

3/4 UNIT MATHEMATICS FORM VI

Time allowed: 2 hours (plus 5 minutes reading)

Exam date: 13th August 2001

Instructions:

- All questions may be attempted.
- All questions are of equal value.
- Part marks are shown in boxes in the left margin.
- All necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

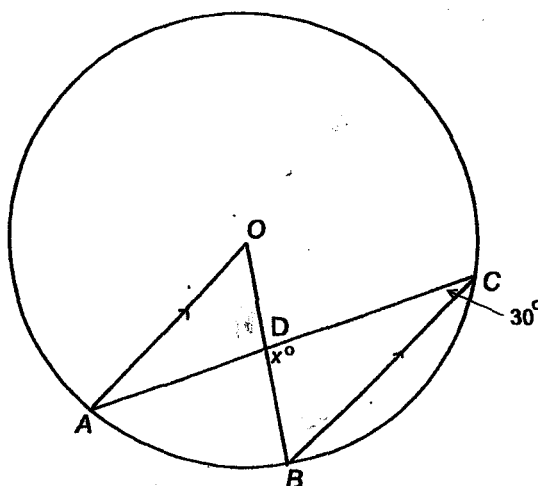
Collection:

- Each question will be collected separately.
- Start each question in a new answer booklet.
- If you use a second booklet for a question, place it inside the first. Don't staple.
- Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

Marks

- 2** (a) Find the coordinates of the point that divides the interval joining the points $(-5, 6)$ and $(4, -3)$ in the ratio $3 : 1$.
- 3** (b) Find the acute angle between the lines $x + 2y = 5$ and $x - 3y = 0$.
- (c)



In the diagram above, O is the centre of the circle, $BC \parallel AO$ and $\angle ACB = 30^\circ$.

- 1** (i) Explain why $\angle AOB = 60^\circ$.
- 2** (ii) Find x , giving reasons.
- (d) Consider the polynomial $P(x) = x^3 - x^2 - 10x - 8$.
- 1** (i) Show that $x = -1$ is a zero of $P(x)$.
- 2** (ii) Express $P(x)$ as a product of three linear factors.
- 1** (iii) Solve $P(x) \leq 0$.

QUESTION TWO (Start a new answer booklet)

Marks

- 1** (a) Sketch the polynomial function $y = x^2(x^2 - 16)$, carefully showing all intercepts.
- 1** (b) (i) Write $x^2 + 4x + 5$ in the form $(x + a)^2 + b$.
- 2** (ii) Hence find $\int \frac{dx}{x^2 + 4x + 5}$.
- 3** (c) Find the general solution of $\cos 2x = \cos x$.
- 2** (d) (i) Sketch the parabola $f(x) = 9 - (x + 2)^2$, showing clearly any intercepts with the axes and the coordinates of the vertex.
- 1** (ii) What is the largest domain containing the value $x = 0$ for which the function has an inverse function?
- 2** (iii) On a separate diagram, sketch the graph of this inverse function, showing all intercepts with the axes.

QUESTION THREE (Start a new answer booklet)

Marks

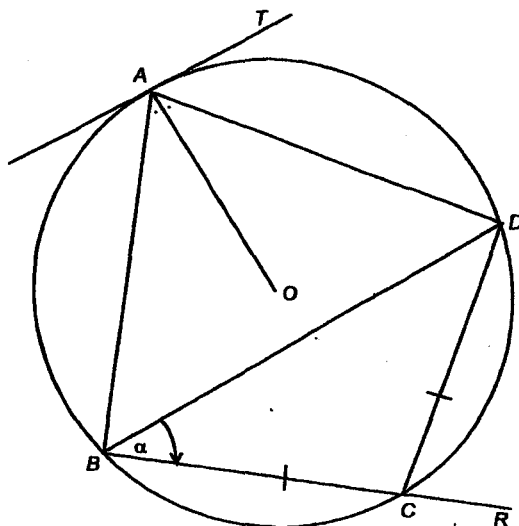
- 2** (a) Evaluate $\lim_{x \rightarrow 0} \left(\frac{\sin 4x}{\tan 2x} \right)$. You must show all working for full marks.
- 2** (b) Find the term independent of x in the expression $\left(x + \frac{1}{x^2} \right)^9$.
- 4** (c) A spherical balloon is expanding so that its volume $V \text{ m}^3$ increases at a constant rate of 72 m^3 per second. What is the rate of increase of the surface area when the radius is 12 metres? You may use the formulae $V = \frac{4}{3}\pi r^3$ for the volume of a sphere and $S = 4\pi r^2$ for its surface area.
- 1** (d) (i) Show that there is a root to the equation $\sin x = x - \frac{1}{2}$ between $x = 0.5$ and $x = 1.8$.
- 3** (ii) Taking $x = 1.2$ as a first approximation to this solution, apply Newton's method once to find a closer approximation to the solution. Give your answer correct to two decimal places.

QUESTION FOUR (Start a new answer booklet)

Marks

- 2** (a) Write $3\sin x + \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$, where $0 \leq \alpha \leq \frac{\pi}{2}$.

(b)



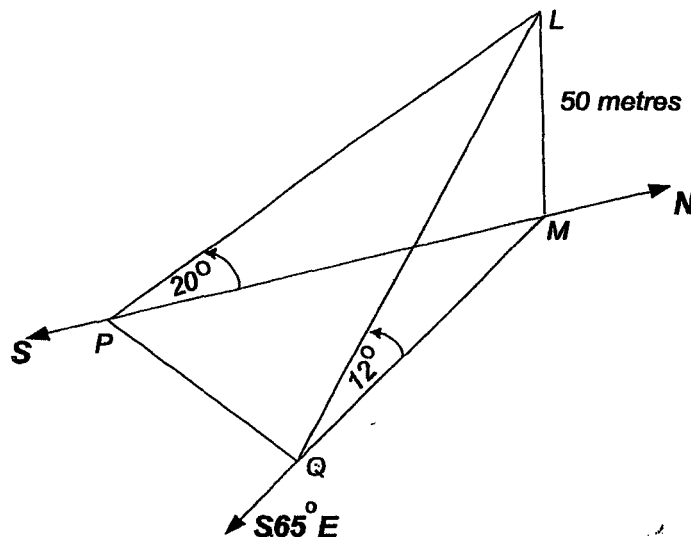
In the diagram above, the points A, B, C and D lie on a circle with centre O . The line TA is a tangent to the circle. The chord BC is produced to R . The interval AO bisects $\angle BAD$ and $BC = CD$.

Let $\angle DBC = \alpha$.

Copy the diagram onto your answer paper.

- 2** (i) Prove that $\angle DCR = 2\alpha$.
- 1** (ii) Show that $\angle OAD = \alpha$.
- 2** (iii) Prove that $\angle ABC$ is a right angle.

(c)



From the top L of a lighthouse 50 metres high a boat is observed at a point P due south at an angle of depression of 20° , as shown in the diagram above. The boat drifts at a constant speed and in a constant direction. After 10 minutes it is again observed from the top of the lighthouse at the point Q at an angle of depression of 12° . The base M of the lighthouse is at sea-level, and the bearing of Q from M is $S65^\circ E$.

- 1 (i) Find an expression for PM .
- 3 (ii) Show that the distance PQ is given by

$$PQ = 50\sqrt{\cot^2 20^\circ + \cot^2 12^\circ - 2 \cot 20^\circ \cot 12^\circ \cos 65^\circ}.$$

- 1 (iii) How fast was the boat drifting? Give your answer in metres per second, correct to two significant figures.

QUESTION FIVE (Start a new answer booklet)

Marks

2 (a) (i) Differentiate $x \cos^{-1} x - \sqrt{1 - x^2}$.

1 (ii) Hence evaluate $\int_0^1 \cos^{-1} x \, dx$.

5 (b) Use the substitution $u = 1 - x$ to evaluate $\int_{-3}^0 \frac{x}{\sqrt{1-x}} \, dx$.

4 (c) By considering the expansion of $(1+x)^{2n} = (1+x)^n(1+x)^n$ in two different ways, show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

THE EXAMINATION PAPER CONTINUES ON THE NEXT PAGE

QUESTION SIX (Start a new answer booklet)

(a) Let $(3 + 2x)^{20} = \sum_{r=0}^{20} a_r x^r$.

Marks

1

(i) Write an expression for a_r .

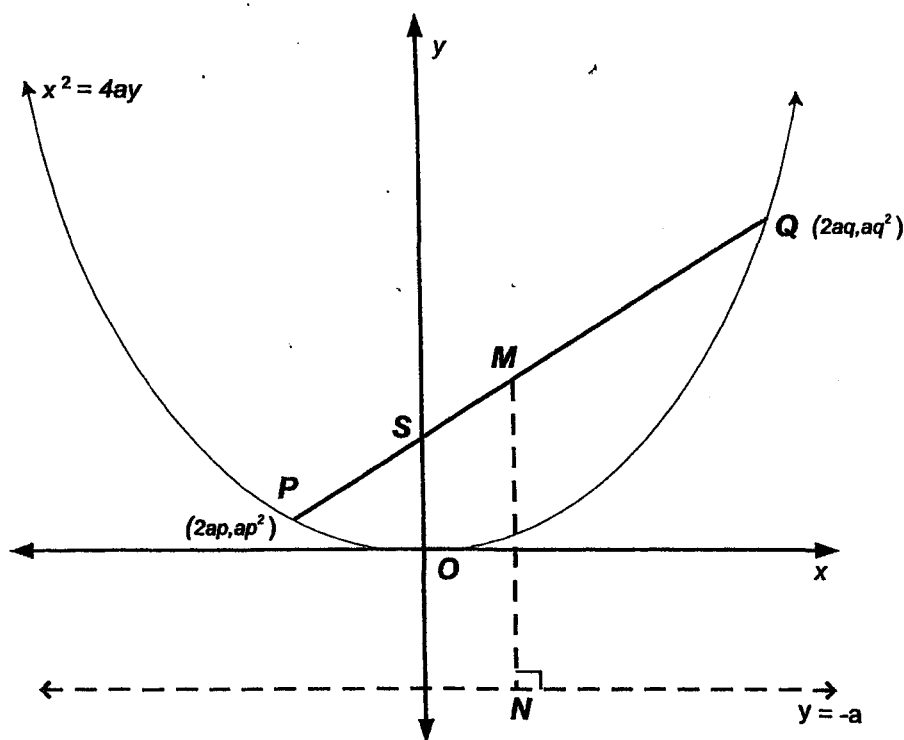
1

(ii) Show that $\frac{a_{r+1}}{a_r} = \frac{40 - 2r}{3r + 3}$.

4

(iii) Hence find the greatest coefficient in the expansion of $(3 + 2x)^{20}$.

(b)



Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be points on the parabola $x^2 = 4ay$, as shown in the above diagram.

1

(i) Show that the equation of the chord PQ is $y = \frac{p+q}{2}x - apq$.

1

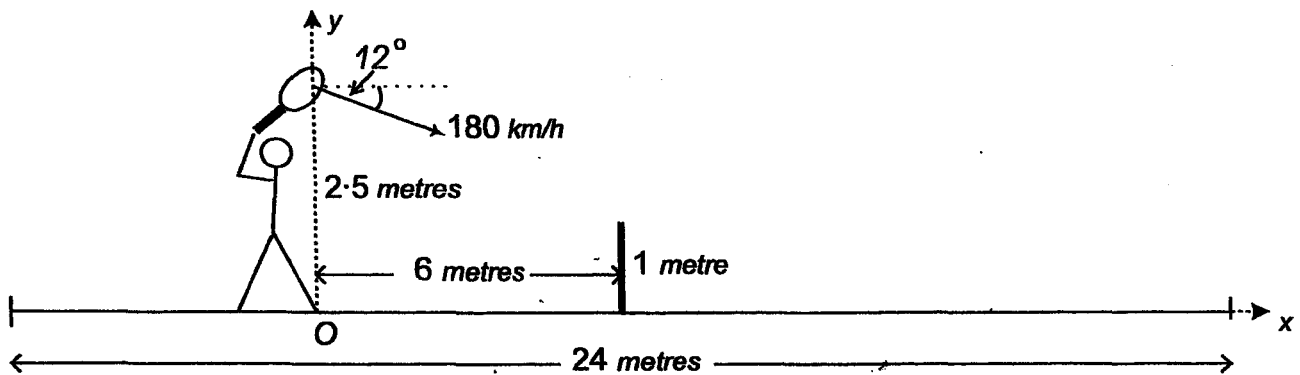
(ii) Show that if the chord PQ passes through the focus $S(0, a)$, then $pq = -1$.

4

(iii) M is the midpoint of the focal chord PQ . N lies on the directrix such that MN is perpendicular to the directrix. T is the midpoint of MN . Find the locus of T .

QUESTION SEVEN (Start a new answer booklet)

(a)



In the diagram above, a tennis court is 24 metres long and has a net one metre high positioned in the middle.

During a match a player standing 6 metres from the net smashes a ball into the opposing court with an initial speed of 180 km/h. The ball is hit parallel to the sideline and is projected with an angle of depression of 12° from a height of 2.5 metres above the ground. Let $g = 10 \text{ m/s}^2$.

Marks

3

- (i) Taking the axes as given on the diagram, show that the horizontal and vertical components of the displacement are given by

$$x = 50t \cos 12^\circ \quad \text{and} \quad y = -5t^2 - 50t \sin 12^\circ + 2.5$$

respectively, where t is the time in seconds and both x and y are measured in metres.

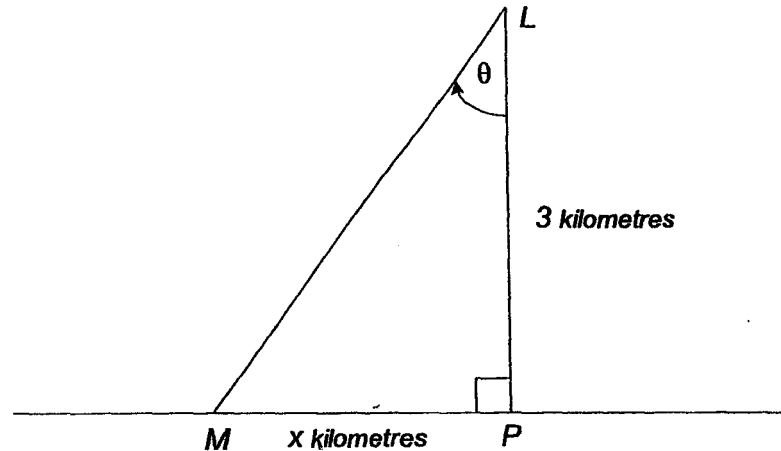
2

- (ii) By what margin does the ball clear the net? Give your answer correct to the nearest centimetre.

2

- (iii) How far from the opposing court's baseline does the ball land? Give your answer correct to the nearest centimetre.

(b)



In the diagram above, a lighthouse L containing a revolving beacon is located out at sea, 3 kilometres from P , the nearest point on a straight shoreline. The beacon rotates clockwise with a constant rotation rate of 4 revolutions per minute and throws a spot of light onto the shoreline.

When the spot of light is at M , x km from P , the angle at L is θ .

- 1 (i) Explain why $\frac{d\theta}{dt} = 8\pi$, where t is the time measured in minutes.
- 2 (ii) How fast is the spot moving when it is at P ?
- 2 (iii) How fast is the spot moving when it is at a point on the shoreline 2 km from P ?

JCM

QUESTION 1

$$(a) \quad x = \frac{3 \times 4 + 1 \times (-5)}{3+1}$$

$$= \frac{7}{4}$$

$$y = \frac{3 \times (-3) + 1 \times 6}{4}$$

$$= -\frac{3}{4}$$

the point is $(\frac{7}{4}, -\frac{3}{4})$

$$(b) \quad m_1 = -\frac{1}{2}, \quad m_2 = \frac{1}{3}$$

let θ be the acute angle

$$\tan \theta = \left| \frac{-\frac{1}{2} - \frac{1}{3}}{1 + (-\frac{1}{2})(\frac{1}{3})} \right|$$

$$\theta = 45^\circ$$

(c) (i) the angle at the centre is equal to twice the angle at the circumference when they are subtended by the same arc. ✓

(ii) $\angle OBC = 60^\circ$ (alternate angles, $AO \parallel BC$) ✓

$x = 90$ (angle sum of $\triangle BCD$) ✓

$$(d) \quad P(x) = x^3 - x^2 - 10x - 8$$

$$(i) \quad P(-1) = -1 - 1 + 10 - 8 = 0$$

so $x = -1$ is a zero of $P(x)$ ✓

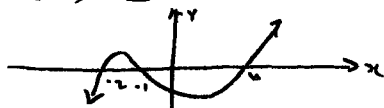
(ii) $(x+1)$ is a factor of $P(x)$

$$\begin{array}{r} x^2 - 2x - 8 \\ x+1 \overline{) x^3 - x^2 - 10x - 8} \\ \underline{x^3 + x^2} \\ -2x^2 - 10x \\ \underline{-2x^2 - 2x} \\ -8x - 8 \\ \underline{-8x - 8} \\ 0 \end{array}$$

$$P(x) = (x+1)(x^2 - 2x - 8)$$

$$= (x+1)(x-4)(x+2)$$

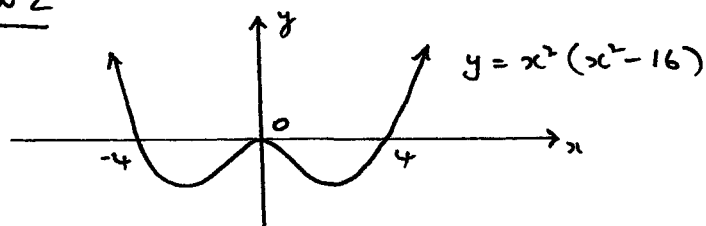
$$(iii) \quad P(x) \leq 0$$



$$x \leq -2 \text{ or } -1 \leq x \leq 4$$

QUESTION 2

(a)



✓

(b) (i) $x^2 + 4x + 5 = (x+2)^2 + 1$

✓

(ii) $\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{1 + (x+2)^2}$

$= \tan^{-1}(x+2) + C$

✓✓

(c) $\cos 2x = \cos x$

$2x = 2n\pi \pm x$

✓

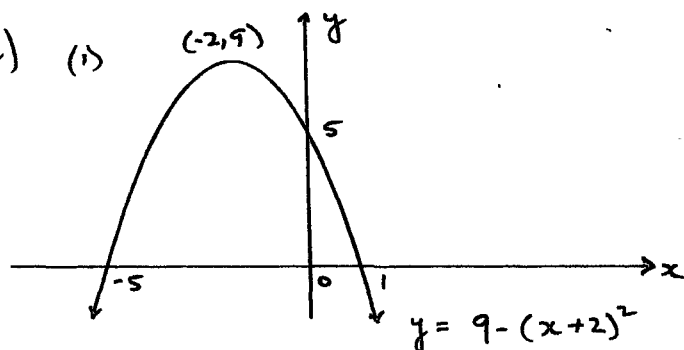
$3x = 2n\pi$ or $x = 2n\pi$

✓

$x = \frac{2n\pi}{3}$ for any integer n

✓

(d) (i)

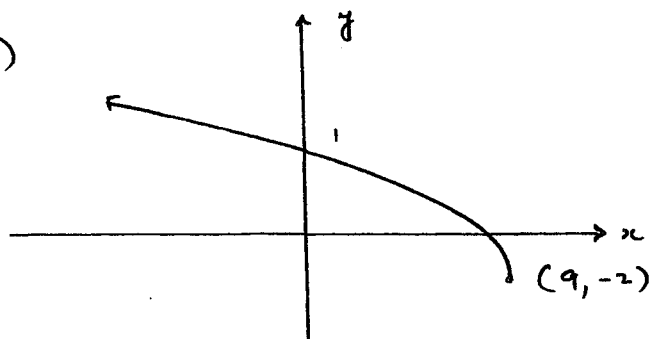


✓✓

(ii) $x \geq -2$

✓

(iii)



✓✓

QUESTION 3

$$(a) \lim_{x \rightarrow 0} \frac{\sin 4x}{\tan 2x} = \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \times \lim_{x \rightarrow 0} \frac{2x \times 2}{\tan 2x} = 2$$

$$(b) \left(x + \frac{1}{x^2}\right)^9$$

$$T_r = {}^9C_r x^{9-r} (x^{-2})^{r-1}$$

$$= {}^9C_r x^{3-r}$$

for the term independent of x

$$3-r=0$$

$$r=3$$

Hence the term is ${}^9C_3 = 84$

$$(c) \frac{dv}{dt} = 72$$

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$$

$$\frac{dv}{dr} = 4\pi r^2, \quad \frac{ds}{dr} = 8\pi r$$

$$\frac{ds}{dt} = \frac{ds}{dr} \times \frac{dr}{dt} \times \frac{dr}{dv}$$

$$= \frac{8\pi r \times 72}{4\pi r^2}$$

$$= \frac{2 \times 72}{r}$$

when $r=12$ $\frac{ds}{dt} = 12 \text{ m}^2/\text{s}$

$$(d)(i) \text{ Consider } f(x) = \sin x - x + \frac{1}{2}$$

$$f(0.5) > 0$$

$$f(1.8) < 0$$

so there is a root between $x=0.5$ and $x=1.8$

$$(ii) f'(x) = \cos x - 1$$

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.2 - \frac{f(1.2)}{f'(1.2)}$$

$$= 1.56 \quad (2 \text{ decimal places})$$

QUESTION 4

$$(a) \quad 3 \sin x + \sqrt{3} \cos x = R \sin(x + \alpha)$$

$$= R \sin x \cos \alpha + R \cos x \sin \alpha$$

$$R \sin \alpha = \sqrt{3}$$

$$R \cos \alpha = 3$$

$$\tan \alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$$R = \sqrt{3^2 + \sqrt{3}^2}$$

$$= 2\sqrt{3}$$

$$3 \sin x + \sqrt{3} \cos x = 2\sqrt{3} \sin\left(x + \frac{\pi}{6}\right)$$

$$(b) \quad (i) \quad \angle BDC = \alpha \quad (\text{base angles of isosceles } \triangle) \quad \checkmark$$

$$\angle DCR = 2\alpha \quad (\text{exterior angle of } \triangle BCD) \quad \checkmark$$

$$(ii) \quad \angle BAD = 2\alpha \quad (\text{exterior angle of cyclic quad. } ABCD) \quad \checkmark$$

$$\therefore \angle OAD = \alpha \quad (OA \text{ bisects } \angle BAD)$$

$$(iii) \quad OA \perp AT \quad (\text{radius is perpendicular to the tangent at the point of contact}) \quad \checkmark$$

$$\text{so, } \angle TAD = 90^\circ - \alpha$$

$$\angle ABD = \angle TAD \quad (\text{alternate segment theorem}) \quad \checkmark$$

$$\text{so, } \angle ABC = (90^\circ - \alpha) + \alpha$$

$$= 90^\circ$$

$$(c) \quad (i) \quad \text{In } \triangle LMP: \quad \tan 20^\circ = \frac{LM}{PM}$$

$$PM = 50 \cot 20^\circ \text{ metres} \quad \checkmark$$

$$(ii) \quad PQ^2 = PM^2 + QM^2 - 2 \cdot PM \cdot QM \cdot \cos \angle PMQ \quad (\text{cosine rule})$$

$$= 50^2 \cot^2 20^\circ + 50^2 \cot^2 12^\circ - 2 \cdot 50 \cot 20^\circ \cdot 50 \cot 12^\circ \cdot \cos 65^\circ \quad \checkmark \checkmark$$

$$\text{so, } PQ = 50 \sqrt{\cot^2 20^\circ + \cot^2 12^\circ - 2 \cot 20^\circ \cot 12^\circ \cos 65^\circ} \quad \checkmark$$

$$(iii) \quad \text{Speed} = \frac{PQ}{10 \times 60}$$

$$= 0.36 \text{ m/s} \quad (2 \text{ sig. fig.}) \quad \checkmark$$

QUESTIONS

$$(a) (i) \frac{d}{dx} (x \cos^{-1} x - \sqrt{1-x^2})$$

$$= \cos^{-1} x - \frac{x}{\sqrt{1-x^2}} - \frac{-\frac{1}{2} 2x}{\sqrt{1-x^2}}$$

$$= \cos^{-1} x$$

✓✓

$$(ii) \int \cos^{-1} x \, dx = [x \cos^{-1} x - \sqrt{1-x^2}]_0^1$$

$$= 1$$

✓

$$(b) u = 1-x \Rightarrow x = 1-u$$

$$du = -dx$$

$$\text{when } x = -3 \quad u = 4$$

$$\text{when } x = 0 \quad u = 1$$

✓

$$I = \int_4^1 \frac{1-u}{\sqrt{u}} \cdot -du$$

✓✓

$$= \int_1^4 u^{-1/2} - u^{1/2} \, du$$

✓

$$= [2u^{1/2} - \frac{2}{3} u^{3/2}]_1^4$$

$$= (4 - \frac{2}{3} \times 4 \times 2) - (2 - \frac{2}{3})$$

$$= -\frac{8}{3}$$

✓

$$(c) \text{ for } (1+x)^{2n}$$

$$\text{the coefficient of } x^n \text{ is } \binom{2n}{n}$$

✓

$$(1+x)^{2n} = (1+x)^n (1+x)^n$$

$$= \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n \right] \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n \right]$$

$$\text{the coefficient of } x^n \text{ is: } \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \dots + \binom{n}{n}\binom{n}{0}$$

✓

$$\text{since } \binom{n}{r} = \binom{n}{n-r} \text{ then the coefficient of } x^n \text{ is}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2$$

✓

$$\text{Equating the coefficients of } x^n \text{ gives}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$$

QUESTION 6

$$(a) (i) (3+2x)^{20} = \sum_{r=0}^{20} {}^{20}C_r 3^{20-r} (2x)^r$$

$$\text{so, } a_r = {}^{20}C_r 3^{20-r} 2^r$$

✓

$$\begin{aligned} (ii) \quad \frac{a_{r+1}}{a_r} &= \frac{{}^{20}C_{r+1} 3^{19-r} 2^{r+1}}{{}^{20}C_r 3^{20-r} 2^r} \\ &= \frac{20-r}{r+1} \times \frac{2}{3} \\ &= \frac{40-2r}{3r+3} \end{aligned}$$

✓

$$(iii) \text{ let } \frac{a_{r+1}}{a_r} > 1$$

$$\text{then, } \frac{40-2r}{3r+3} > 1$$

✓

$$40-2r > 3r+3$$

$$5r < 37$$

$$r < 7\frac{2}{3}$$

✓

$$\text{when } r=7 : a_8 > a_7$$

$$r=6 : a_7 > a_6$$

⋮

$$r=0 : a_1 > a_0$$

$$\text{i.e. } a_8 > a_7 > a_6 > \dots > a_0$$

$$\text{if } \frac{a_{r+1}}{a_r} < 1 \text{ then } a_8 > a_9 > \dots > a_{20}$$

} ✓

So the greatest co-efficient is $a_8 = {}^{20}C_8 3^{12} 2^8$ ✓

$$(b)(i) m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$= \frac{p+q}{2}$$

$$\text{equation of PQ: } y - ap^2 = \frac{p+q}{2} (x - 2ap)$$

$$\text{so, } y = \frac{p+q}{2} x - apq$$

$$(ii) \text{ If } S \in PQ \text{ then when } x=0, y=a$$

$$\text{i.e. } a = 0 - apq$$

$$\text{so, } pq = -1$$

$$(iii) M \text{ is } (a(p+q), \frac{ap^2 + aq^2}{2})$$

$$N \text{ is } (a(p+q), -a)$$

$$\text{so } T \text{ is } (a(p+q), \frac{ap^2 + aq^2 - 2a}{4})$$

The locus of T is

$$x = a(p+q) \quad \text{--- (1)}$$

$$y = \frac{a}{4} (p^2 + q^2 - 2) \quad \text{--- (2)}$$

$$\text{from (ii) } pq = -1, \quad y = \frac{a}{4} (p^2 + q^2 + 2pq)$$

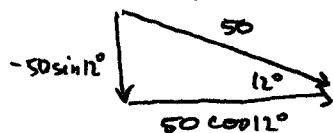
$$\text{i.e. } y = \frac{a}{4} (p+q)^2$$

$$\text{so, } y = \frac{a}{4} \frac{x^2}{a^2} \quad \text{from (1)}$$

$$\text{i.e. } x^2 = 4ay$$

QUESTION 7

(a)(i) $180 \text{ km/h} = 50 \text{ m/s}$ ✓



$$\dot{x} = 50 \cos 12^\circ$$

$$x = 50t \cos 12^\circ + C_1$$

when $t=0$, $x=0$

so, $x = 50t \cos 12^\circ$

$$\dot{y} = -10$$

$$y = -10t + C_2$$

when $t=0$, $y = -50 \sin 12^\circ$

so, $y = -10t - 50 \sin 12^\circ$

$$y = -5t^2 - 50t \sin 12^\circ + C_3$$

when $t=0$, $y = 2.5$

so, $y = -5t^2 - 50t \sin 12^\circ + 2.5$

(ii) when $x=6$, $t = \frac{6}{50 \cos 12^\circ}$ ✓

when $t = \frac{6}{50 \cos 12^\circ}$, $y = 1.149 \dots$

so the ball clears the net by 15cm. ✓

(iii) when $y=0$, $5t^2 + 50t \sin 12^\circ - 2.5 = 0$

$$t = \frac{-50 \sin 12^\circ \pm \sqrt{(50 \sin 12^\circ)^2 + 50}}{10}$$
 ✓

when $t = \frac{-50 \sin 12^\circ + \sqrt{(50 \sin 12^\circ)^2 + 50}}{10}$

$$x = 10.6468 \dots$$

So it lands 7.35 metres from the base line ✓

(b) (i) $4 \text{ revs/min} = 8\pi \text{ rad/min}$

so, $\frac{d\theta}{dt} = 8\pi$ ✓

(ii) $\tan \theta = \frac{x}{3}$

$x = 3 \tan \theta$ ✓

$\frac{dx}{d\theta} = 3 \sec^2 \theta$

$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$

$= 3 \sec^2 \theta \cdot 8\pi$

$= 24\pi \sec^2 \theta$ ✓

at P $\theta = 0$

so $\frac{dx}{dt} = 24\pi \text{ km/min}$

(iii) when $x=2$, $\cos \theta = \frac{3}{\sqrt{13}}$ ✓

so, $\frac{dx}{dt} = \frac{24\pi}{\left(\frac{3}{\sqrt{13}}\right)^2}$

$= \frac{104\pi}{3} \text{ km/min}$ ✓