3}, \frac{17}{3}\right)$

$\int \frac{-x}{\sqrt{1-x^2}}\,dx = \sqrt{1-x^2} + c$

$\int \frac{3\,dx}{16+x^2} = 3\int \frac{dx}{16+x^2}$

$= \frac{3}{4}\tan^{-1}\frac{x}{4} + c$

...stion 2

$\ldots x + \sin 2x = 0$

$\ldots x + 2\sin x \cos x = 0$

$\ldots x(1 + 2\cos x) = 0$

$\ldots x = 0 \quad \text{or} \quad 1 + 2\cos x = 0$

$x = 0^\circ, 180^\circ, 360^\circ \quad 2\cos x = -1$

$\cos x = -\frac{1}{2}$

$x = \underline{120^\circ}$

$\ldots$

c) Let $f(x) = x^2 - 37 = 0$

a root of which is exactly $\sqrt{37}$

Taking $x_1 = 6$ as a first approx.

then $x_2 = x_1 - \frac{f(x)}{f'(x)}$

$f'(x) = 2x$

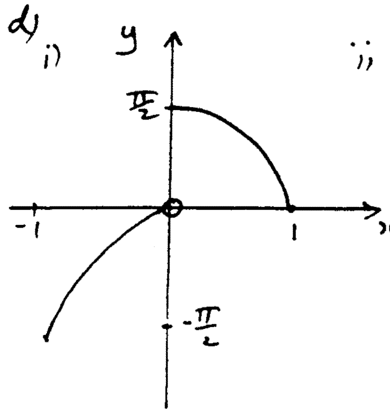
$\therefore f(6) = -1$ and $f'(x) = 12$

$x_2 = 6 - \frac{-1}{12}$

$= 6\frac{1}{12}$

$= 6.08$ (2 dec pl)

d) i)



ii) $f(\frac{1}{2}) + 2f(0) - f(-\frac{1}{2})$

$\cos^{-1}\frac{1}{2} + 2\cos^{-1}0 - \sin^{-1}-\frac{1}{2}$

$\frac{\pi}{3} + 2.\frac{\pi}{2} - (-\frac{\pi}{6})$

$\frac{\pi}{3} + \pi + \frac{\pi}{6}$

$\frac{3\pi}{2}$

Question 3

i) $t = 2x^2 - 4x + 3$

$\therefore \frac{dt}{dx} = 4x - 4$

Now $v = \frac{dx}{dt} = \frac{1}{\frac{dt}{dx}} = \frac{1}{4x-4}$

ii) $a = v\frac{dv}{dx}$ $\quad \left[a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}\right]$

$= \frac{1}{4x-4} \cdot \frac{-1}{(4x-4)^2} \cdot 4$

$= \frac{-4}{(4x-4)^3}$

iii) a. when $x = 2$

$v = \frac{1}{4\times 2-4} = \frac{1}{4}$ cm/sec

$a = \frac{-4}{(8-4)^3} = \frac{-4}{64} = -\frac{1}{16}$ cm/sec^2

b) when $t = 9$

$2x^2 - 4x + 3 = 9$

$2x^2 - 4x - 6 = 0$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$$y = a(p^2+pq+q^2+2)$$

$pq=-1$ $\quad x = a(p+q)$

$$y = a(p^2+q^2+1)$$

$$x^2 = a^2(p^2+2pq+q^2)$$

$$= a^2(p^2+q^2-2)$$

$$q^2 = \frac{x^2}{a^2}+2 = \frac{y}{a} = -1$$

eqⁿ of locus of N is

$= ay - 3a^2$

$x^2 = a(y-3a)$

is a parabola with vertex $(0,3a)$ and focus $(0, \frac{13}{4}a)$

Question 5

$$\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2}$$

$\cos 2A = \cos^2 A - \sin^2 A$
$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$
$= 1-2\sin^2\frac{A}{2}$

$$\frac{1-\left(1-2\sin^2\frac{\theta}{2}\right)}{1+\left(2\cos^2\frac{\theta}{2}-1\right)}$$

$$\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$\tan^2\frac{\theta}{2}$$

$= RHS$

$B\hat{C}D = E\hat{A}F$ (ext ∠ of cyclic quad equals interior opp)

$B\hat{C}D = D\hat{F}E$ (alternate ∠'s $BC \parallel EF$)

$E\hat{A}F = D\hat{F}E$

∆'s AFE + DFE

$E\hat{A}F = D\hat{F}E$ (proved)

$F\hat{E}D$ is common

$\triangle AFE \parallel\!| \triangle FDE$ (equiangular)

$\therefore \frac{EF}{EA} = \frac{ED}{EF}$ (ratios of corresponding sides)

$\therefore EF^2 = EA \cdot ED.$

c)

If roots are α, β, α

then $\alpha\beta = \gamma$

$$\alpha+\beta+\alpha = \frac{-14}{24}$$

$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{-11}{24}$$

$$\alpha\beta\gamma = \frac{6}{24} = \frac{1}{4} \Rightarrow \gamma^2 = \frac{1}{4}$$

$$\therefore \gamma = \pm\frac{1}{2}$$

testing $A(\frac{1}{2}) = -5$ $\quad A(-\frac{1}{2}) = 0$

$\therefore 2x+1$ is a factor

$$A(x) = (2x+1)(12x^2+x-6)$$

$$= (2x+1)(3x-2)(4x+3)$$

$\therefore$ the roots are $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}$

Question 6

a) Let $u = \sin x$ $\qquad x=0 \quad u=0$
$\qquad\qquad\qquad\qquad x=\frac{\pi}{2} \quad u=1$

$$\frac{du}{dx} = \cos x$$

$$dx = \frac{du}{\cos x}$$

$$= \sec x \, du$$

$$\int_0^1 \frac{\sec^2 u \; \sec x \, du}{\sec x}$$

$$\Big[\tan u\Big]_0^1$$

$\tan 1 - \tan 0$

1.5574 (4 D.P.

b) $7\cos\alpha - \sin\alpha = R\cos\alpha\cos\beta - R\sin\alpha\sin\beta$

$\therefore R\cos\beta = 7 \quad -①$

$R\sin\beta = 1 \quad -②$

sq. ① + ② and add

$R^2(\cos^2\beta+\sin^2\beta) = 50$

$R = \sqrt{50}$ (take +ve root only)

From ① + ②

$\tan\beta = \frac{1}{7}$, (since $\cos\beta$ + $\sin\beta$ +ve must lie in 1st quad)

$\beta = 8°8'$

$\therefore 7\cos\alpha - \sin\alpha = \sqrt{50}\cos(\alpha + 8°8') = 5$

i) $T = A + Ce^{-kt}$

$\frac{dT}{dt} = -kCe^{-kt}$

$= -k(T-A)$

$A = 21°$ & when $t=0$ $T = 35°$

$35 = 21 + Ce^{-k0}$

$\therefore C = 14$

n $t = 60$ $T = 34$

$34 = 21 + 14e^{-60k}$

$13 = 14e^{-60k}$

$\frac{13}{14} = e^{-60k}$

$\frac{13}{14} = -60k \ln e$

$\boxed{\frac{\ln\frac{13}{14}}{-60} = k}$

$A + Ce^{-kt}$

$= 21 + 14e^{-kt}$

$= 14e^{-kt}$

$= e^{-kt}$

$\frac{16}{14} = t$

$\doteqdot 108$ mins

under committed

48 mins before 2am

12.12 a.m.

Question 7

$V = \pi \int y^2 dx$

$= \pi \int_0^{\pi} (1+\sin x)^2 dx$

$= \pi \int_0^{\pi} (1 + 2\sin x + \sin^2 x)\, dx$

$= \pi \left[x - 2\cos x + \frac{x}{2} - \frac{\sin 2x}{2}\right]_0^{\pi}$

$= \pi \left[\pi + 2 - \frac{\pi}{2} - 0 - 0 - 2 + 0 - 0\right]^{\pi}$

b)

i) $\dot{x} = -20 \sin\left(4t + \frac{\pi}{2}\right)$

$\ddot{x} = -80 \cos\left(4t + \frac{\pi}{2}\right)$

$= -16\left(5\cos\left(4t + \frac{\pi}{2}\right)\right)$

$= -16x$

ii) amp $= 5$

iii) Max speed occurs at $x = 0$

$\dot{x}^2 = n^2(a^2 - x^2)$

$= 16(25 - 0)$

Max speed 20 cm/sec.

or max value $\sin \theta = 1$

$\dot{x} = -20 \times 1$

c. i) Vertical Motion

$\frac{d^2y}{dx^2} = -g$

$\frac{dy}{dx} = -gt + c$

initial vertical motion $\frac{dy}{dx} = V\sin\alpha$ when $t =$

$\therefore V\sin\alpha = -gt + c$

i.e. $c = V\sin\alpha$

$\therefore \frac{dy}{dt} = -gt + V\sin\alpha$

$y = V\sin\alpha t - \frac{1}{2}gt^2 + k$

initially $y = 3d$ when $t = 0$ & $\alpha = 0$

$\therefore k = 3d$

$\therefore y = 3d - \frac{1}{2}gt^2$

Horizontal motion

$\frac{dx}{dt} = V\cos\alpha$

$x = V\cos\alpha t + k$

when $t = 0$ $x = 0$ $k = 0$

$\therefore x = V\cos\alpha t$

initially $\alpha = 0$

$x = Vt.$

) leak 2 $y = 2d - \frac{1}{2}gt^2$

$x = Vt$

leak 3

$y = d - \frac{1}{2}gt^2$

$x = Vt$

) leak 1 $h = d$

$V^2 = kd$

$V = \sqrt{kd}$

$x = Vt$
$= \sqrt{kd}\,t$ $\quad y = 3d - \frac{1}{2}gt^2$

when $y = 0$ $\quad gt^2 = 6d$

$t = \sqrt{\frac{6d}{g}}$

$\therefore x = \sqrt{kd} \cdot \sqrt{\frac{6d}{g}}$

$= \sqrt{\frac{6kd^2}{g}}$

leak 3 $h = 3d$

$V^2 = kh$

$V = \sqrt{3kd}$

$x = Vt \quad y = d - \frac{1}{2}gt^2$

when $y = 0$ $\quad gt^2 = 2d$

$t^2 = \frac{2d}{g}$

$t = \sqrt{\frac{2d}{g}}$

$x = \sqrt{3kd} \times \sqrt{\frac{2d}{g}}$

$= \sqrt{\frac{6kd^2}{g}}$

$\therefore$ Water from leak 1 & 3 reach the ground level at the same pt. i.e. where $x = \sqrt{\frac{6kd^2}{g}}$

leak 2 $h = 2d$

$V^2 = 2kd$

$V = \sqrt{2kd}$

$y = 2d - \frac{1}{2}gt^2$

when $y = 0$ $\quad gt^2 = 4d$

$t = \sqrt{\frac{4d}{g}}$

water reaches ground when $t = \sqrt{\frac{4d}{g}}$

$V^2 = \dot{x}^2 + \dot{y}^2$ $\quad \dot{y} = -gt$

$= -g\sqrt{\frac{4d}{g}} = -\sqrt{4dg}$

$\dot{x} = V = \sqrt{2kd}$

$\therefore V^2 = 2kd + 4dg$

v) Max value of x

$x = Vt \quad y = l - \frac{1}{2}gt^2$

water hits ground when $t = \sqrt{\frac{2l}{g}}$

$\therefore x = \sqrt{k(4d-l)} \times \sqrt{\frac{2l}{g}}$

$= \sqrt{\frac{8lkd - 2kl^2}{g}}$

x is max when $\dot{x} = 0$ & when x^2 is m.

$x^2 = \frac{8lkd - 2kl^2}{g}$

$\frac{dx^2}{dl} = \frac{8kd - 4kl}{g}$

$\frac{dx^2}{dl} = 0$ when $8kd = 4kl$

i.e. $2d = l$

$\frac{d^2x^2}{dl^2} = -\frac{4k}{g}$ $-$ve