SYDNEY GRAMMAR SCHOOL

NAME

MATHS MASTER

CANDIDATE NUMBER

2025 Trial Examination

Form VI Mathematics Extension 1

Monday 18th August, 2025
12:50pm

General Instructions

- Reading time — 10 minutes
- Working time — 2 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Fourteen Questions — 70 Marks

Section I (10 marks) Questions 1 – 10

- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (60 marks) Questions 11 – 14

- Each question is worth 15 marks.
- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- Your name and master should only be written on this page.
- Write your candidate number on this page, on each booklet and on the multiple choice sheet.
- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.

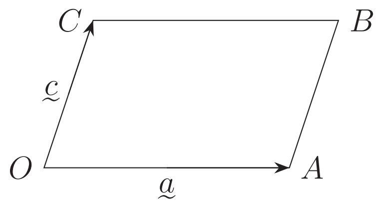
Checklist

- Reference sheet
- Multiple-choice answer sheet
- 4 booklets per boy
- Candidature: 143 pupils

Writer: BLR

Section I

1. The diagram below shows the parallelogram $OABC$ with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. **1**



Which expressions correctly represent the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$?

(A) $\overrightarrow{OB} = \underset{\sim}{a} - \underset{\sim}{c}$ and $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{c}$

(B) $\overrightarrow{OB} = \underset{\sim}{c} - \underset{\sim}{a}$ and $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{c}$

(C) $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$ and $\overrightarrow{AC} = \underset{\sim}{a} - \underset{\sim}{c}$

(D) $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$ and $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

2. Given $y = \sin^{-1}(e^{2x})$, what is $\dfrac{dy}{dx}$? **1**

(A) $\dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$

(B) $\dfrac{e^{2x}}{\sqrt{1-e^{4x}}}$

(C) $\dfrac{2e^{2x}}{\sqrt{1+e^{4x}}}$

(D) $\dfrac{2e^{2x}}{\sqrt{e^{4x}-1}}$

3. Which of the following is the correct expression for the coefficient of x^3 in the expansion of $(4x+5)^9$? **1**

(A) ${}^9C_6 \times 4^3 \times 5^6 \times x^3$

(B) ${}^9C_6 \times 4^6 \times 5^3 \times x^3$

(C) ${}^9C_6 \times 4^3 \times 5^6$

(D) ${}^9C_6 \times 4^6 \times 5^3$

4. When a fish market has a supply of x kilograms of salmon available for sale on a given day, it charges y dollars per kilogram according to the equation $y = 32 - 8\tan^{-1}\dfrac{x}{100}$. If the daily supply is decreasing at a rate of 5 kilograms per day, at what rate is the price changing when the supply is 100 kilograms? **1**

   (A) decreasing at \$0.04 per day

   (B) decreasing at \$0.20 per day

   (C) increasing at \$0.04 per day

   (D) increasing at \$0.20 per day

5. In how many ways can the letters of the word SASSAFRAS be arranged in a line if the letters F and R must be adjacent? **1**

   (A) $\dfrac{8!}{3!\,4!}$

   (B) $\dfrac{8!\,2!}{3!\,4!}$

   (C) $\dfrac{9!}{3!\,4!}$

   (D) $\dfrac{9!\,2!}{3!\,4!}$

6. The function $f(x)$ has an inverse function, $f^{-1}(x)$. Which of the following is always a correct expression for the inverse of $y = f(3x)$? **1**

   (A) $f^{-1}\left(\dfrac{x}{3}\right)$

   (B) $f^{-1}(3x)$

   (C) $\dfrac{1}{3}f^{-1}(x)$

   (D) $3f^{-1}(x)$

7. Which expression is **not** equivalent to $\displaystyle\int \sin x \cos x \, dx$? **1**

   (A) $-\dfrac{\cos^2 x}{2} + c$

   (B) $\dfrac{\sin^2 x}{2} + c$

   (C) $-\dfrac{\cos(2x)}{4} + c$

   (D) $-\dfrac{\cos(2x)}{2} + c$

8. Which of the following statements is **always** true, given that $\underset{\sim}{u} \neq \underset{\sim}{0}$ and $\underset{\sim}{p} \neq \underset{\sim}{0}$? **1**

   (A) If $\text{proj}_{\underset{\sim}{p}}\underset{\sim}{u} = \text{proj}_{\underset{\sim}{p}}(-\underset{\sim}{u})$, then $\underset{\sim}{u} \cdot \underset{\sim}{p} = 0$.

   (B) If $\text{proj}_{\underset{\sim}{p}}\underset{\sim}{u} = \underset{\sim}{u}$, then $\underset{\sim}{u} = \underset{\sim}{p}$.

   (C) If $\text{proj}_{\underset{\sim}{p}}\underset{\sim}{u} = \text{proj}_{\underset{\sim}{p}}\underset{\sim}{w}$, then $\underset{\sim}{u} = \underset{\sim}{w}$.

   (D) If $\text{proj}_{\underset{\sim}{p}}\underset{\sim}{u} = \text{proj}_{\underset{\sim}{q}}\underset{\sim}{u}$, then $\underset{\sim}{p} = \underset{\sim}{q}$.

9. Consider the range, time of flight, and maximum height of a projectile. Which of these quantities will not change if the initial vertical velocity of the projectile is fixed, but the initial horizontal velocity is allowed to vary? **1**

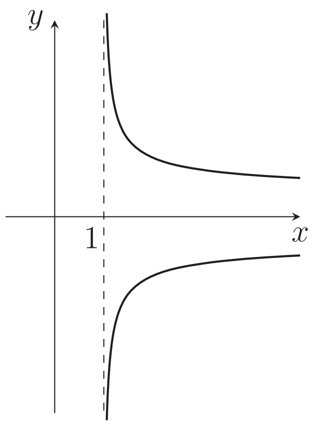
   (A) maximum height only

   (B) time of flight only

   (C) maximum height and time of flight

   (D) range and maximum height

10. If $f(x) = \ln x$, which of the following best describes the equation of the graph below? **1**



   (A) $y^2 = \dfrac{1}{f(x)}$

   (B) $y = \dfrac{1}{f(x)}$

   (C) $|y| = \sqrt{f(x)}$

   (D) $y = \sqrt{f(x)}$

End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (15 marks) Start a new answer booklet.

(a) For the vectors $\underset{\sim}{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, calculate each of the following.

(i) $\underset{\sim}{u} - 5\underset{\sim}{v}$ **1**

(ii) $\underset{\sim}{u} \cdot \underset{\sim}{v}$ **1**

(b) Expand and simplify $(x-2)^3$. **2**

(c) Solve $\dfrac{4}{x-1} \leq 9$. **3**

(d) Consider the polynomial $P(x) = 2x^3 - 7x^2 - 12x + 45$.

(i) Show that $x = 3$ is a zero with multiplicity of at least two. **2**

(ii) Hence, find the third zero of $P(x)$. **1**

(e) Write down the domain and range of $y = 2\cos^{-1}\left(\dfrac{x}{3}\right)$. **2**

(f) Use the substitution $u = x + 6$ to find $\displaystyle\int x\,(x+6)^5\; dx$. **3**

The paper continues on the next page

QUESTION TWELVE (15 marks) Start a new answer booklet.

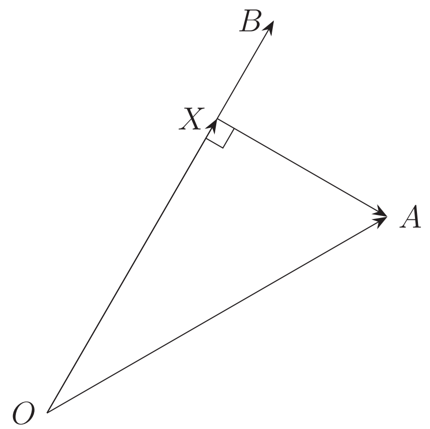
(a) (i) Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$ and $0 \leq \alpha \leq \dfrac{\pi}{2}$. **2**

(ii) Hence, solve $\sin x - \sqrt{3}\cos x = 2$ for $0 \leq x \leq \pi$. **1**

(b) Evaluate $\displaystyle\int_0^2 \frac{dx}{4 + 3x^2}$. **3**

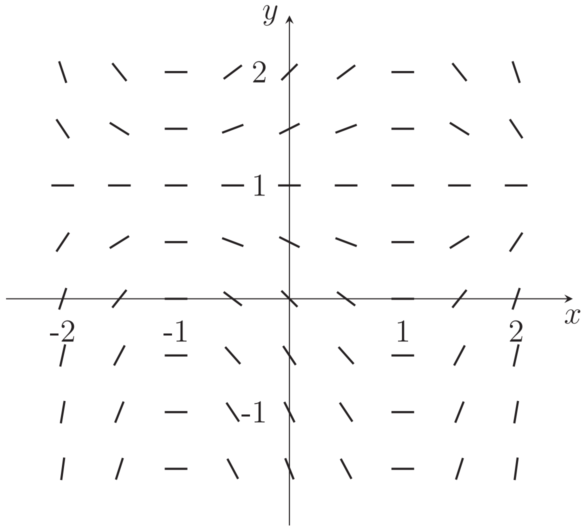
(c) Prove by mathematical induction that $3^{2n} - 1$ is divisible by 8 for all integers $n \geq 1$. **3**

(d) The diagram below shows the position vectors $\overrightarrow{OA} = \begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix}$. The point X is the foot of the perpendicular from A to OB. **2**

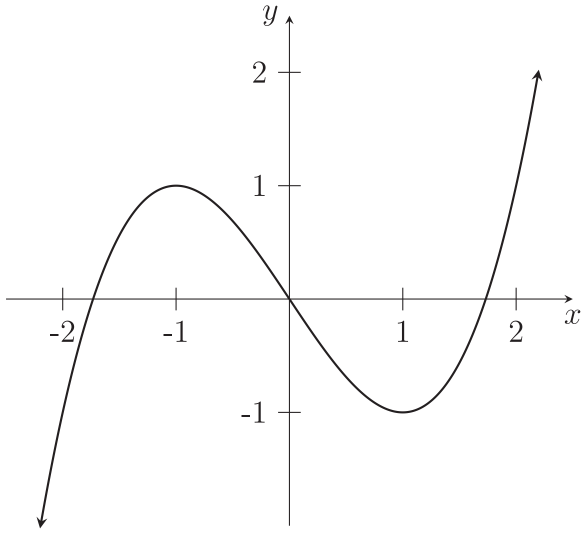


Find $\overrightarrow{OX}$ by calculating the projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$.

(e) Consider the differential equation $\dfrac{dy}{dx} = -\left(x^2 - 1\right)(y - 1)$ and its slope field, which has been graphed below.



(i) By referring to the slope field for the differential equation, explain why a solution could not have the graph shown below. **1**

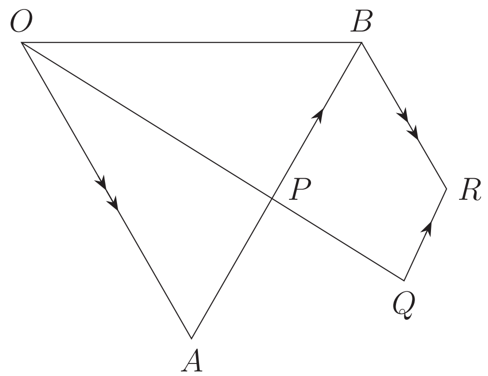


(ii) Find the particular solution of the differential equation that passes through the point $(0, 2)$. **3**

The paper continues on the next page

QUESTION THIRTEEN (15 marks) Start a new answer booklet.

(a) The points O, A, and B form an equilateral triangle and the points B, P, Q, and R form a trapezium, as shown in the diagram below. The points O, P, and Q are collinear, $\overrightarrow{BR} = \frac{1}{2}\overrightarrow{OA}$, and $\overrightarrow{QR} = \lambda\overrightarrow{AB}$ for some constant λ.



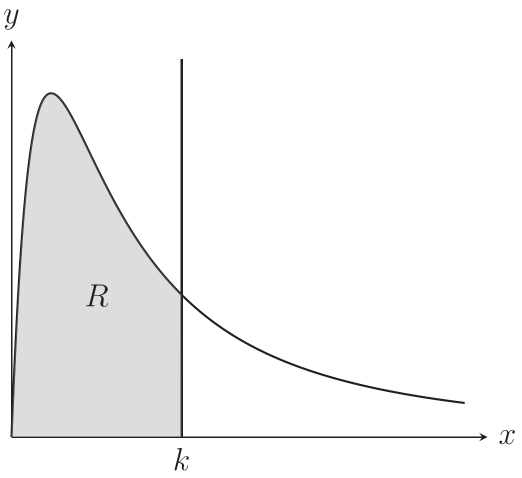
Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

(i) Show that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \frac{1}{2}\left|\underset{\sim}{a}\right|^2$. **1**

(ii) Write down expressions for $\overrightarrow{AB}$ and $\overrightarrow{OQ}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$, and λ. **2**

(iii) Given that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{OQ}$, find the value of λ. **2**

(b) The diagram below shows the graph of the function $f(x) = \dfrac{\pi}{\pi x + 1}\sin\left(\dfrac{\pi}{\pi x + 1}\right)$ for $x \geq 0$. The shaded region R is bounded by the graph of $y = f(x)$, the x-axis, and the vertical line $x = k$, where $k > 0$.



(i) Show that the volume V of the solid generated when R is rotated about the x-axis can be found by evaluating the integral: **2**

$$V = \frac{\pi^3}{2}\int_0^k \frac{1}{(\pi x + 1)^2}\left[1 - \cos\left(\frac{2\pi}{\pi x + 1}\right)\right] dx.$$

(ii) Use the substitution $u = \dfrac{2\pi}{\pi x + 1}$ to show that $V = \dfrac{\pi}{4}\left[\sin\left(\dfrac{2\pi}{\pi k + 1}\right) + \dfrac{2\pi^2 k}{\pi k + 1}\right]$. **3**

(iii) What does the volume V approach as $k \to \infty$? **1**

(c) (i) Prove that $\dfrac{1}{2}\cot\theta - \dfrac{1}{2}\tan\theta = \cot 2\theta$, where $\theta \neq \dfrac{k\pi}{2}$ for any integer k. **1**

(ii) Hence, use mathematical induction to prove that, for all positive integers n, **3**

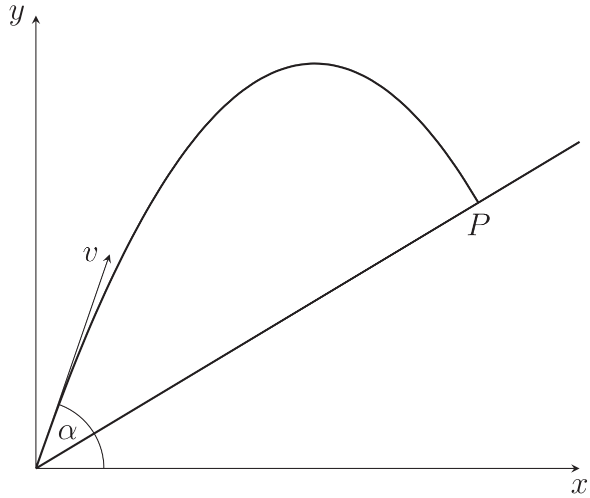
$$\frac{1}{2}\tan\left(\frac{x}{2}\right) + \frac{1}{2^2}\tan\left(\frac{x}{2^2}\right) + \frac{1}{2^3}\tan\left(\frac{x}{2^3}\right) + \cdots + \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - \cot x$$

where $x \neq k\pi$ for any integer k.

The paper continues on the next page

QUESTION FOURTEEN (15 marks) Start a new answer booklet.

(a) As part of avalanche control programs, small explosives are fired into mountainsides. An explosive is fired upwards along a mountainside that has a gradient of 0.6. The explosive is launched with an initial speed of $v\,\mathrm{ms}^{-1}$ at an angle of α to the horizontal, and hits the mountainside at the point of impact P, as shown in the diagram below.



The position of the projectile after t seconds is given by

$$\underset{\sim}{r} = \begin{bmatrix} vt\cos\alpha \\ vt\sin\alpha - 5t^2 \end{bmatrix} \qquad \text{Do NOT prove this.}$$

(i) Show that the Cartesian equation of the path of the projectile is given by **1**

$$y = x\tan\alpha - \frac{5x^2}{v^2}\sec^2\alpha.$$

(ii) Show that the horizontal distance of P, in metres, from the point of projection is given by $\frac{1}{5}v^2\cos^2\alpha\,(\tan\alpha - 0.6)$. **2**

(iii) Given that the path of the explosive is perpendicular to the mountainside at the point of impact, show that the angle of projection is approximately 71°, correct to the nearest degree. **2**

(iv) Hence, determine the impact speed as a percentage of the initial speed. Give your answer correct to the nearest percent. **3**

(b) (i) The differential equation $\dfrac{dx}{dt} = kx$, where k is a constant, has initial condition $x = x_0$ when $t = 0$. **1**

Show that $x = x_0 e^{kt}$ satisfies the differential equation and the initial condition.

(ii) The rate at which new Wikipedia articles are written can be modelled by the differential equation:

$$\frac{dN}{dt} = 0.17N \ln\left(\frac{7.2}{N}\right),$$

where N is the number of Wikipedia articles in millions, and t is the number of years after the launch of Wikipedia at the beginning of January 2001.

(α) Find any constant solutions of the differential equation. **1**

Consider the substitution $u = \ln\left(\dfrac{7.2}{N}\right)$.

(β) Initially, at the beginning of January 2001, there were 3600 Wikipedia articles. That is, $N = 0.0036$ when $t = 0$. Show that the initial value of u, correct to two significant figures, is $u_0 = 7.6$. **1**

(γ) Use the given substitution to find the solution of the differential equation $N(t)$ that satisfies the initial condition. **4**

END OF PAPER

Extension 1 Trial, 2025

1. $\vec{OB} = \underset{\sim}{a} + \underset{\sim}{c}$

$\vec{AC} = \underset{\sim}{c} - \underset{\sim}{a}$ D

2. $y = \sin^{-1}(e^{2x})$

$y' = \dfrac{2e^{2x}}{\sqrt{1-(e^{2x})^2}}$ A

3. $(4x+5)^9$

$= {}^9C_0(4x)^9 + {}^9C_1(4x)^8(5) + \dots + {}^9C_6(4x)^3(5)^6$

coefficient is ${}^9C_6 \times 4^3 \times 5^6$

C

4. $\dfrac{dy}{dx} = -\dfrac{8 \times \frac{1}{100}}{1+\left(\frac{x}{100}\right)^2}$

$= \dfrac{-800}{100^2 + x^2}$

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$

when $x = 100$,

$= \dfrac{-800}{2 \times 100^2} \times (-5)$

$= 0.2$ D

5. SSSSAAA$\boxed{FR}$

$\dfrac{8!}{4! \times 3!} \times 2!$ B

6. inverse: $x = f(3y)$

$3y = f^{-1}(x)$

$y = \frac{1}{3}f^{-1}(x)$ C

7. $-\frac{1}{2}\cos^2 x + c = -\frac{1}{2}(1-\sin^2 x) + c$

$= \frac{1}{2}\sin^2 x + d$

$= \frac{1}{2}\left[\frac{1}{2}(1-\cos 2x)\right] + d$

$= -\frac{1}{4}\cos 2x + e$

options A, B, C all equivalent D

8. $\left.\begin{array}{l} \text{proj}_{\underset{\sim}{p}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\hat{\underset{\sim}{p}})\hat{\underset{\sim}{p}} \\ \text{proj}_{\underset{\sim}{p}}(-\underset{\sim}{u}) = -(\underset{\sim}{u}\cdot\hat{\underset{\sim}{p}})\hat{\underset{\sim}{p}} \end{array}\right\}$ equal when $\underset{\sim}{u}\cdot\underset{\sim}{p} = 0$

A

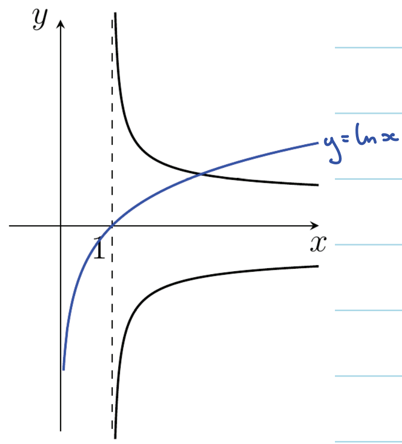
9. max. height when $\dot{y} = 0$ ✓ ind. of $\dot{x}$

time of flight: when $y = k$, where k is the vertical displacement at impact ✓ ind. of $\dot{x}$

range is x value at time of impact, x is found by integrating $\dot{x}$ ✗ ind. of $\dot{x}$

C

10.



symmetric across x-axis, so not B or D

$y \to \pm\infty$ as $x \to 0$, so A

Question 11

a) i) $\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 5\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -14 \\ -12 \end{bmatrix}$

ii) $1\times 3 \quad -2\times 2$

$= -1$

b) $(x-2)^3$

$= x^3 - 3x^2(2) + 3x(2)^2 - (2)^3$

$= x^3 - 6x^2 + 12x - 8$

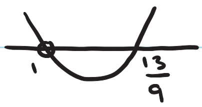
c) $\dfrac{4}{x-1} \leq 9$, note that $x \neq 1$

$4(x-1) \leq 9(x-1)^2$

$0 \leq 9(x-1)^2 - 4(x-1)$

$(x-1)[9(x-1) - 4] \geq 0$

$(x-1)(9x-13) \geq 0$



$x < 1, \quad x \geq \frac{13}{9}$

Comments and Criteria

✓ correct answer

very well done

✓ correct answer

very well done

✓ binomial coefficients

✓ signs, powers of x, powers of 2

generally well done

most common error: simple algebraic mistakes

✓ multiplying throughout by the square of the denominator

✓ factorised quadratic inequality

✓ solution

there were two common mistakes:

- missing that $x \neq 1$, which gave an incorrect answer of $x \leq 1$, $x \geq \frac{13}{9}$
- errors in sketching and solving the quadratic inequality, giving an incorrect answer of $1 < x \leq \frac{13}{9}$

d) i) $P(3) = 2 \times 3^3 - 7 \times 3^2 - 12 \times 3 + 45$

$= 0$

✓ showing that $x = 3$ is a zero of $P(x)$, with clear supporting working

$P'(x) = 6x^2 - 14x - 12$

$P'(3) = 6 \times 3^2 - 14 \times 3 - 12$

$= 0$

✓ showing that $x = 3$ is a zero of $P'(x)$, with clear supporting working

$\therefore x = 3$ is a double root

marks were commonly lost for:

- not showing the substitution of $x = 3$ into $P(x)$ and $P'(x)$, or other working of equivalent merit, to justify the claim that $P(3) = 0$ and $P'(3) = 0$
- showing that $P'(3) = 0$ but not also showing that $x = 3$ is a zero of $P(x)$

ii) let the roots be $3, 3, \alpha$

sum of the roots $= -\frac{b}{a}$

$6 + \alpha = \frac{7}{2}$

$\alpha = -\frac{5}{2}$

✓ finding the third zero

e) Domain: $-3 \le x \le 3$

Range: $0 \le y \le 2\pi$

✓ domain

✓ range

done well by most candidates

f) $\int x(x+6)^5 \, dx$ let $u = x + 6$

$du = dx$

$= \int (u-6) u^5 \, du$

✓ finding u, du, and making the substitution

$= \int u^6 - 6u^5 \, du$

$= \frac{1}{7} u^7 - u^6 + c$

✓ finding primitive

$= \frac{1}{7}(x+6)^7 - (x+6)^6 + c$

✓ rewriting in terms of x

generally done well

most common error: skipping the final step of rewriting in terms of x

Question 12

a) i) let $\sin x - \sqrt{3}\cos x = R\sin(x-\alpha)$

$= R\sin x\cos\alpha - R\cos x\sin\alpha$

equating coefficients of $\sin x$, $\cos x$:

$1 = R\cos\alpha$ ① $\sqrt{3} = R\sin\alpha$ ②

$①^2 + ②^2$ $② \div ①$

$1 + 3 = R^2(\cos^2\alpha + \sin^2\alpha)$ $\tan\alpha = \sqrt{3}$

$R = 2$, as $R > 0$ $\alpha = \frac{\pi}{3}$

$\therefore \sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$

generally well answered

several candidates used an incorrect expression for the tangent ratio, writing $\frac{\cos\alpha}{\sin\alpha}$ instead of the correct $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

✓ value of R, with supporting working

✓ value of α, with supporting working

ii) $2\sin\left(x - \frac{\pi}{3}\right) = 2$

$\sin\left(x - \frac{\pi}{3}\right) = 1$

$x - \frac{\pi}{3} = \frac{\pi}{2}$

$x = \frac{5\pi}{6}$

most candidates did well

most common mistake: simple arithmetic errors

✓ finding x-value

b) $\displaystyle\int_0^2 \frac{dx}{4 + 3x^2}$

$= \displaystyle\frac{1}{3}\int_0^2 \frac{dx}{\left(\frac{2}{\sqrt{3}}\right)^2 + x^2}$

$= \displaystyle\frac{1}{3 \times \frac{2}{\sqrt{3}}}\left[\tan^{-1}\frac{\sqrt{3}x}{2}\right]_0^2$

$= \displaystyle\frac{\sqrt{3}}{6}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right)$

$= \displaystyle\frac{\sqrt{3}}{6}\left(\frac{\pi}{3} - 0\right)$

$= \displaystyle\frac{\pi\sqrt{3}}{18}$

some candidates had difficulty with recognizing that this is one of the standard integrals on the reference sheet

✓ for any two, or ✓✓ for all three

- primitive is of the form $a\tan^{-1}bx$
- $a = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$
- $b = \frac{\sqrt{3}}{2}$

✓ evaluating definite integral using primitive found in working

leaving the result in terms of inverse tan is not sufficient

c) step 1: prove true for $n=1$

$3^{2\times1}-1=8$, which is divisible by 8

✓ base case

step 2: assume true for $n=k$, $k \in \mathbb{Z}^+$

$3^{2k}-1=8A$, where $A \in \mathbb{Z}^+$

$3^{2k}=8A+1$

step 3: prove true for $n=k+1$

$3^{2(k+1)}-1$

$=3^{2k+2}-1$

$=3^2\times3^{2k}-1$

$=9(8A+1)-1$, from step 2

✓ using assumption

$=72A+9-1$

$=72A+8$

$=8(9A+1)$, which is divisible by 8 as $(9A+1)\in\mathbb{Z}^+$

✓ completed proof

step 4: true by mathematical induction for all positive integers n

induction question mostly well handled

some candidates lacked practice, resulting in incomplete or inefficient proofs, and are encouraged to study the provided solution to understand the expected structure and level of detail

d) $\vec{OA}\cdot\vec{OB}=3(2)+\sqrt{3}(2\sqrt{3})$

$=12$

$|\vec{OB}|^2=2^2+(2\sqrt{3})^2$

$=16$

$$\text{proj}_{\vec{OB}}\vec{OA}=\frac{\vec{OA}\cdot\vec{OB}}{|\vec{OB}|^2}\vec{OB}$$

$$=\frac{12}{16}\begin{bmatrix}2\\2\sqrt{3}\end{bmatrix}$$

$$=\frac{3}{2}\begin{bmatrix}1\\\sqrt{3}\end{bmatrix}$$

✓ for any one, or ✓✓ for all three:

- correctly calculating dot product anywhere in working
- correctly calculating $|\vec{OB}|$ or $|\vec{OB}|^2$ anywhere in working
- showing correct use of projection formula and simplifying

many candidates were unfamiliar with the projection formula and are encouraged to memorize it, as it is not on the reference sheet

e) i) the slope field shows that the solution should have a gradient of 0 when $y = 1$, but the graph does not have this feature

✓ a correct explanation that mentions behaviour of slope field and graph at $y = 1$

most common mistake: incorrectly stating that the point $(-1, 1)$ contradicts the slope field

ii) $\frac{dy}{dx} = -(x^2 - 1)(y - 1)$

$\int \frac{dy}{y-1} = \int -(x^2 - 1)\,dx$

✓ separation of variables

$\ln|y - 1| = -\left(\frac{x^3}{3} - x\right) + c$

✓ primitive

$y - 1 = \pm e^{x - \frac{1}{3}x^3 + c}$

$y = Ae^{x - \frac{1}{3}x^3} + 1$, where $A = \pm e^c$

when $x = 0$, $y = 2$:

$2 = Ae^0 + 1$

$A = 1$

$y = e^{x - \frac{1}{3}x^3} + 1$

✓ particular solution

most common mistakes: incorrect separation of variables and introducing errors when finding the particular solution

Question 13

a) i) ΔOAB is equilateral

$\therefore |\underset{\sim}{a}| = |\underset{\sim}{b}|$, angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $60°$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$
$$= |\underset{\sim}{a}||\underset{\sim}{a}|\cos 60°$$
$$= \frac{1}{2}|\underset{\sim}{a}|^2$$

ii) $\vec{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$$\vec{OQ} = \vec{OB} + \vec{BR} + \vec{RQ}$$
$$= \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a} - \lambda(\underset{\sim}{b} - \underset{\sim}{a})$$
$$= (\tfrac{1}{2} + \lambda)\underset{\sim}{a} + (1 - \lambda)\underset{\sim}{b}$$

iii) $\vec{OQ} \cdot \vec{AB} = 0$

$$[(\tfrac{1}{2} + \lambda)\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}] \cdot [\underset{\sim}{b} - \underset{\sim}{a}] = 0$$
$$(\tfrac{1}{2}+\lambda)\underset{\sim}{a}\cdot\underset{\sim}{b} - (\tfrac{1}{2}+\lambda)\underset{\sim}{a}\cdot\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}\cdot\underset{\sim}{b} - (1-\lambda)\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$$
$$\tfrac{1}{2}(\tfrac{1}{2}+\lambda)|\underset{\sim}{a}|^2 - (\tfrac{1}{2}+\lambda)|\underset{\sim}{a}|^2 + (1-\lambda)|\underset{\sim}{b}|^2 - \tfrac{1}{2}(1-\lambda)|\underset{\sim}{a}|^2 = 0$$
$$-\tfrac{1}{2}(\tfrac{1}{2}+\lambda)|\underset{\sim}{a}|^2 + (1-\lambda)|\underset{\sim}{a}|^2 - \tfrac{1}{2}(1-\lambda)|\underset{\sim}{a}|^2 = 0$$
$$-\tfrac{1}{2}(\tfrac{1}{2}+\lambda)|\underset{\sim}{a}|^2 + \tfrac{1}{2}(1-\lambda)|\underset{\sim}{a}|^2 = 0$$
$$-(\tfrac{1}{2}+\lambda) + (1-\lambda) = 0$$
$$-\tfrac{1}{2} - \lambda + 1 - \lambda = 0$$
$$-2\lambda + \tfrac{1}{2} = 0$$
$$2\lambda = \tfrac{1}{2}$$
$$\lambda = \tfrac{1}{4}$$

b) i) $$V = \pi\int_0^k \left[\frac{\pi}{\pi x+1}\sin\left(\frac{\pi}{\pi x+1}\right)\right]^2 dx$$

$$= \pi\int_0^k \frac{\pi^2}{(\pi x+1)^2}\sin^2\left(\frac{\pi}{\pi x+1}\right)dx$$

$$= \frac{\pi^3}{2}\int_0^k \frac{1}{(\pi x+1)^2}\left(1 - \cos\frac{2\pi}{\pi x+1}\right)dx$$

a) i) ii) both done well overall

✓ showing use of properties of equilateral triangles in the dot product

✓ expression for $\vec{AB}$ and some progress on an expression for $\vec{OQ}$

✓ expression for $\vec{OQ}$ in terms of $\underset{\sim}{a}, \underset{\sim}{b}, \lambda$

✓ correctly expanding the dot product

some expanded $(\underset{\sim}{b} - \underset{\sim}{a})\cdot(\underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{a} - \lambda\underset{\sim}{b} + \lambda\underset{\sim}{a})$ without collecting $\underset{\sim}{b}, \underset{\sim}{a}$ vectors, making the expansion harder and sometimes leading to errors

most common mistake: missing $\underset{\sim}{a}\cdot\underset{\sim}{b}$ and doing $(\underset{\sim}{b} - \underset{\sim}{a})\cdot[(1-\lambda)\underset{\sim}{b} + (\tfrac{1}{2}+\lambda)\underset{\sim}{a}] = 1(1-\lambda) - (\tfrac{1}{2}+\lambda)$

✓ solving for λ using expression for dot product found earlier in working

✓ correct expression for the volume

✓ with correct supporting working

most did well

ii) let $u = \dfrac{2\pi}{\pi x + 1}$

$$\frac{du}{dx} = \frac{-2\pi^2}{(\pi x + 1)^2}$$

$$\frac{-du}{2\pi^2} = \frac{dx}{(\pi x + 1)^2}$$

x	0	k
u	2π	$\dfrac{2\pi}{\pi k + 1}$

$$V = \frac{\pi^3}{2}\int_{2\pi}^{\frac{2\pi}{\pi k+1}} -(1 - \cos u)\frac{du}{2\pi^2}$$

$$= \frac{\pi}{4}\int_{2\pi}^{\frac{2\pi}{\pi k+1}} (\cos u - 1)\, du$$

$$= \frac{\pi}{4}\Big[\sin u - u\Big]_{2\pi}^{\frac{2\pi}{\pi k+1}}$$

$$= \frac{\pi}{4}\left[\sin\frac{2\pi}{\pi k+1} - \frac{2\pi}{\pi k+1} - (\sin 2\pi - 2\pi)\right]$$

$$= \frac{\pi}{4}\left[\sin\frac{2\pi}{\pi k+1} + \frac{-2\pi + 2\pi(\pi k+1)}{\pi k+1}\right]$$

$$= \frac{\pi}{4}\left[\sin\frac{2\pi}{\pi k+1} + \frac{2\pi^2 k}{\pi k+1}\right] \text{ units}^2$$

iii) $$\lim_{k\to\infty} \frac{\pi}{4}\left[\sin\frac{2\pi}{\pi k+1} + \frac{2\pi^2}{\pi + \frac{1}{k}}\right]$$

$$= \frac{\pi}{4}\left[\sin 0 + \frac{2\pi^2}{\pi + 0}\right]$$

$$\frac{\pi^2}{2} \text{ units}^2$$

ii) generally done well

✓ finding du in terms of x and dx

most common error: not using the chain rule when finding $\frac{du}{dx}$

✓ correct substitution (indefinite integral first also ok)

✓ substituting limits and simplifying

✓ limiting value

some candidates used calculators to estimate and couldn't get the right answer

c) i) $LHS = \frac{1}{2}\cot\theta - \frac{1}{2}\tan\theta$ $RHS = \cot 2\theta$

$= \frac{1}{2\tan\theta} - \frac{\tan\theta}{2}$

$= \frac{1-\tan^2\theta}{2\tan\theta}$

$= \cot 2\theta = RHS$

i) generally done well

✓ showing use of double angle identity

ii) step 1: prove true for $n=1$

$LHS = \frac{1}{2}\tan\frac{x}{2}$ $RHS = \frac{1}{2}\cot\left(\frac{x}{2}\right) - \cot x$

$= \frac{1}{2}\cot\left(\frac{x}{2}\right) - \left[\frac{1}{2}\cot\frac{x}{2} - \frac{1}{2}\tan\frac{x}{2}\right]$, from (i) where $\theta = \frac{x}{2}$

$= \frac{1}{2}\tan\frac{x}{2}$

$= LHS$

ii) many did well

✓ base case

step 2: assume true for $n=k$

$\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots + \frac{1}{2^k}\tan\frac{x}{2^k} = \frac{1}{2^k}\cot\frac{x}{2^k} - \cot x$

step 3: prove true for $n=k+1$. aim is to show:

$\frac{1}{2}\tan\frac{x}{2} + \ldots + \frac{1}{2^k}\tan\frac{x}{2^k} + \frac{1}{2^{k+1}}\tan\frac{x}{2^{k+1}} = \frac{1}{2^{k+1}}\cot\frac{x}{2^{k+1}} - \cot x$

$RHS = \frac{1}{2^{k+1}}\cot\frac{x}{2^{k+1}} - \cot x$

$LHS = \frac{1}{2}\tan\frac{x}{2} + \ldots + \frac{1}{2^k}\tan\frac{x}{2^k} + \frac{1}{2^{k+1}}\tan\frac{x}{2^{k+1}}$

$= \frac{1}{2^k}\cot\frac{x}{2^k} - \cot x + \frac{1}{2^{k+1}}\tan\frac{x}{2^{k+1}}$, from step 2

✓ using assumption

$= \frac{1}{2^k}\left(\frac{1}{2}\cot\frac{x}{2^{k+1}} - \frac{1}{2}\tan\frac{x}{2^{k+1}}\right) - \cot x + \frac{1}{2^{k+1}}\tan\frac{x}{2^{k+1}}$, from part (i)

$= \frac{1}{2^{k+1}}\cot\frac{x}{2^{k+1}} - \frac{1}{2^{k+1}}\tan\frac{x}{2^{k+1}} - \cot x + \frac{1}{2^{k+1}}\tan\frac{x}{2^{k+1}}$

$= \frac{1}{2^{k+1}}\cot\frac{x}{2^{k+1}} - \cot x$

$= RHS$

✓ completed proof showing clear use of part (i)

step 4: true by mathematical induction $\forall n \in \mathbb{Z}^+$

a few solved without using part (i), which was more time-consuming

Question 14

a) i) $x = vt\cos\alpha$, $y = vt\sin\alpha - 5t^2$ ②

$t = \dfrac{x}{v\cos\alpha}$ ①

sub ① into ②:

$$y = v\sin\alpha \times \frac{x}{v\cos\alpha} - 5 \times \left(\frac{x}{v\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{5x^2}{v^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{5x^2}{v^2}\sec^2\alpha$$

✓ clearly showing elimination of t

ii) mountainside has equation $y = 0.6x$

point of intersection:

$$0.6x = x\tan\alpha - \frac{5x^2}{v^2}\sec^2\alpha$$

✓ equating y-values of mountanside and path of explosive equations

$$\frac{5x^2\sec^2\alpha}{v^2} + (0.6 - \tan\alpha)x = 0$$

$$x\left(\frac{5x}{v^2}\sec^2\alpha + 0.6 - \tan\alpha\right) = 0$$

$x = 0$ is the point of projection

← can still receive full marks if this is missing

explosive hits mountainside at:

$$\frac{5x}{v}\sec^2\alpha + 0.6 - \tan\alpha = 0$$

$$\frac{5x}{v^2}\sec^2\alpha = \tan\alpha - 0.6$$

$$x = \frac{v^2(\tan\alpha - 0.6)}{5\sec^2\alpha}$$

$$= \frac{1}{5}v^2\cos^2\alpha(\tan\alpha - 0.6)$$

✓ solving for x

iii) gradient of mountain $= 0.6 = \frac{3}{5}$

$$\frac{dy}{dx} = \tan\alpha - \frac{10x^2}{v^2}\sec^2\alpha$$

at point of impact, $\dfrac{dy}{dx} = -\dfrac{5}{3}$

$$-\frac{5}{3} = \tan\alpha - \frac{2v^2\cos^2\alpha(\tan\alpha - 0.6)\sec^2\alpha}{v^2}$$

✓ equating $\frac{dy}{dx}$ (or $\frac{\dot{y}}{\dot{x}}$) to $-\frac{5}{3}$

$$-\frac{5}{3} = \tan\alpha - 2(\tan\alpha - 0.6)$$

$$-\frac{5}{3} = -\tan\alpha + 1.2$$

$$\tan\alpha = \frac{43}{15}$$

$$\alpha \doteqdot 71°$$

✓ solving for α, showing clear working

iv) $t = \frac{x}{v\cos\alpha}$

$$= \frac{1}{v\cos\alpha}\left[\frac{1}{5}v^2\cos^2\alpha\left(\tan\alpha - \frac{3}{5}\right)\right]$$

$$= \frac{1}{5}v\cos\alpha\left(\tan\alpha - \frac{3}{5}\right)$$

✓ value of t at impact

$\dot{x} = v\cos\alpha$, $\dot{y} = v\sin\alpha - 10t$

when $\alpha = 71°$, $t = \frac{1}{5}v\cos 71°\left(\tan 71° - \frac{3}{5}\right)$

$\dot{x} = v\cos 71°$, $\dot{y} = v\sin 71° - 2v\cos 71°\left(\tan 71° - \frac{3}{5}\right)$

✓ $\dot{x}$, $\dot{y}$ at impact in terms of v

$$= v\sin 71° - 2v\sin 71° + \frac{6v}{5}\cos 71°$$

$$= v\left(\frac{6}{5}\cos 71° - \sin 71°\right)$$

impact speed $= \sqrt{v^2\cos^2 71° + v^2\left(\frac{6}{5}\cos 71° - \sin 71°\right)^2}$

$$= v\sqrt{\cos^2 71° + \left(\frac{6}{5}\cos 71° - \sin 71°\right)^2}$$

$$= v \times 0.643302...$$

approximately 64% of initial speed

✓ impact speed as a percentage of initial speed (ignore rounding errors)

b) i) $x = x_0 e^{kt}$

sub $t = 0$:

$x = x_0 e^0$

$= x_0$, which satisfies initial condition

most common error: not showing that the equation satisfies the initial conditions

differentiating x with respect to t:

$$\frac{dx}{dt} = k \times x_0 e^{kt}$$

$$= kx, \text{ as } x = x_0 e^{kt}$$

which satisfies the differential equation

✓ showing DE and initial conditions are satisfied using any reasonably sound method

ii) $0 = 0.17N \times \ln\left(\frac{7.2}{N}\right)$

$\ln\frac{7.2}{N} = 0$, as $N \neq 0$

← can still receive the mark if this is missing

$N = 7.2$

✓ finding constant solution

iii) when $t = 0$, $N = 0.0036$

$$u = \ln\left(\frac{7.2}{0.0036}\right)$$

$$= 7.6009\ldots$$

$$\doteqdot 7.6$$

✓ with correct supporting working

iv) $\frac{dN}{dt} = 0.17N\ln\left(\frac{7.2}{N}\right)$

$$u = \ln\left(\frac{7.2}{N}\right)$$

$$= \ln 7.2 - \ln N$$

$$\frac{du}{dN} = -\frac{1}{N}$$

✓ differentiating u

$$\frac{dN}{dt} = \frac{dN}{du} \times \frac{du}{dt}$$

$$0.17N\ln\left(\frac{7.2}{N}\right) = -N \times \frac{du}{dt}$$

✓ some correct progress using the chain rule

$$\frac{du}{dt} = -0.17\ln\left(\frac{7.2}{N}\right)$$

$$= -0.17u$$

$u = 7.6e^{-0.17t}$, from (i) and (iii)

✓ solving for u

$$\ln\left(\frac{7.2}{N}\right) = 7.6e^{-0.17t}$$

$$\frac{7.2}{N} = e^{7.6e^{-0.17t}}$$

$$\frac{N}{7.2} = e^{-7.6e^{-0.17t}}$$

$$N = 7.2e^{-7.6e^{-0.17t}}$$

✓ solving for N