Trinity Grammar School

Mathematics Department

2014

TRIAL EXAMINATION

HSC ASSESSMENT TASK 4

Year 12

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Only approved calculators for this course are allowed in this task
- A table of Standard Integrals is supplied
- Show all necessary working
- Write your Board of Studies Student Number and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating **N/A** and your Student Number or Name.

- **HSC Assessment Weighting:** 40%

Board of Studies Student Number
(Year 12 only)

Class Teacher:

……………………………………………………………………

Name:
(Year 11 only)

……………………………………………………………………

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

Date of examination:

Friday, 15 August 2014

Total marks – 70

Section I

10 marks

- Attempt Questions 1-10
- Allow about **15** minutes for this section

Section II

60 marks

- Attempt Questions 11-14
- Allow about **1** hour **45** minutes for this section

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Section I 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section
- Shade the correct response on the multiple-choice answer sheet for Questions 1 – 10
- Each question is worth 1 mark

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1 What is the domain and range of $y=\frac{1}{2}\cos^{-1}(\frac{x}{2})$?

(A) Domain: $-2\le x\le 2$ Range: $0\le y\le \pi$

(B) Domain: $-1\le x\le 1$ Range: $0\le y\le \pi$

(C) Domain: $-2\le x\le 2$ Range: $0\le y\le \frac{\pi}{2}$

(D) Domain: $-1\le x\le 1$ Range: $0\le y\le \frac{\pi}{2}$

2 A particle is moving in a simple harmonic motion with displacement *x*. Its acceleration $\ddot{x}$ is given by: $\ddot{x}=-4x+3$. What is the centre and period of the motion?

(A) $x=3, T=\frac{\pi}{2}$

(B) $x=-3, T=\pi$

(C) $x=\frac{3}{4}, T=\pi$

(D) $x=\frac{3}{4}, T=\frac{\pi}{2}$

3 Which of the following is an expression for $\int \frac{x}{(2-x^2)^3}dx$?

Use the substitution $u=2-x^2$.

(A) $\frac{1}{2(2-x^2)^2}+C$

(B) $\frac{1}{4(2-x^2)^2}+C$

(C) $\frac{1}{4(2-x^2)^4}+C$

(D) $\frac{1}{8(2-x^2)^4}+C$

4 What is the exact value of the definite integral $\int_0^1 \frac{1}{x^2+1}dx$?

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$

(D) π

5 The term independent of *x* in the expansion $\left(x+\frac{3}{x}\right)^4$ is?

(A) 54
(B) 3
(C) 6
(D) 18

6 Find the acute angle between the lines $y=2x$ and $x+y-5=0$. Answer correct to the nearest degree.

(A) 18°
(B) 32°
(C) 45°
(D) 72°

7 What is the exact value of the definite integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1+\sin^2 x)dx$?

(A) $\frac{\pi}{4}+\frac{1}{2}$

(B) $\frac{\pi}{8}+\frac{1}{8}$

(C) $\frac{\pi}{4}-\frac{\sqrt{3}-2}{4}$

(D) $\frac{\pi}{8}-\frac{\sqrt{3}-2}{8}$

8 What are the coordinates of the point that divides the interval joining the points $A(-1,2)$ and $B(3,5)$ externally in the ratio 3:1?

(A) $(5,6.5)$

(B) $(2.5,6.5)$

(C) $(5,4.25)$

(D) $(2.5,4.25)$

9 What is the solution to the inequality $\dfrac{2x-5}{x-4} \geq x$?

(A) $x \leq -1$ and $4 \leq x \leq 5$

(B) $x \leq -1$ and $4 < x \leq 5$

(C) $x \leq 1$ and $4 \leq x \leq 5$

(D) $x \leq 1$ and $4 < x \leq 5$

10 The velocity of a particle moving in a straight line is given by $v = 2x + 5$, where x metres is the distance from fixed point O and v is the velocity in metres per second. What is the acceleration of the particle when it is 1 metre from O?

(A) $a = 7\ \text{ms}^{-2}$

(B) $a = 12\ \text{ms}^{-2}$

(C) $a = 14\ \text{ms}^{-2}$

(D) $a = 24\ \text{ms}^{-2}$

End of Section I

Section II commences on the next page

Section II 60 marks

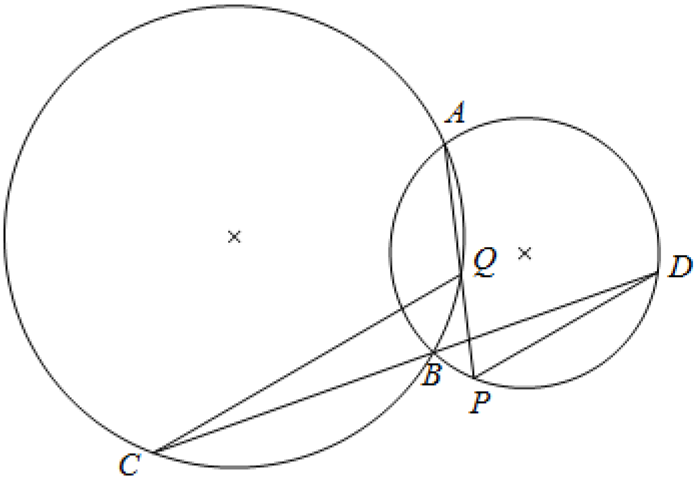
- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section
- Answer each question in a SEPARATE writing booklet or answer sheet. Extra writing booklets are available.
- In Questions 11 – 14, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Write out the expansion of $(1+x)^n$ in ascending powers of x. Hence or otherwise show that: **3**

$$\sum_{k=0}^{n}\binom{n}{k}=2^n$$

(b) Two circles meet at A and B. CBD is a double chord and AQP is a chord of the smaller circle. Prove that CQ is parallel to PD. **2**

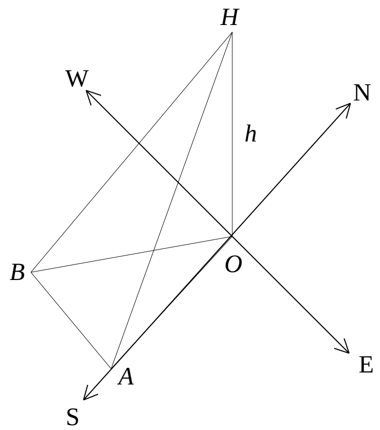


(c) Prove the following identity **3**

$$\frac{\sin\theta}{\sin\theta+\cos\theta}+\frac{\sin\theta}{\cos\theta-\sin\theta}=\tan 2\theta$$

(d) Find: $\dfrac{d}{dx}\left(2\sin^{-1}5x\right)$ **2**

(e) Point A is due south of a hill and the angle of elevation from A to the top of the hill is 35º. Another point B is a bearing 200º from the hill and the angle of elevation from B to the top of the hill is 46º. The distance AB is 220 m.



(i) Copy the diagram and label the given information. **1**

(ii) Express OA and OB in terms of h. **2**

(iii) Calculate the height h of the hill correct to three significant figures. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) The tangent at the point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ cuts the x-axis at A and the y-axis at B.

(i) Find the coordinates of M, the midpoint of A and B in terms of P. **2**

(ii) Show that the locus of M is a parabola. **1**

(iii) Find the coordinates of the focus of this parabola and the equation of the directrix. **1**

(b) Use the principle of mathematical induction to prove that for all positive integers n: **3**

$$1+2+4+...+2^{n-1} = 2^n - 1$$

(c) Find the exact value of $\sin\left[\cos^{-1}\frac{2}{3}+\tan^{-1}\left(-\frac{3}{4}\right)\right]$ **2**

(d) Find all the angles θ with $0 \le \theta \le 2\pi$ for which $\sin\theta + \cos\theta = 1$. **3**

(e) The function $f(x)$ is given by $f(x) = \sin^{-1}x + \cos^{-1}x$, $0 \le x \le 1$.

(i) Find $f'(x)$. **1**

(ii) Sketch the graph of $y = f(x)$. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) (i) Show that the function $f(x) = xe^x - 1$ has a zero between $x = 0$ and $x = 1$. **1**

(ii) Using $x = 0.5$ as the first approximation, use Newton's Method to obtain a second approximation. Answer correct to 2 decimal places. **2**

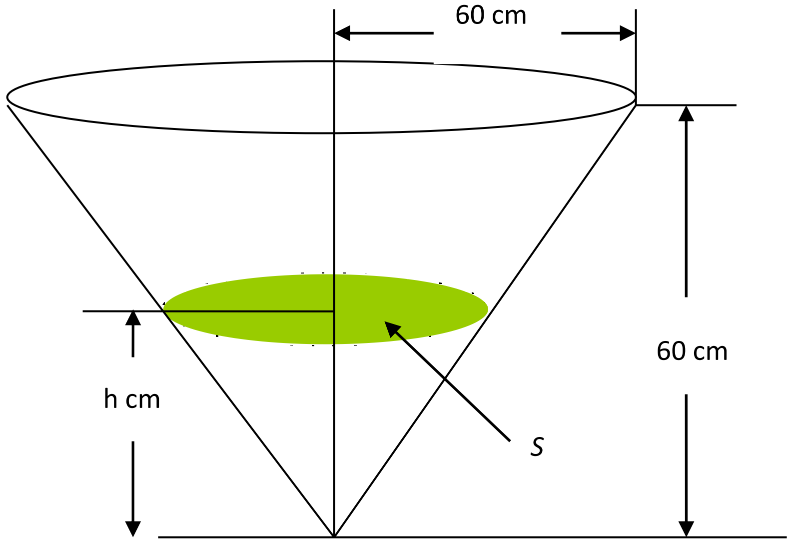
(b) A golfer hits a golf ball to clear a 6 metres high tree. The tree is 20 metres away on level ground. The golfer uses a golf club that produces an angle of projection of 40°. Take $g = 10 \text{ ms}^{-1}$.

(i) Derive the expressions for the vertical and horizontal components of the displacement of the ball from the point of projection. **3**

(ii) What is the Cartesian equation of the flight path? **2**

(iii) Calculate the speed in km/h at which the ball must leave the ground to just clear the tree. Answer correct to one decimal place. **3**

(c)



Not to Scale

Water is poured into a conical vessel at a constant rate of 30 cm^3 per second. The depth of water is h cm at any time t seconds. What is the rate of increase of the area of the surface S of the liquid when the depth is 24cm? **4**

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) The quadratic equation $x^2 - 4x + 9 = 0$ has roots $\tan A$ and $\tan B$.
Find the value(s) of angle $A + B$, noting that $0^0 \leq A + B \leq 360^0$. **3**
(leave your answer correct to the nearest degree)

(b) A particle moves in a straight line and its position at any time is given by:

$$x = 3\cos 2t + 4\sin 2t$$

(i) Prove that the motion is simple harmonic. **1**

(ii) Calculate the particle's greatest speed. **2**

(c) Water at a temperature of 24˚C is placed in a freezer maintained at a temperature of -12°C. After time *t* minutes the rate of change of temperature *T* of the water is given by the formula:

$$\frac{dT}{dt} = -k(T + 12)$$

where *t* is the time in minutes and *k* is a positive constant.

(i) Show that $T = Ae^{-kt} - 12$ is a solution of this equation, where *A* is a constant. **1**

(ii) Find the value of *A*. **1**

(iii) After 15 minutes the temperature of the water falls to 9˚C. Find to the nearest minute the time taken for the water to start freezing. (Freezing point of water is 0˚C). **2**

(d) When a projectile is fired with velocity $V\ ms^{-1}$ at an angle θ above the horizontal, the horizontal and vertical displacements (in metres) from the point of projection at time t seconds are given by:

$x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$ respectively (where g is the acceleration due to gravity).

(i) A projectile is fired horizontally with velocity V ms^{-1} from the top of a cliff. The projectile hits a fixed target in the water after T seconds. **2**

Show that the height of the cliff is given by $H = \frac{1}{2}gT^2$ metres.

(ii) It is found that the target can also be hit by firing a projectile from the top of the cliff at an angle α above the horizontal with the same initial speed (V ms^{-1}). **3**

Show that $\tan\alpha = \dfrac{2V}{gT}$.

End of Section II

End of Examination

Trinity Grammar School
Mathematics Department
2014
Year 12 Mathematics Extension 1
TRIAL EXAMINATION

HSC Assessment Task 4
Section I
Multiple Choice Answer Sheet

NSW Board of Studies Student Number
(Year 12 only)

Name: Solutions
(Year 11 only)

Class Teacher: ______________________

- Write your answers for **Section I** on this answer sheet.
- ***Shade completely only ONE answer for each question.***
- To change an answer, erase your previous mark completely and then record your new answer.

1	A	B	~~C~~	D
2	A	B	~~C~~	D
3	A	B	C	D ✓
4	~~A~~	B	C	D
5	~~A~~	B	C	D
6	A	B	C	~~D~~
7	A	B	C	~~D~~
8	~~A~~	B	C	D
9	A	B	C	~~D~~
10	A	B	~~C~~	D

A table of Standard Integrals, if required, is printed on the reverse side of this answer sheet

HSC Mathematics Extension 1:

1.) Domain & range of $y = \frac{1}{2}\cos^{-1}\left(\frac{x}{2}\right)$?

Domain: $-1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$

Range: $0 \le \cos^{-1}x \le \pi$

$0 \le \frac{1}{2}\cos^{-1}x \le \frac{\pi}{2}$

$0 \le y \le \frac{\pi}{2}$

$\therefore$ 1) C ✓

2.) The acceleration $\ddot{x} = -4x + 3 = -4\left(x - \frac{3}{4}\right)$
The acceleration at the centre is 0, that is, $x - \frac{3}{4} = 0$
so the centre is at $x = \frac{3}{4}$. Also, $n^2 = 4$ that is
$n = 2$ as $n > 0$. Therefore the period $= \frac{2\pi}{2} = \pi$

2) C ✓

3.) $\int \frac{x}{(2-x^2)^3}\,dx$ let $u = 2 - x^2$

$\frac{du}{dx} = \frac{d}{dx}(2 - x^2)$

$du = -2x \cdot dx$

$-\frac{du}{2} = x \cdot dx$

$= -\frac{1}{2}\int u^{-3}\,du$

$= -\frac{1}{2}\left[\frac{u^{-2}}{-2}\right]$

$= +\frac{1}{4}\left(\frac{1}{(2-x^2)^2}\right) + C$

All options given 1 mark

3) B ✓

4. $\int_0^1 \frac{1}{x^2+1}\,dx$

$= \left[\tan^{-1}x\right]_0^1$

$= \tan^{-1}1 - \tan^{-1}0$

$= \frac{\pi}{4} - 0$

4) A ✓

5mm A4

5.) $\left(x+\frac{3}{x}\right)^4$

$T_{k+1} = {}^nC_k\, a^{n-k} b^k$

$= {}^4C_k (x)^{4-k}\left(\frac{3}{x}\right)^k$

$= {}^4C_k\, x^{4-k} \cdot 3^k \cdot x^{-k}$

$= {}^4C_k\, x^{4-k-k} \cdot 3^k$

$x^0 = x^{4-2k}$

$\therefore 0 = 4-2k$

$2k = 4$

$k=2$

$\therefore {}^4C_2 \times 3^2$

$= 6 \times 9$

$= 54$

$(a+b)^1$ 1 1

$(a+b)^2$ 1 2 1

$(a+b)^3$ 1 3 3 1

$(a+b)^4$ 1 4 6 4 1

1 5 10 10 5 1

5) A ✓

6.) $y=2x$ $\quad y=-x+5$

$\therefore m_1 = 2 \quad m_2 = -1$

$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{-1-2}{1+(2)(-1)}\right|$

$= \left|\frac{-3}{-1}\right|$

$\tan\theta = 3$

$\theta = 72°$

6) D ✓

7) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 x\, dx$

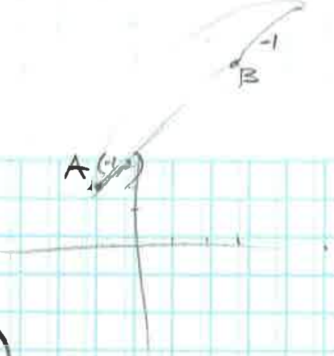
$= \left[x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1-\cos 2x)\, dx$

$= \frac{\pi}{3} - \frac{\pi}{4} + \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \frac{\pi}{3} - \frac{\pi}{4} + \frac{\pi}{6} - \frac{\pi}{8} - \frac{1}{4}\left(\frac{\sqrt{3}}{2} - 1\right)$

$= \frac{\pi}{8} - \frac{\sqrt{3}-2}{8}$

7) D ✓



8.) $A(-1, 2) \quad B(3, 5)$

$3 : -1$

$$= \left(\frac{(-1)(-1) + (3)(3)}{2} ; \frac{(2)(-1) + (5)(3)}{2} \right)$$

$$= \left(\frac{10}{2} ; \frac{13}{2} \right)$$

$$= (5, 6.5)$$

8) A ✓

9.) $\frac{2x-5}{x-4} \geq x$

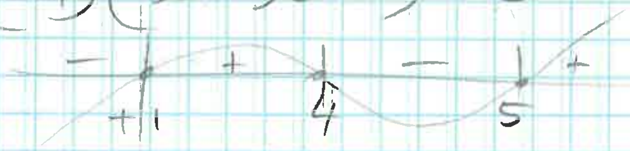
$$(x-4)(2x-5) \geq x(x-4)^2$$

$$(x-4)[(2x-5) - x(x-4)] \geq 0$$

$$(x-4)(2x-5-x^2+4x) \geq 0$$

$$-(x-4)(x^2-6x+5) \geq 0$$

$$(x-4)(x-5)(x-1) \leq 0$$



$x \leq 1;\ 4 < x \leq 5$

9) D ✓

10.) $v = 2x + 5$

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$v^2 = 4x^2 + 20x + 25$$

$$\frac{1}{2}v^2 = 2x^2 + 10x + \frac{25}{2}$$

$$\frac{d}{dx}\left(2x^2 + 10x + \frac{25}{2}\right)$$

$= 4x + 10$ let $x = 1$

$= 14\ m/s^2$

10) C ✓

Question 11:

a) $(1+x)^n = {}^nC_0 1^n + {}^nC_1 x^1 + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots {}^nC_n x^n$ ✓

LHS: let $x = 1$ ✓

$(1+1)^n$

$= 2^n$

RHS: (let $x=1$) $= {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots {}^nC_n$

$= \sum_{k=0}^{n} {}^nC_k$ ✓

$= \sum_{k=0}^{n} \binom{n}{k}$

$\therefore$ LHS = RHS.

(3)

b) R.T.P: $CQ \parallel PD$

Proof: Construct AB

In larger circle $\angle RCQ = \angle BAQ = x$ ($\angle$'s subtended at the circumference by same arc are =)

sim In smaller circle $\angle PDB = \angle BAP$ " " ✓

$\therefore \angle BCQ = PDB$

$\therefore$ $CQ \parallel PD$ (alternate angles are equal). ✓

(2)

c) LHS:

$$\frac{\sin\theta}{\sin\theta+\cos\theta}+\frac{\sin\theta}{\cos\theta-\sin\theta}$$

$$\frac{\sin\theta(\cos\theta-\sin\theta)+\sin\theta(\cos\theta+\sin\theta)}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)} \checkmark$$

$$=\frac{\sin\theta.\cos\theta-\sin^2\theta+\sin\theta\cos\theta+\sin^2\theta}{(\cos^2\theta-\sin^2\theta)}$$

$$=\frac{2\sin\theta.\cos\theta}{\cos2\theta} \checkmark$$

$$=\frac{\sin2\theta}{\cos2\theta}$$

$$=\tan2\theta$$

$$=\text{RHS}$$

(2)

d) $\frac{d}{dx}\left(2\sin^{-1}5x\right)$

$$=\frac{2}{\sqrt{1-(5x)^2}}\times 5 \checkmark$$

$$=\frac{10}{\sqrt{1-25x^2}} \checkmark$$

(2)

e) i) $\tan 35° = \frac{h}{OA}$

$\therefore OA = \frac{h}{\tan 35°}$ ✓

$\tan 46° = \frac{h}{OB}$

$\therefore OB = \frac{h}{\tan 46°}$ ✓

(2)

ii) $AB^2 = OA^2 + OB^2 - 2 \times OA \times OB \times \cos 20°$

$$220^2 = \left(\frac{h}{\tan 35°}\right)^2 + \left(\frac{h}{\tan 46°}\right)^2 - 2\left(\frac{h}{\tan 35°}\right) \times \left(\frac{h}{\tan 46°}\right) \times \cos 20°$$ ✓

$$= h^2 \left[\left(\frac{1}{\tan^2 35°}\right) + \left(\frac{1}{\tan^2 46°}\right) - 2 \times \frac{\cos 20°}{(\tan 35°)(\tan 46°)}\right]$$

$$h^2 = 220^2 \div \left(\frac{1}{\tan^2 35°} + \frac{1}{\tan^2 46°} - \frac{2 \times \cos 20°}{\tan 35° \times \tan 46°}\right)$$

$$= 127296.7453\ldots$$

$h = 356.7866944\ldots$ ✓

$h \approx 357 \text{m}$

(2)

f) $f(x) = x^3 + 3x^2 - 9x + 5$

$f(-5) = (-5)^3 + 3(-5)^2 - 9(-5) + 5$

$= -125 + 75 + 45 + 5$

$= 0$

$\therefore (x+5)$ is a factor ✓

$$\begin{array}{r|l} & x^2 - 2x + 1 \\ x+5 & x^3 + 3x^2 - 9x + 5 \\ & -x^3 \mp 5x^2 \\ \hline & -2x^2 - 9x \\ & \pm 2x^2 \pm 10x \\ \hline & 1x + 5 \\ & x + 5 \\ \hline \end{array}$$

$\therefore x^3 + 3x^2 - 9x + 5$

$= (x+5)(x-1)(x-1)$ ✓

$= (x-1)^2(x+5)$

(2)

Question 12:

a) i) Given $x^2 = 4ay$

$\therefore y = \frac{1x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$\frac{dy}{dx} = \frac{x}{2a} = \frac{2ap}{2a} = p$

$\therefore y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$ ✓

Xint $(y = 0)$

$px = ap^2$

$x = ap$ $\therefore A(ap, 0)$

Yint $(x = 0)$

$y = -ap^2$ $B(0, -ap^2)$

Midpt: $M\left(\frac{ap}{2}, \frac{-ap^2}{2}\right)$ ✓ (2)

ii) To find the locus of M eliminate p from co-ordinates of M.

Now $x = \frac{ap}{2}$ (1) and $y = \frac{-ap^2}{2}$ (2)

from (1) $p = \frac{2x}{a}$ and subst into equation (2)

$$y = \frac{-a\left(\frac{2x}{a}\right)^2}{2} = \frac{-a}{2} \times \frac{4x^2}{a^2} = -\frac{2x^2}{a}$$

$y = -\frac{2x^2}{a}$ OR $x^2 = -\frac{1}{2}ay$ (parabola) ✓ (1)

iii) $\therefore x^2 - \frac{1}{2}ay = 4 \times \left(-\frac{1}{8}a\right) \times y$

$\therefore$ Focus is $\left(0, -\frac{1}{8}a\right)$ and the equation of directrix is $y = \frac{1}{8}a$ ✓ (1)

b) Prove true for $n=1$

LHS: 1 RHS: $2^1 - 1$

$= 1$

$\therefore$ true for $n=1$

Assume true for $n=k$

ie $1+2+4+\ldots+2^{k-1} = 2^k - 1$ *

Prove true for $n=k+1$

ie Prove $1+2+4+\ldots 2^{k-1}+2^k = 2^k \cdot 2 - 1$

LHS: from assumption *

$$\underbrace{1+2+4+\ldots 2^{k-1}}+2^k$$

$$= 2^k - 1 + 2^k$$

$$= 2 \cdot 2^k - 1$$

$$= 2^{k+1} - 1$$

$$= \text{RHS}$$

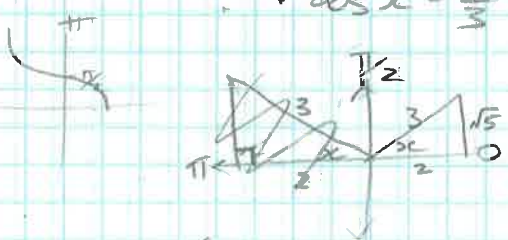
(3)

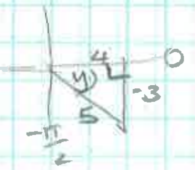
$\therefore$ Proven true for $n=1$ and if true for $n=k$, true for $n=k+1$ $\therefore$ by the principle of mathematical induction, true for all positive integers n

c) $\sin\left[\cos^{-1}\left(\frac{2}{3}\right) + \tan^{-1}\left(\frac{3}{4}\right)\right]$

let $\cos^{-1}\left(\frac{2}{3}\right) = x$ $\quad$ let $\tan^{-1}\left(-\frac{3}{4}\right) = y$

$\therefore \cos x = \frac{2}{3}$ $\quad$ $\therefore \tan y = -\frac{3}{4}$





$\sin(x+y)$

$= \sin x \cos y + \cos x \sin y$

$= \left(\frac{\sqrt{5}}{3}\right)\left(\frac{4}{5}\right) + \left(\frac{2}{3}\right)\left(\frac{-3}{5}\right)$

$= \frac{4\sqrt{5}-6}{15}$

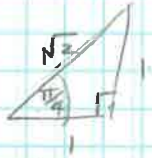
(2)

d) $1\sin\theta + 1\cos\theta = 1$

$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = 1$



$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ ✓

$\left(\theta + \frac{\pi}{4}\right) = \frac{\pi}{4}$ OR $\left(\theta + \frac{\pi}{4}\right) = \pi - \frac{\pi}{4}$

$\underline{\theta = 0}$

$\left(\theta + \frac{\pi}{4}\right) = \frac{3\pi}{4}$

$\theta = \frac{\pi}{2}$ ✓

✓

$\theta = 0, \frac{\pi}{2}, 2\pi$ (3)

e) $f(x) = \sin^{-1}x + \cos^{-1}x \quad 0 \leq x \leq 1$

i) $\frac{d}{dx}\left(\sin^{-1}x + \cos^{-1}x\right)$

$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$ ✓

$= 0$ (1)

$\therefore$ $\sin^{-1}x + \cos^{-1}x$ is a horizontal line ie cuts y axis

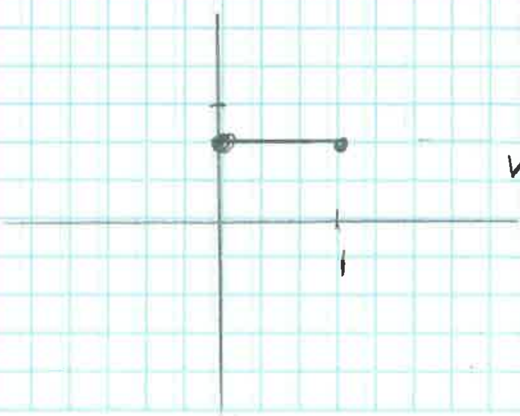
ii) y int let $x = 0$

$y = \sin^{-1}0 + \cos^{-1}0$

$= 0 + \frac{\pi}{2}$

$= \frac{\pi}{2}$

$\therefore \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ ✓



✓

(2)

Question 13:

a i) $f(x) = xe^{x} - 1$

$f(0) = 0e^{0} - 1$
$= -1$

$f(1) = (1)e^{(1)} - 1$
$= e^{1} - 1$
$= 1.718$

Since $f(0)$ and $f(1)$ have opposite signs, the root lies between 0 and 1 ✓ (1)

ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_1 = 0.5$

$f(0.5) = (0.5)e^{0.5} - 1$
$= -0.175639$

$f'(x_1) = 1e^{x} + e^{x}.x$

$f'(0.5) = e^{0.5}(0.5+1)$
$= 1.5.e^{0.5}$

✓ finds $f(0.5)$ and $f'(0.5)$

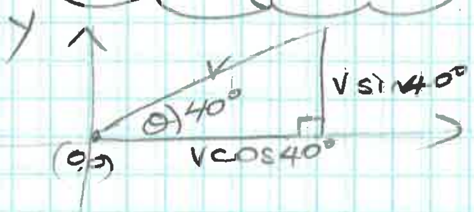
$x_2 = 0.5 - \left(\frac{0.5e^{0.5} - 1}{1.5e^{0.5}}\right) = 0.5710204398 \approx 0.57$ ✓ (2)

b i) $\boxed{\ddot{x} = 0}$ at $t=0$ $\dot{x} = v\cos 1$

$\dot{x} = c_1$

$\boxed{\dot{x} = V\cos 40°}$ (1)

Initially $\dot{x} = v\cos\theta$
$\dot{y} = v\sin\theta$



$x = vt\cos\theta + c_2$

at $t = 0$ $x = 0$

$\therefore c_2 = 0$

$\therefore x = vt\cos 40°$

$\boxed{x = vt\cos 40°}$ (2) ✓

$\boxed{\ddot{y} = -10}$

at $t=0$ $\dot{y} = v\sin 40°$

$\dot{y} = \int -10\, dt$

$\dot{y} = -10t + c_3$

$\therefore c_3 = v\sin 40°$

$\boxed{\dot{y} = -10t + v\sin 40°}$ (3)

at $t=0$ $y=0$

$y = \int (-10t + v\sin 40°)\, dt$

$\boxed{y = -5t^2 + vt\sin 40°}$ (4) ✓

✓ (2 marks for deriving either the horizontal or vertical equations of motion) (3)

ii) from ②

$t = \frac{x}{V\cos 40°}$

subst into ④

$y = -5\left(\frac{x}{V\cos 40°}\right)^2 + V\left(\frac{x}{V\cos 40°}\right)\sin 40°$

$= \frac{-5x^2}{V^2\cos^2 40°} + x\tan 40°$

$= \frac{-5x^2}{V^2}\sec^2 40° + x\tan 40°$

$y = \frac{-5x^2}{V^2}(1+\tan^2 40°) + x\tan 40°$ (2)

iii) Find V if $x = 20$ and $y = 6$

$(6) = \frac{-5(20)^2}{V^2}(1+\tan^2 40°) + 20\tan 40°$

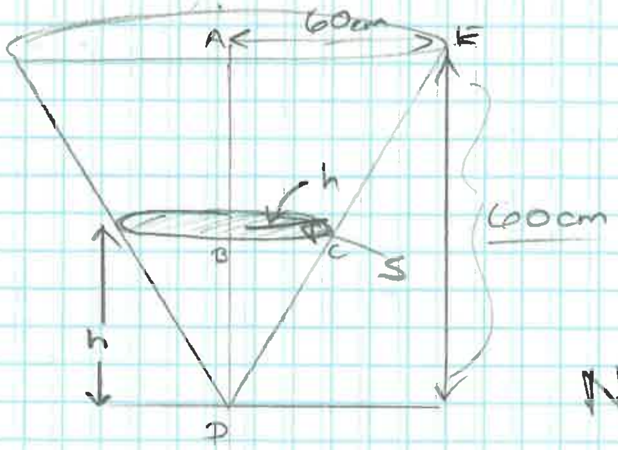
$6V^2 = -3408.176 + 20\tan 40° V^2$

$3408.176 = V^2(20\tan 40° - 6)$

$316.084 = V^2$

$17.779 = V$

$17.8\,m/s = V$ $\therefore V = 64.0\,km/h$ (3)

d) 

$\boxed{\frac{dV}{dt} = 30\,cm^3/s}$

In $\triangle ADE$ and $\triangle BDC$

$\frac{AD}{BD} = \frac{AE}{BC}$

$\frac{60}{h} = \frac{60}{r}$

$\therefore h = r$

Now $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(h)^2 h$

$V = \frac{1}{3}\pi h^3$ $\therefore \left(\frac{dV}{dh} = \pi h^2\right)$

Now $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$

$\frac{dV}{dt} = \pi h^2 \cdot \frac{dh}{dt}$

$30 = \pi(24)^2\cdot\frac{dh}{dt}$

$\frac{5}{96\pi} = \frac{dh}{dt}$

$S = \pi r^2$

πh^2 $\left(\frac{ds}{dh} = 2\pi h\right)$

$\frac{ds}{dt} = \frac{ds}{dh}\cdot\frac{dh}{dt}$

$\frac{ds}{dt} = 2\pi(24)\times\frac{5}{96\pi}$

$\frac{ds}{dt} = \frac{5}{2} = 2.5\,cm^2/s$ (4)

Question 14:

a) $\tan A + \tan B = -\frac{b}{a} = 4$

$\tan A . \tan B = \frac{c}{a} = 9$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{4}{1-9}$

$= -\frac{1}{2}$

angle $(A+B) = 153°26' = 153°$

OR

$= 333°26' = 333°$

(3)

bi) $x = 3\cos 2t + 4 \sin 2t$

$\dot{x} = -3(2\sin 2t) + 4 \times 2\cos 2t$

$= -6\sin 2t + 8\cos 2t$

$\ddot{x} = -3 \times 2^2 \cos 2t - 4 \times 2^2 \sin 2t$

$= -12\cos 2t - 16\sin 2t$

$= -2^2(3\cos 2t + 4\sin 2t)$

$= -2^2 x$

OR

$x = 5\cos(2t - \tan^{-1}(\frac{4}{3}))$

$\dot{x} = -10\sin(2t - \tan^{-1}(\frac{4}{3}))$

$\ddot{x} = -20\sin(2t - \tan^{-1}(\frac{4}{3}))$

$= -4(5\cos(2t - \tan^{-1}(\frac{4}{3})))$

$\ddot{x} = -4x$

(1)

ii) Max speed occurs when $\ddot{x} = 0$ OR $x = 0$ (centre of motion)

$x = 3\cos 2t + 4\sin 2t = 0$

$4\sin 2t = -3\cos 2t$

$\tan 2t = -\frac{3}{4}$

$2t = -0.6435011088... + 0, \pi, 2\pi ...$

$(2t = \tan^{-1}(-0.75) + n\pi$, where n is an integer$)$

OR $2t = -36.87 + 180°$

Smallest positive value of t for max speed:

$t = \frac{1}{2}(-0.6435... + \pi) = 1.249045....$ subst $t = 71.5...$

$\therefore \dot{x} = -3(2\sin(2 \times 1.249045...)) + 4 \times 2\cos(2 \times 1.249045...)$

$= -10$ $\therefore$ Max speed is 10

(2)

14c i) $T = Ae^{-kt} - 12$ ie $Ae^{-kt} = T + 12$

$\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T+12)$ (1)

ii) Initially : at $t = 0$, $T = 24$

$T = Ae^{-kt} - 12$

$24 = Ae^{-k \times 0} - 12$

$\underline{A = 36}$ (1)

iii) Also $t = 15$ when $T = 9$

$9 = 36e^{-k \times 15} - 12$

$e^{-15k} = \frac{21}{36} = \frac{7}{12}$

$-15k = \log_e \frac{7}{12}$

$k = -\frac{1}{15} \times \log_e \left(\frac{7}{12}\right)$

$\underline{k = 0.0359331000 5 \ldots}$

We need to find t when $T = 0$

$\therefore 0 = 36e^{-kt} - 12$

$e^{-kt} = \frac{12}{36} = \frac{1}{3}$

$-kt = \log_e \frac{1}{3}$

$t = -\frac{1}{k} \log_e \frac{1}{3}$

$= 30.57382 \ldots$

$t \approx 31$ minutes

$\therefore$ It will take about 31 minutes for the water to cool to $0°C$. (2)

14d i) $x = Vt\cos\theta$

$y = Vt\sin\theta - \frac{1}{2}gt^2$ fired

$\theta = 0$ (horizontally $\therefore \theta = 0$)

$y = Vt\sin(0^\circ) - \frac{1}{2}gt^2$

$y = -\frac{1}{2}gt^2$

H = height of cliff, when $t = T$

$\therefore H = -\frac{1}{2}g(T)^2$ (2)

ii) fired at angle of α

$\therefore x = Vt\cos\alpha$ (1)

$y = Vt\sin\alpha - \frac{1}{2}gt^2$ (2)

from (1) $t = \frac{x}{V\cos\alpha}$ (3)

Subst into (2)

$$y = V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{V^2}\sec^2\alpha$$

Now when $\theta = 0$, $x = Vt$ let $t = T$ $\therefore x = VT$ ... (3)

and $y = -\frac{1}{2}gt^2$ let $t = T$ $y = -\frac{1}{2}gT^2$ ... (4)

$$\therefore -\frac{1}{2}gT^2 = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$$

(target is ~~fixed~~ hit at the same time $\therefore t = T$ ie $x = Vt$

$x = Vt$ $\therefore x = VT$)

$$\therefore -\frac{1}{2}gT^2 = Vt\tan\alpha - \frac{g(Vt)^2}{2V^2}\sec^2\alpha$$

$$-\frac{1}{2}gT^2 = Vt\tan\alpha - \frac{gt^2}{2}(1 + \tan^2\alpha)$$

$$-\frac{1}{2}gT^2 = VT\tan\alpha - \frac{gT^2}{2} - \frac{gT^2\tan^2\alpha}{2}$$

$$0 = VT\tan\alpha - \frac{gT^2\tan^2\alpha}{2}$$

$$0 = T\tan\alpha\left(V - \frac{gT}{2}\tan\alpha\right)$$ (3)

$\therefore T\tan\alpha = 0$ **OR** $\boxed{\tan\alpha = \frac{2V}{gT}}$

5mm A4