Trinity Grammar School

Mathematics Department

2013

TRIAL EXAMINATION

HSC ASSESSMENT TASK 4

Year 12

Mathematics Extension 1

Board of Studies Student Number
(Year 12 only)

Class Teacher:

………………………………………………………………………

Name:
(Year 11 only)

………………………………………………………………………

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

Date of examination: 9/08/2013

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Only approved calculators for this course are allowed in this task
- A table of Standard Integrals is supplied
- Show all necessary working
- Write your Board of Studies Student Number (Year 12 HSC) or Name (Year 11) and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating **N/A** and your Student Number or Name.

- **HSC Assessment Weighting:** 40%

Total marks – 70

Section I

10 marks

- Attempt all Questions
- Allow about **15** minutes for this section

Section II

60 marks

- Attempt all Questions
- Allow about **1** hour **45** minutes for this section

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Section I **10 marks**

- Shade the correct response on the multiple-choice answer sheet for Questions 1 – **10**
- Each question is worth 1 mark

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1 Lines L_1 and L_2 have gradients $-\frac{1}{2}$ and $\frac{4}{5}$ respectively.

If θ is the acute angle between the lines L_1 and L_2 then, to the nearest degree, θ equals?

(A) 12°

(B) 25°

(C) 65°

(D) 78°

2 If $u = x^3 + 1$ then $\int_0^1 x^2 (x^3 + 1)^3 \; dx$ is equivalent to:

(A) $\frac{1}{3}\int_0^1 u^3 \; du$

(B) $3\int_0^1 u^3 \; du$

(C) $\frac{1}{3}\int_1^2 u^3 \; du$

(D) $3\int_1^2 u^3 \; du$

Question 3 commences on the next page

3 If α, β and γ are the roots of $2x^3 + 12x^2 - 6x + 1 = 0$, what is the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$?

(A) 12

(B) 6

(C) 2

(D) $\frac{1}{6}$

4 What is the domain of the function $f(x) = \cos^{-1} 2x$?

(A) $-2 \le x \le 2$

(B) $-\frac{1}{2} \le x \le \frac{1}{2}$

(C) $0 \le x \le 2\pi$

(D) $0 \le x \le \frac{\pi}{2}$

5 The number N of animals in a population at time t years is given by $N = 100 + Ae^{kt}$ for constants $A > 0$ and $k > 0$.

Which of the following is the correct differential equation?

(A) $\frac{dN}{dt} = k(N - 100)$

(B) $\frac{dN}{dt} = -k(N + 100)$

(C) $\frac{dN}{dt} = -k(N - 100)$

(D) $\frac{dN}{dt} = k(N + 100)$

6 Which of the following parametric equations could represent the Cartesian equation $(x+2)^2+(y-3)^2=1$?

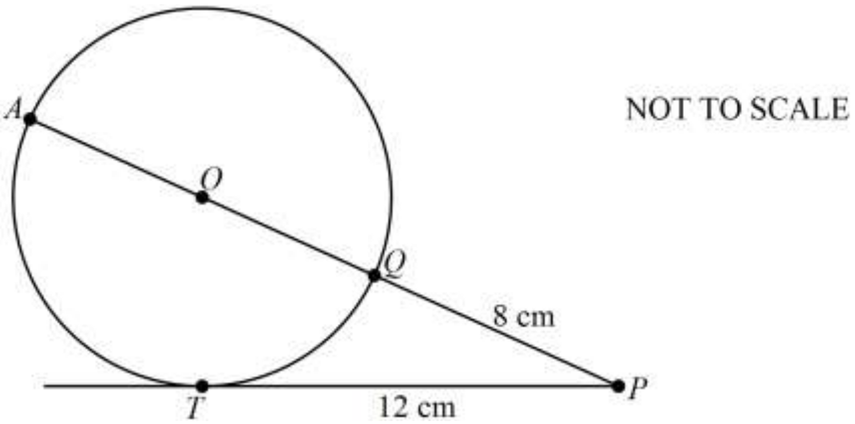
(A) $x=\tan\theta+2$ and $y=\sec^2\theta-3$

(B) $x=\tan\theta-2$ and $y=3+\sin\theta$

(C) $x=\cos\theta+2$ and $y=\sin\theta-3$

(D) $x=\cos\theta-2$ and $y=3+\sin\theta$

7



In the diagram, PT is a tangent to the circle, centre at O. If PQ = 8 cm and PT = 12 cm, then the radius of the circle, in cm, is

(A) 5

(B) 6

(C) 7

(D) 9

8 The trigonometric expression $\sqrt{3}\cos\theta + \sin\theta$ may be written in the form $R\cos\ \theta - \alpha$.

The values of R and α could be?

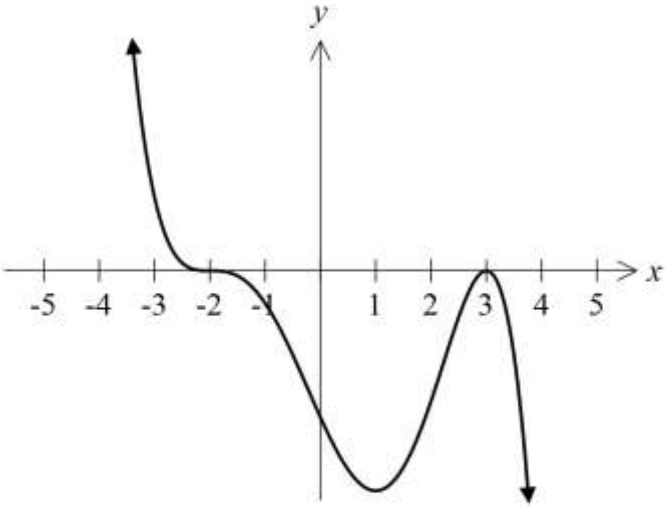
(A) $R = 4$ and $\alpha = \dfrac{\pi}{3}$

(B) $R = 4$ and $\alpha = \dfrac{\pi}{6}$

(C) $R = 2$ and $\alpha = \dfrac{\pi}{3}$

(D) $R = 2$ and $\alpha = \dfrac{\pi}{6}$

9



A graph of $y = P(x)$ where $P(x)$ is a polynomial of degree 5 is shown above.

Which of the following could be correct?

(A) $P(x) = -\ \ x+3\ ^{2}\ \ x-2\ ^{3}$

(B) $P(x) = \ \ x+3\ ^{2}\ \ x-2\ ^{3}$

(C) $P(x) = -\ \ x-3\ ^{2}\ \ x+2\ ^{3}$

(D) $P(x) = \ \ x-3\ ^{2}\ \ x+2\ ^{3}$

10 Given $f(x)=\dfrac{1}{(x+3)(x-2)}$, which of the following statements is true for the graph of $y=f(x)$.

(A) There is no intercept on the y-axis.

(B) It intersects the x-axis at $x=2$ and $x=-3$

(C) It has a maximum turning point at $\left(-\dfrac{1}{2},-\dfrac{4}{25}\right)$

(D) The lines $x=-2$ and $x=3$ are vertical asymptotes.

End of Section I

Section II commences on the next page

Section II **60 marks**

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section
- Answer each question in a SEPARATE writing booklet or answer sheet. Extra writing booklets are available.
- In Questions 11 – 14, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 **(15 marks)** Use a SEPARATE writing booklet

(a) Solve $\frac{2}{x-1} > x$. **3**

(b) Find the slope of the tangent to $y = \tan^{-1}\frac{x}{2}$ at the point where $x = 1$. **3**

(c) Use the Table of Standard Integrals to find $\int \frac{dx}{\sqrt{x^2-4}}$. **1**

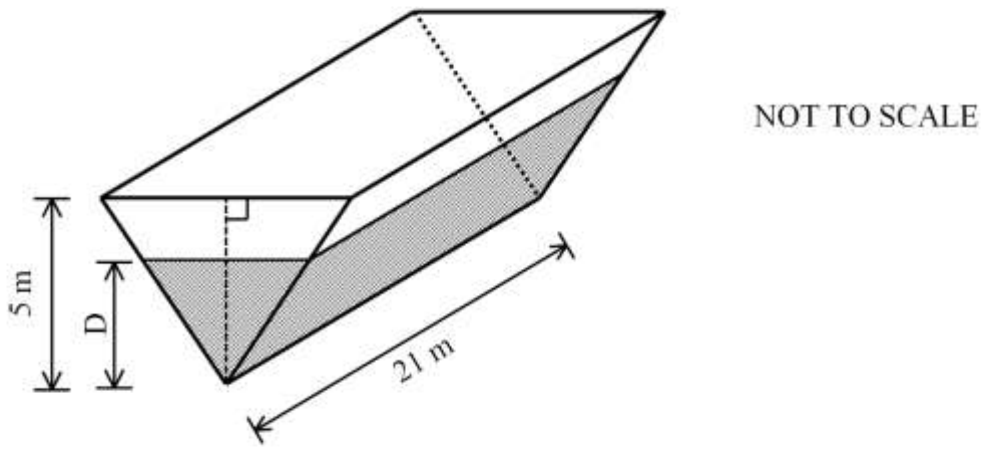
(d) Find $\int \frac{3}{\sqrt{4-x^2}}\,dx$. **1**

(e) Simplify $\tan\left[\cos^{-1}\left(-\frac{2}{3}\right)\right]$, leaving your final answer in simplest exact form. **2**

Question 11 continues on the next page

Question 11 (continued)

(f)



A water trough has a vertical cross section in the shape of an **equilateral** triangle as shown in the diagram above. The water trough is initially empty and is being filled with water at the rate of 4 cubic metres per hour.

(i) Given that the trough is 5 metres high and 21 metres long, show that the volume of water V, in the trough at depth D metres ($0 \le D \le 5$) is given by $V = 7\sqrt{3}D^2$. **3**

(ii) Find the rate at which the water level is rising when the water has a depth of 1.5 metres? **2**

Question 12 commences on the next page

Question 12 **(15 marks)** Use a SEPARATE writing booklet

(a) (i) Show that $P(x) = 2x - e^x + 3$ has a zero between $x = 0.5$ and $x = 2$. **2**

(ii) Taking $x = 1.75$ as a first approximation, use one application of Newton's method to find an approximation of the zero of $P(x)$, correct to two decimal places. **2**

(iii) Explain why $x = \ln 2$ would not be a good choice as a first approximation in part (ii)? **1**

(b) A particle moving along the x-axis has velocity $v\,ms^{-1}$ given by $v^2 = 8 - 2x - x^2$, where x is the displacement of the particle from a fixed point O.

(i) Prove that the particle is moving in simple harmonic motion. **2**

(ii) Find the end points of the motion. **2**

(iii) Find the maximum speed of the particle. **2**

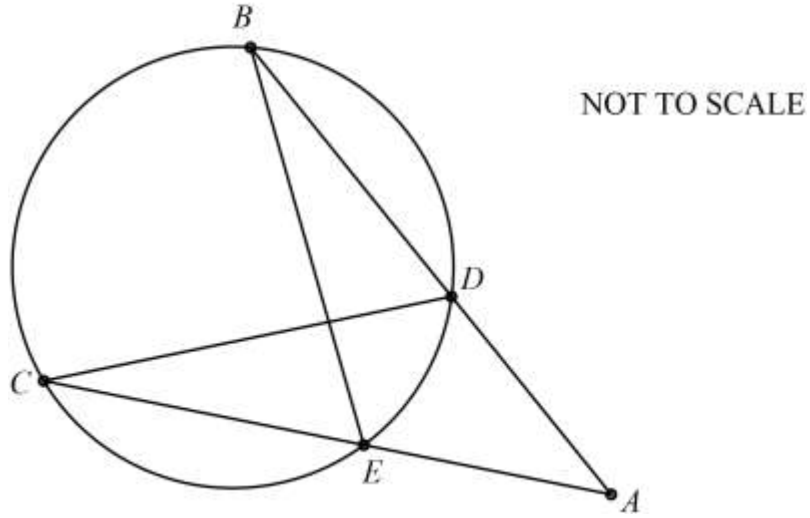
(c) (i) Sketch the graph of $y = 1 + \cos x$, for $-\pi \le x \le \pi$. **1**

(ii) The region bounded by the graph of $y = 1 + \cos x$ and the x-axis, for $-\pi \le x \le \pi$, is rotated about the x-axis to generate a solid. **3**

Find the volume of the solid formed.

Question 13 commences on the next page

Question 13 (15 marks) Use a SEPARATE writing booklet



(a) In the above circle chords BD and CE produced meet at A as shown.
The length of the **arc** BD is equal to the length of the **arc** CE.

Copy this diagram into your writing booklet.

(i) Prove that $\Delta DBC \equiv \Delta ECB$. **2**

(ii) Prove that $AB = AC$. **2**

(b) Find the ratio in which the line $2x + y - 8 = 0$ internally divides the interval joining the points $A(2, 3)$ and $B(4, 5)$. **3**

Question 13 continues on the next page

Question 13 (continued)

(c) Newton's Law of cooling states that the rate of change of the temperature, T, of an object is given by $\frac{dT}{dt} = -k\left(T - T_0\right)$, where T_0 is the temperature of the surrounding medium. Both k and T_0 are constants. It is known that $T = T_0 + Ae^{-kt}$, where A is a constant, satisfies Newton's law of cooling (DO NOT PROVE THIS).

A packet of meat with initial temperature $20°C$ is placed in a refrigerator. The temperature of the refrigerator is kept at a constant $4°C$. It takes 15 minutes for the temperature of the meat to drop to $15°C$.

(i) Write down the value of T_0. **1**

(ii) Write down the value of A. **1**

(iii) Show that $k = -\frac{1}{15}\ln\left(\frac{11}{16}\right)$. **1**

(iv) How much **additional** time is required for the temperature of the meat to drop to $10°C$? Give your final answer correct to the nearest minute. **2**

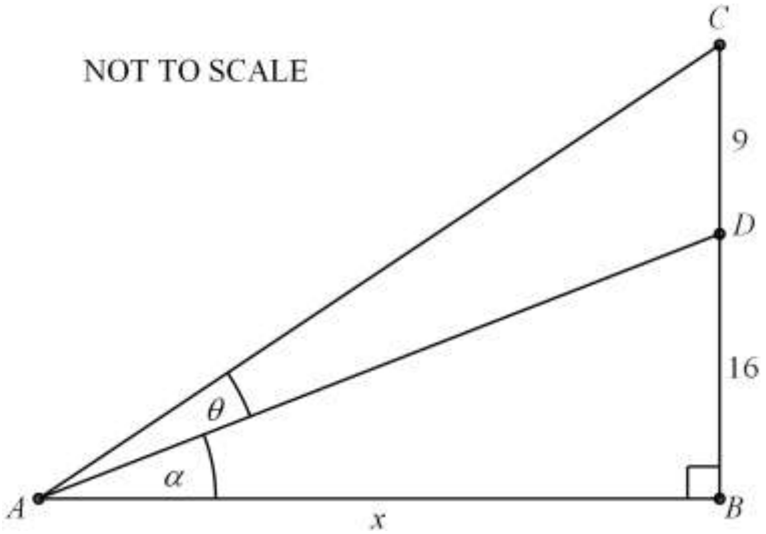
(d) Use the principle of mathematical induction to prove for all positive integers n, that $5^n + 9^n + 2$ is divisible by 4. **3**

Question 14 commences on the next page

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) (i) Show that $\tan^{-1} A - \tan^{-1} B = \tan^{-1}\left(\dfrac{A-B}{1+AB}\right)$, where $AB \neq -1$. **2**

(ii)



In right ΔABC above, $\angle CAD = \theta$, $\angle DAB = \alpha$, $AB = x, CD = 9$ and $BD = 16$. **3**

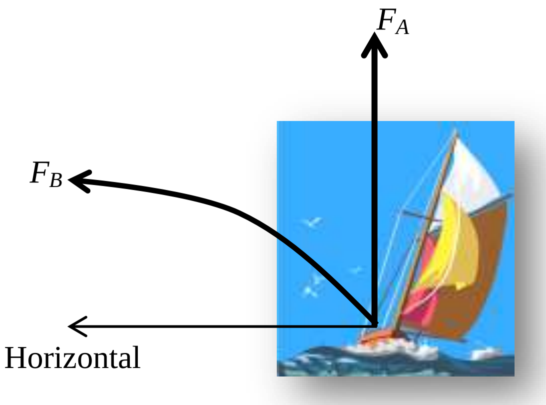
Show that $\theta = \tan^{-1}\left(\dfrac{9x}{x^2+400}\right)$.

(iii) Hence, or otherwise, find the value of x such that θ is a maximum. **2**

Question 14 continues on the next page

Question 14 (continued)

(b)



Two flares are fired from a boat at the same time and with the same velocity, V .

Flare *A* (F_A) is fired vertically upwards.

Flare *B* (F_B) is fired at an angle of inclination to the horizontal of θ .

(i) Ignoring air resistance, and taking g , to be the acceleration due to gravity, show that the height (y_A) of **Flare *A*** at any time $t \geq 0$ is given by: **2**

$$y_A = -\frac{1}{2}gt^2 + Vt$$

(ii) Show that the **difference** (*D*) in the **maximum** height of the two flares is **3**

$$D = \left| \frac{V^2}{2g}\cos^2\theta \right|$$

You may **assume** (WITHOUT PROOF) that the height of **Flare *B*** is given by:

$$y_B = -\frac{1}{2}gt^2 + Vt\sin\theta .$$

(iii) It is known that the best chance for the two flares to be seen is for the ratio of the range of **Flare *B*** to the difference (*D*) in the maximum height of the two flares, to be $4:\sqrt{3}$. **3**

Find the exact value of θ that gives the best chance of the two flares being seen.

End of Section II
End of Paper

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Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x \;, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

2013 TGS Year 12 Mathematics Extension 1 TRIAL HSC Examination
Suggested Marking Criteria

SECTION I

Section I (1 mark each)

Question 1

C $\tan\theta = \left|\dfrac{\frac{4}{5}+\frac{1}{2}}{1-\frac{2}{5}}\right| = \left|\dfrac{\frac{13}{10}}{\frac{3}{5}}\right| = \dfrac{13}{6} \quad \therefore \theta \approx 65^o$

Question 2

C $u = x^3+1 \to du = 3x^2dx, x=1 \to u=2, x=0 \to u=1$

$$\therefore \int_0^1 x^2\left(x^3+1\right)^3 dx = \frac{1}{3}\int_0^2 u^3 du$$

Question 3

B $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma} = \dfrac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = \dfrac{-3}{-\frac{1}{3}} = 6$

Question 4

B $-1 \le 2x \le 1 \to -\dfrac{1}{2} \le 2x \le \dfrac{1}{2}$

Question 5

A $\dfrac{dN}{dt} = k(N-100)$

Question 6

D

$$x+2=\cos\theta \Rightarrow \left(x+2\right)^2 = \cos^2\theta$$
$$y-3=\sin\theta \Rightarrow \left(y-3\right)^2 = \sin^2\theta$$
$$\left(x+2\right)^2+\left(y-3\right)^2 = 1$$

Question 7

A

$$PT^2 = PA.PQ$$
$$144 = (AQ+8).8$$
$$AQ = 10\ cm$$
$$radius = 5\ cm$$

Question 8

D $R\cos\alpha=\sqrt{3}$ and $R\sin\alpha=1\rightarrow R^2=4\therefore R=2$

$\tan\alpha=\dfrac{1}{\sqrt{3}}\therefore\alpha=\dfrac{\pi}{6}$

Question 9

C $P(x)=-\left(x-3\right)^2\left(x+2\right)^3$

Question 10

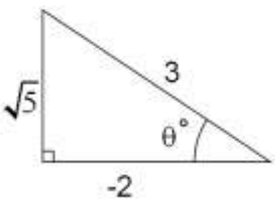
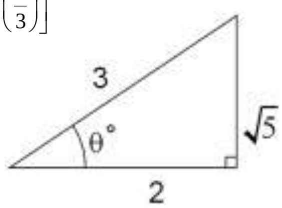
C $y=x^2+x-6$

has a minimum turning point at $\left(-\dfrac{1}{2},-\dfrac{25}{4}\right)$

$\therefore y=\dfrac{1}{f(x)}$ has a maximum turning point at $\left(-\dfrac{1}{2},-\dfrac{4}{25}\right)$

Section II commences on the next page

SECTION II: **Question 11: (15 Marks)**

Suggested Solution	Marking Criteria	Comments
(a) $\frac{2}{x-1} > x$ $2(x-1) > x(x-1)^2$ $(x-1)\left[2-x(x-1)\right] > 0$ $(x-1)(-x^2+x+2) > 0$ $(x-1)(x^2-x-2) < 0$ $(x-1)(x-2)(x+1) < 0$ $\therefore x < -1 or 1 < x < 2$	• **[3]** Correct solution • **[2]** Significant progress toward a **correct** solution and gives either $x < -1 or 1 < x < 2$ • **[1]** Getting a fully factorised expression that leads to the correct answer such as $(x-1)(x-2)(x+1) < 0$	Award equivalent marks if different methods are used such as graphical, cases, critical value(s) etc…
(b) $\frac{d}{dx}\tan^{-1}\left(\frac{x}{2}\right) = \frac{1}{1+\left(\frac{x}{2}\right)^2} \times \frac{1}{2}$ $= \frac{2}{4+x^2}$ Slope of the tangent at x = 1, is $\frac{2}{4+1^2} = \frac{2}{5}$	• **[3]** Correct solution • **[2]** Significant progress toward a **correct** solution, including correct derivative and substitution of x = 1. • **[1]** correct (un-simplified) derivative or correct answer of $\frac{2}{5}$ without working	If the derivative is incorrect and involves a quadratic expression in the denominator award [1], and then award a maximum of [2] marks provided the final answer on FT is correct.
(c) $\int \frac{dx}{\sqrt{x^2-4}} = \ln\left(x+\sqrt{x^2-4}\right)+c$	• **[1]** correct answer only	Do not penalise for omission of +c
(d) $\int \frac{3}{\sqrt{4-x^2}}\,dx = 3\sin^{-1}\frac{x}{2}+c$	• **[1]** correct answer	Do not penalise for omission of +c
(e) Let $\theta = \cos^{-1}\left(-\frac{2}{3}\right)$ $-\frac{2}{3} = \cos\theta,\ 0 \le \theta \le \pi$  *This means θ is in Quad 2 form* From the diagram, $\tan\theta = -\frac{\sqrt{5}}{2}$ **ALTERNATIVELY** $\tan\left[\cos^{-1}\left(-\frac{2}{3}\right)\right] = \tan\left[\pi - \cos^{-1}\left(\frac{2}{3}\right)\right]$ $= \frac{\tan\pi - \tan\left[\cos^{-1}\left(\frac{2}{3}\right)\right]}{1+\tan\pi\tan\left[\cos^{-1}\left(\frac{2}{3}\right)\right]}$ $= -\tan\left[\cos^{-1}\left(\frac{2}{3}\right)\right]$ Let $\theta = \cos^{-1}\left(\frac{2}{3}\right)$ $\therefore \cos\theta = \frac{2}{3},\quad 0 \le \theta \le \pi$ 	• **[2]** Correct solution • **[1]** correct answer only (with a negative sign)	Award equivalent marks if different (correct) methods are used

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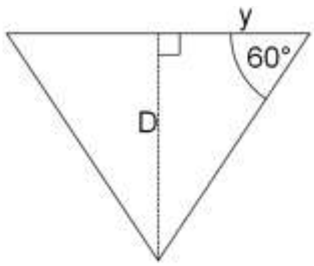
(i)



$$\tan 60^o = \frac{D}{y}$$

$$\Rightarrow \quad y = \frac{D}{\sqrt{3}}$$

Area of cross section is given by:

$$A = \frac{1}{2}(2y)D = \frac{1}{2}\left(\frac{2D}{\sqrt{3}}\right)D = \frac{\sqrt{3}D^2}{3}$$

$$\text{Volume, } V = \frac{\sqrt{3}D^2}{3} \times 21 = 7\sqrt{3}D^2$$

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- **[3]** Correct proof/solution
- **[2]** Significant progress toward a **correct** solution, including **and** must see evidence of calculation of the cross-sectional area $A = \frac{\sqrt{3}D^2}{3}$ and $y = \frac{D}{\sqrt{3}}$
- **[1]** Correct expression $y = \frac{D}{\sqrt{3}}$ or equivalent, or using trigonometry or equivalent method to get the base length of the cross-section or correct expression of cross-sectional area without working.

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<td>Award equivalent marks if different (correct) methods are used</td>
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(f)
(ii)

$$\frac{dD}{dt} = \frac{dD}{dv} \times \frac{dv}{dt} = \frac{1}{14\sqrt{3}D} \times 4 = \frac{2}{7\sqrt{3}D}$$

$When\ ,\ D = 1.5$

$$\frac{dD}{dt} = \frac{2\sqrt{3}}{21} \times \frac{2}{3} = \frac{4\sqrt{3}}{63} m/h \quad (or\ 0.10997...)$$

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- **[2]** correct solution
- **[1]** attempts to use the chain rule correctly (or its equivalent e.g. $\frac{dD}{dt} = \frac{dD}{dv} \times \frac{dv}{dt}$ is seen), or $\frac{dD}{dt} = \frac{2}{7\sqrt{3}D}$ or equivalent

</td>
<td>A correct decimal approximation in the final answer is acceptable.</td>
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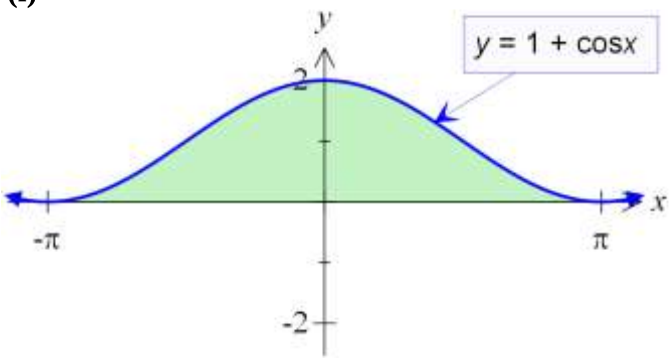
SECTION II: Question 12: (15 Marks)

Suggested Solution	Marking Criteria	Comments
(a) **(i)** $P(0.5)=2.351..>0$ $P(2)=-0.3890...<0$ $P(x)$ is continuous for $0.5\le x\le 2$ $\therefore P(x)$ has a zero in the interval $0.5\le x\le 2$.	• **[2]** correct solution including reference to change in sign in the values of $P(0.5)$ and $P(2)$ • **[1]** correct calculation of $P(0.5)$ or $P(2)$	Should include reference to continuity in the specified interval but is not to be penalised in this instance.
(a) **(ii)** $P(1.75)=0.745....$ $P'(x)=2-e^x$ $\therefore P'(1.75)=-3.754......$ $x_1=x_0-\dfrac{P(x)}{P'(x)}$ $=1.75-\dfrac{0.745...}{-3.754...}$ $=1.948...$ $=1.95\,(2\,decimal\ places)$	• **[2]** Correct solution **with** a correctly rounded solution. • **[1]** correct solution, or correct evaluation of either $P(0.5)$ and $P(2)$, $P'(x)$, or substitution into Newton's formula $x_1=1.75-\dfrac{0.745...}{-3.754...}$	This is the only question in the paper where a penalty for accuracy is awarded.
(a) **(iii)** It is not a good choice since $P'(\ln 2)=0$.	• **[1]** Correct reasoning must refer too **and show that** the derivative is zero for this chosen root.	
(b) **(i)** $\ddot{x}=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ $=\dfrac{d}{dx}\left(4-x-\dfrac{1}{2}x^2\right)$ $=-1-x$ $=-1(x+1)$ *This is in the form* $\ddot{x}=-n^2(x-k)$ $\therefore$ *the particle moves in SHM*	• **[2]** correct proof (needs to write the expression in the form $\ddot{x}=-n^2(x-k)$ • **[1]** attempts to use a correct expression for acceleration in terms of *x*.	Could use $\ddot{x}=v\dfrac{dv}{dx}$
(b) **(ii)** The end points of the motion of a particle moving in SHM will be when v = 0 (or v^2 = 0). $8-2x-x^2=0$ $(2-x)(x+4)=0$ $\therefore x=2\ or\ -4$	• **[2]** correct solution • **[1]** v = 0 (or v^2 = 0), or $8-2x-x^2=0$.	

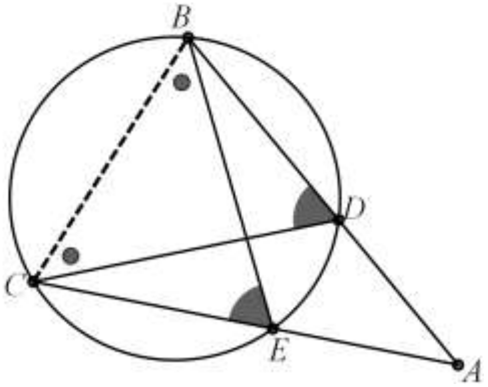
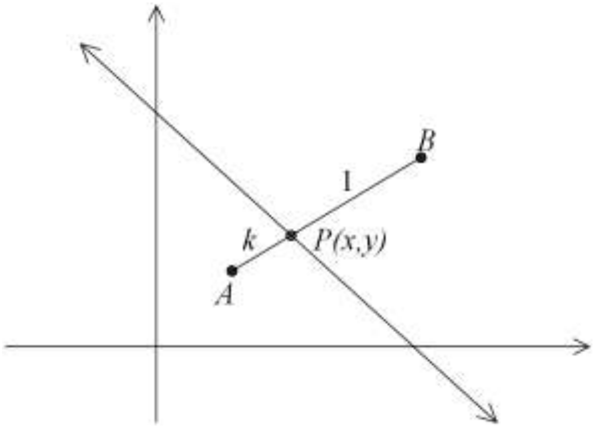
<table>
<tr><td>(b)
(iii)
From (ii) centre of motion is at −1 (or from part (i) $\ddot{x}=0$)

$v^2=9$
$v=\pm 3$
$\therefore\ Speed\ is\ 3\,ms^{-1}$</td><td>• [2] correct solution
• [1] centre of motion at x = 1, or $v^2=9$ or $v=\pm 3$</td><td></td></tr>
<tr><td>(c)
(i)

</td><td>• [1] correct graph (at least one ½ period is shown and at least must include the required interval) but it may extend beyond the given domain.</td><td></td></tr>
<tr><td>(c)
(ii)
$V=\pi\int_{-\pi}^{\pi}(1+\cos x)^2\quad dx$
$=\pi\int_{-\pi}^{\pi}1+2\cos x+\cos^2 x\quad dx$
$=\pi\left[x+2\sin x+\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)\right]_{-\pi}^{\pi}$
$=\pi\left[\left(\pi+2\sin\pi+\frac{1}{2}\left(\pi+\frac{1}{2}\sin 2\pi\right)\right)-\left(-\pi+2\sin(-\pi)+\frac{1}{2}\left(-\pi+\frac{1}{2}\sin(-2\pi)\right)\right)\right]$

$V=\pi\left[\pi+\frac{\pi}{2}+\pi+\frac{\pi}{2}\right]$
$=3\pi^2\ cubic\ units$
(or 29.61 is acceptable to two decimal places or equivalent)</td><td>• [3] correct solution including substitution and evaluation of limits
• [2] correct primitive of
$V=\pi\int(1+\cos x)^2\quad dx$
or its equivalent.

• [1] correct integral statement with limits
$V=\pi\int_{-\pi}^{\pi}(1+\cos x)^2\quad dx$</td><td>Symmetry can be used for example the correct answer can be obtained by evaluating
$V=2\pi\int_{0}^{\pi}(1+\cos x)^2\quad dx$</td></tr>
</table>

SECTION II: Question 13: (15 Marks)

Suggested Solution	Marking Criteria	Comments
(a) **(i)**  In the triangles DBC and ECB 1. $\angle BDC = \angle BEC$ (standing on the ***same*** arc BC) 2. $BD = CE$ (arc BD = arc CE) 3. $\angle CBE = \angle BCD$ (standing on ***equal*** arcs *CE and BD*) BC is common so $\Delta DBC = \Delta ECB$ (*ASA test for congruency*)	• **[2]** correct solution with supporting reasons • **[1]** at least two correct geometrical statements and supporting reason(s) relevant to the question	The angle must be the included angle between the sides that are equal to use the SAS test for congruency, therefore needed to prove $\angle CBD = \angle BCD$, it could not be assumed if this method was used.
(a) **(ii)** $\angle DBC = \angle ECB$, *corresponding angles in congruent triangles* Now $\angle DCE = \angle DBE$ (standing on ***same*** arc DE) so $\angle CBD = \angle BCE$, which are base angles of ΔABC. $\therefore \Delta ABC$ is isosceles, sides opposite equal angles are equal in an isosceles triangle, hence $AB = AC$	• **[2]** correct reasoning and solution • **[1]** recognising either $\angle DCE = \angle DBE$ **Or** proving isosceles triangle exists	
 **(b)** The general coordinates of P are given by: $P(x,y) = \left(\frac{kx_1 + lx_1}{k+l}, \frac{ky_1 + ly_1}{k+l}\right)$. Let the ratio of internal interval division be k : 1 (**note** the one).	• **[3]** correct solution with the ratio in the form *a*:*b* and both *a* and *b* are integers and positive • **[2]** correct solution with the ratio in any form or significant progress towards the correct solution (such as finding the point of intersection of the given line with *AB, and calculation of* distances *AP* and *PB*) • **[1]** attempts to find the point of division by using the ratio formula or some equivalent method (such as finding the point of intersection of the given line with *AB*)	There are alternative methods for example, find the equation of the *AB*, find the point of intersection with the given line, and then calculate the ratio of the lengths with the intersection point on the given interval. ***Another*** could be the use of the *perpendicular distance from a point to a line* formula and *similar triangles* to work out the corresponding ratio.

Therefore, using the coordinates of $A(2, 3)$ and $B(4, 5)$ (already given): $P(x,y)=\left(\frac{kx_1+x_1}{k+1},\frac{ky_1+y_1}{k+1}\right)$ $=\left(\frac{4k+2}{k+1},\frac{5k+3}{k+1}\right)$ This point must also lie on the given line: *i.e.* $2\left(\frac{4k+2}{k+1}\right)+\frac{5k+3}{k+1}-8=0$ Solving this equation: $8k+4+5k+3-8(k+1)=0$ $5k-1=0$ $k=\frac{1}{5}$ Hence the required ratio is 1:5.	**COMMON METHOD:** *Equation of* AB: $y-3=1(x-2)$ $y=x+1$.......(1) *Solving* simultaneously: $y=x+1$.......(1) $y=-2x+8$...(2) $(x,y)=\left(\frac{7}{3},\frac{10}{3}\right)$ $d_{AP}=\frac{\sqrt{2}}{3}$ $d_{PB}=\frac{5\sqrt{2}}{3}$ $\therefore AP:PB=1:5$	
(c) **(i)** $T_o=4$	• **[1]** correct answer	
(c) **(ii)** $20=4+A\to A=16$	• **[1]** correct answer	
(c) **(iii)** $T=4+16e^{-kt}$ $15=4+16e^{-15k}$ $\frac{11}{16}=e^{-15k}$ $\therefore k=-\frac{1}{15}\ln\left(\frac{11}{16}\right)$	• **[1]** correct **solution** at least to the step involving $\frac{11}{16}=e^{-15k}$ (or equivalent)	
(c) **(iv)** $4+16e^{-kt}=10$ $t=-\frac{1}{k}\ln\left(\frac{3}{8}\right)$ $\therefore t=39.265$..... An additional 24 minutes (to the nearest minutes) is needed.	• **[2]** correct solution and must have 24 minutes in the final answer • **[1]** correctly find t or correct substitution of variables with values that lead to an answer	

Question 13 is continued on the next page

(d) Let $S(n)$ be the statement $5^n + 9^n + 2$ **Prove** $S(1)$ *is true* **.** $S(1) = 5^1 + 9^1 + 2$ $= 16$ 16 is divisible by 4 because $16 = 4 \times 4$ **Assume** $S(k)$ *is true*, $1 \leq k \leq n$, *where* k, n *are positive* integers ***i.e.*** $5^k + 9^k + 2 = 4M$ (M *an* integer) $5^k = 4M - 9^k - 2$ **Prove** $S(k+1)$ *is true* **,** ***i.e.* show** $5^{k+1} + 9^{k+1} + 2 = 4N$ (N *is an* integer) $5^{k+1} + 9^{k+1} + 2 = 5.5^k + 9^{k+1} + 2$ $= 5(\underbrace{4M - 9^k - 2}_{\text{from the assumption}}) + 9^{k+1} + 2$ $= 5(4M) - 5(9^k) - 8 + 9^{k+1}$ $= 4(5M - 2) - 9^k(5 - 9)$ $= 4(5M - 2) + 4.9^k$ $= 4(5M - 2 + 9^k)$ $= 4N$ $S(n)$ is true for $n = 1$. Whenever $S(k)$ is true $S(k+1)$ is also true for all positive integers n greater than or equal to 1.	• **[3]** Correct solution with all the key steps of induction incorporating the assumption correctly. Conclusion not required • **[2]** Shows $S(1)$ is true **and** writes down a correct statement involving the assumption $S(k)$ **and** incorporates the assumption in the induction hypothesis $S(k+1)$ • **[1]** Correctly shows $S(1)$ is true **or** writes a correct statement involving $S(k)$ **and** the hypothesis $S(k+1)$.	Alternative methods awarded marks on merit.

End of Question 13

SECTION II: **Question 14: (15 Marks)**

Suggested Solution	Marking Criteria	Comments
(a) **(i)** Let $\alpha=\tan^{-1}A \rightarrow \tan\alpha=A$ Let $\beta=\tan^{-1}B \rightarrow \tan\beta=B$ Now $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\dfrac{A-B}{1+AB}$ $\therefore \alpha-\beta=\tan^{-1}\left(\dfrac{A-B}{1+AB}\right)$ ***i.e.*** $\tan^{-1}A-\tan^{-1}B=\tan^{-1}\left(\dfrac{A-B}{1+AB}\right)$	• **[2]** correct solution • **[1]** attempt to use a correct trigonometric identity that could complete the proof or evidence of using the intended result such as $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$ (a valid substitution must be seen)	Allow for alternative methods, if possible.
(a) **(ii) From the diagram it can be seen that:** $\tan(\theta+\alpha)=\dfrac{25}{x}$ $\therefore \theta+\alpha=\tan^{-1}\left(\dfrac{25}{x}\right)$........(1) $also:\ \tan\alpha=\dfrac{16}{x}$ $\therefore \alpha=\tan^{-1}\left(\dfrac{16}{x}\right)$...............(2) *From* (1) $\theta=\tan^{-1}\left(\dfrac{25}{x}\right)-\alpha$ *From* (2) $\theta=\tan^{-1}\left(\dfrac{25}{x}\right)-\tan^{-1}\left(\dfrac{16}{x}\right)$ **Now from part (i)** $\theta=\tan^{-1}\left(\dfrac{{}^{25}/_{x}-{}^{16}/_{x}}{1+{}^{25}/_{x}\times{}^{16}/_{x}}\right)$..........(3) *BUT* $\dfrac{{}^{25}/_{x}-{}^{16}/_{x}}{1+{}^{25}/_{x}\times{}^{16}/_{x}}=\dfrac{25x-16x}{x^2+400}$ $=\dfrac{9x}{x^2+400}$ From (3) $\therefore$ $\theta=\tan^{-1}\left(\dfrac{9x}{x^2+400}\right)$	• **[3]** correct solution **at least** to (3) in the suggested solution or equivalent • **[2]** identifying $\tan(\theta+\alpha)=\dfrac{25}{x}$, expanding LHS **and** incorporating $\tan\alpha=\dfrac{16}{x}$ for example, $\theta=\tan^{-1}\left(\dfrac{25}{x}\right)-\tan^{-1}\left(\dfrac{16}{x}\right)$ • **[1]** identifying $\tan(\theta+\alpha)=\dfrac{25}{x}$ **and** $\tan\alpha=\dfrac{16}{x}$ **or** attempts to use part (i) with either $\tan(\theta+\alpha)=\dfrac{25}{x}$ **or** $\tan\alpha=\dfrac{16}{x}$	Allow for alternative methods, if possible. For example: $\tan(\theta+\alpha)=\dfrac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$ $\dfrac{25}{x}=\dfrac{\tan\theta+\dfrac{16}{x}}{1-\dfrac{16}{x}\tan\theta}$ $i.e.\ \tan\theta=\dfrac{9x}{x^2+400}$

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(a)
(iii)
θ will be a maximum when $f(x)=\dfrac{9x}{x^2+400}$ is a maximum. This is because in the interval $0<x\le 20$ (see below why the upper bound), $\tan^{-1} f(x)$ is an increasing continuous function, so whenever $f(x)$ is maximum so too is $\tan^{-1} f(x)$.
Let $y=\dfrac{9x}{x^2+400}$
$\dfrac{dy}{dx}=\dfrac{9\left(x^2+400\right)-9x\times 2x}{\left(x^2+400\right)^2}$
$=\dfrac{3600-9x^2}{\left(x^2+400\right)^2}$
$=\dfrac{9(400-x^2)}{\left(x^2+400\right)^2}$
$\dfrac{dy}{dx}=0$ when $3600-9x^2=0$
$\therefore x=20,\ x>0$
<table>
<tr><td>x</td><td>19</td><td>20</td><td>21</td></tr>
<tr><td>y'</td><td>+0.46</td><td>0</td><td>-0.44</td></tr>
</table>
$\therefore \theta$ is a maximum when $x=20$
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[2] correct solution includes a correct derivative, test and answer
[1] a correct derivative (e.g. $\dfrac{d\theta}{dx}=\dfrac{3600-9x^2}{(x^2+400)^2+81x^2}$ or $\dfrac{dy}{dx}=\dfrac{3600-9x^2}{\left(x^2+400\right)^2}$ as shown in the suggested solution (un-simplified is acceptable) or a correct answer from a correct derivative

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A correct answer without supporting (valid) working receives 0 marks.

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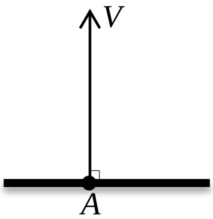
(b)
(i) Starting with:


$\ddot{y}_A=-g$
Integrating :
$\dot{y}_A=-gt+c_1$
when $t=0, \dot{y}_A=V\sin 90°, \dot{x}_A=V\cos 90°$
i.e. $\dot{y}_A=V$
$\Rightarrow$ $c_1=V$
$\therefore$ $\dot{y}_A=-gt+V$
$y_A=-\dfrac{1}{2}gt^2+Vt+c_2$
when $t=0, y_A=0$
$\Rightarrow$ $c_2=0$
$\therefore y_A=-\dfrac{1}{2}gt^2+Vt$ (already given)
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[2] Correct solution involving integration, finding c_1 and c_2 as suggested in the solution. If a formula is stated or implied such as $\dot{y}_A=-gt+V\sin\alpha$, and $y_A=-\dfrac{1}{2}gt^2+\left(V\cos\alpha\right)t$ deduces (or explains) that $t=0, \sin\alpha=1, \cos\alpha=0$ $\because \alpha=\dfrac{\pi}{2}$ then this qualifies for 2 marks
[1] finding correctly (by integration) an expression for $\dot{y}_A$ (c_1 need not be found for this mark)

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Award 0 marks if a student differentiates the given equation to get $\ddot{y}_A=-g$ for this approach is a verification not a "proof" with assumptions made.
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(b) (ii) Maximum height occurs at $\dot{y}=0$ for both flares. **Flare A:** $\dot{y}_A=0\to t=\frac{V}{g}$ $y_A=-\frac{1}{2}g\left(\frac{V}{g}\right)^2+V\left(\frac{V}{g}\right)$ $\therefore y_A=\frac{V^2}{2g}$ **Flare B:** $y_B=-\frac{1}{2}gt^2+Vt\sin\theta$ $\therefore \quad \dot{y}_B=V\sin\theta-gt$ $\dot{y}_B=0\to t=\frac{V\sin\theta}{g}$ $y_B=-\frac{1}{2}\left(\frac{V\sin\theta}{g}\right)^2+V\left(\frac{V\sin\theta}{g}\right)\sin\theta$ $\therefore y_B=\frac{V^2\sin^2\theta}{2g}$ Difference (D) in maximum height is given by $\lvert y_A-y_B\rvert=\left\lvert\frac{V^2}{2g}-\frac{V^2\sin^2\theta}{2g}\right\rvert$ $D=\left\lvert\frac{V^2}{2g}\left(1-\sin^2\theta\right)\right\rvert$ $\therefore D=\left\lvert\frac{V^2}{2g}\cos^2\theta\right\rvert \quad$ {*already given*}	• **[3]** correct solution (absolute value not required) • **[2]** for finding the maximum height of each flare in terms of V and g (and θ where appropriate) • **[1]** find t_A or t_B that gives maximum height (when $\dot{y}_A=0$ or $\dot{y}_B=0$) **or** finding either $y_{A(\max)}$, or $y_{B(\max)}$ **or** recognising maximum height when either $\dot{y}_A=0$ or $\dot{y}_B=0$	Alternative methods may exist, for example the maximum height for flare B would occur when $x=\frac{Range\ B}{2}$ etc…
(b) (iii) Range of Flare B (WITHOUT PROOF) is: $R=\frac{V^2\sin 2\theta}{g}$ **So** $\frac{V^2\sin 2\theta}{g}:\frac{V^2}{2g}\cos^2\theta=\frac{4}{\sqrt{3}}$ *i.e.* $\frac{V^2\sin 2\theta}{g}\times\frac{2g}{V^2\cos^2\theta}=\frac{4}{\sqrt{3}}$ $\frac{4\sin\theta\cos\theta}{\cos^2\theta}=\frac{4}{\sqrt{3}}$ $\tan\theta=\frac{1}{\sqrt{3}}$ $\therefore \quad \theta=\frac{\pi}{6}\quad\left(NB:\quad 0\le\theta<\frac{\pi}{2}\right)$	• **[3]** correct solution and answer • **[2]** finding range of B and establishing $\tan\theta=\frac{1}{\sqrt{3}}$ by using equating $R_B:D=\frac{4}{\sqrt{3}}$ • **[1]** Writing down range of B *(exclusive of time t)* or using the ratio $\tan\theta=\frac{1}{\sqrt{3}}$.	