Trinity Grammar School

Mathematics Department

2015

TRIAL EXAMINATION

HSC ASSESSMENT TASK 4

Year 12

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Only approved calculators for this course are allowed in this task
- A table of Standard Integrals is supplied
- Show all necessary working
- Write your Board of Studies Student Number and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating **N/A** and your Student Number or Name.

- **HSC Assessment Weighting:** 40%

Board of Studies Student Number
(Year 12 only)

Class Teacher:

………………………………………………………………………
…

Name:
(Year 11 only)

………………………………………………………………………
…

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

Date of examination:

Wednesday, 19 August 2015

Total marks – 70

Section I

10 marks

- Attempt Questions 1-10
- Allow about **15** minutes for this section

Section II

60 marks

- Attempt Questions 11-14
- Allow about **1** hour **45** minutes for this section

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Section I 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section
- Shade the correct response on the multiple-choice answer sheet for Questions 1 – 10
- Each question is worth 1 mark

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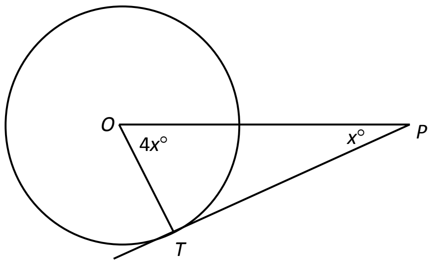
1 The diagram shows a circle with centre *O*. The line *PT* is tangent to the circle at the point *T*. $\angle TOP = 4x°$ and $\angle TPO = x°$.

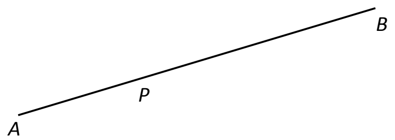


What is the value of *x*?

(A) 9
(B) 18
(C) 36
(D) 72

2 The point *P* divides the interval *AB* in the ratio 3:7.

In what external ratio does *A* divide the interval *PB*?



(A) 3:10
(B) 3:4
(C) 7:3
(D) 10:3

3 The parabola $y = f(x)$ has a zero between $x = 0$ and $x = 1$, and a zero between $x = 1$ and $x = 2$.

Which set of three points could lie on the graph of $y = f(x)$?

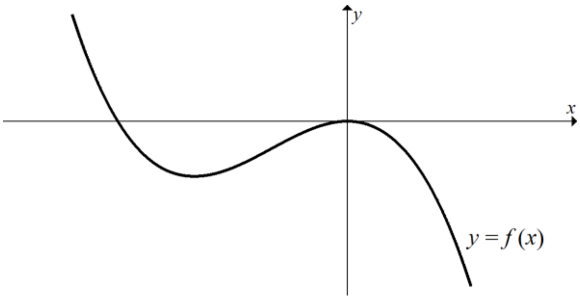
(A) (0, 2), (1, 4) and (2, –1)
(B) (0, –2), (1, –4) and (2, 1)
(C) (0, 2), (1, 0) and (2, –1)
(D) (0, –2), (1, 4) and (2, –1)

4 The temperature of an object is given by $T = A - Be^{-kt}$ where T is temperature (measured in °C) and t is time (measured in minutes). A, B and k are positive constants and $A > B$. As $t \to \infty, T \to T_\infty$.

What is the value of T_∞?

(A) 0°C
(B) B°C
(C) A°C
(D) $(A - B)$°C

5 The diagram shows the graph of a cubic function $y = f(x)$.



What is a possible equation of this function?

(A) $f(x) = -x(x-2)(x+2)$
(B) $f(x) = x^2(x-2)$
(C) $f(x) = -x^2(x-2)$
(D) $f(x) = -x^2(x+2)$

6 Given that $t = \tan\frac{\theta}{2}$, which expression is equal to $\tan\theta - \tan\frac{\theta}{2}$?

(A) t

(B) $\frac{t(1+t^2)}{1-t^2}$

(C) $\frac{t}{1-t^2}$

(D) $\frac{t(3-t^2)}{1-t^2}$

7 What is the coefficient of x^3 in the binomial expansion of $\left(x^2 - \frac{4}{x}\right)^9$?

(A) ${}^9C_4 4^5$

(B) ${}^9C_4 4^4$

(C) $-{}^9C_4 4^5$

(D) $-{}^9C_4 4^4$

Commented [WJ1]:

8 Disregard this question.

9 A ball is projected into the air from the ground with a velocity of 40 m/s at an angle of 45°.

After how many seconds will the ball reach its greatest height?

(A) $\frac{40\sqrt{2}}{g}$ s

(B) $\frac{20\sqrt{2}}{g}$ s

(C) $\frac{160\sqrt{2}}{g}$ s

(D) $80\sqrt{2}g$ s

10. The parametric equations of a curve are $x = p+1, y = p^2 - 1$.
The Cartesian equation of the curve is?

(A) $x^2 = 4y$

(B) $y = x^2 - 2x$

(C) $y = x^2 - 2$

(D) $y = (x-2)^2$

End of Section I
Section II commences on the next page

Section II 60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section
- Answer each question in a SEPARATE writing booklet or answer sheet. Extra writing booklets are available.
- In Questions 11 – 14, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Solve $\dfrac{x^2-4}{x}<0$. **3**

(b) Differentiate $\tan^{-1}\left(\dfrac{x^2}{9}\right)$. **2**

(c) Determine the domain and range of the function $f\left(x\right)=2\pi\cos^{-1}\left(\dfrac{x}{\pi}\right)$. **2**

(d) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin^2 x+\cos^2 x\ dx$ **3**

(e) A particle moves with acceleration $\ddot{x}=\dfrac{1}{16\sqrt{x}}$. Initially, the particle is at the origin and its velocity is -4 m/s. **3**

Show that the particle is never at rest.

(f) Solve $\left(\log_e x\right)^2-e^2\left(1+e\right)\log_e x+e^5=0$. **2**

End of Question 11

Question 12 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) $P\left(2ap, ap^2\right)$ and $Q\left(-2ap, ap^2\right)$ are points on the parabola $x^2 = 4ay$.

(i) Tangents to the parabola at P and Q meet at the point A.
Find the coordinates of A. **3**

(ii) M is the midpoint of PQ. State the coordinates of M. **1**

(iii) Show that the origin O is the midpoint of AM. **1**

(b) Use mathematical induction to prove that $\left(n+1\right)^2 + n - 1$ is divisible by 2 for all integers $n \geq 1$. **3**

(c)

(i) Show that $P\left(x\right) = -4x^5 + 5x^4 - 0.7$ has a zero between $x = 0$ and $x = 1$ **1**

(ii) Explain why Newton's method fails when $x = 1$ is used as the first approximation. **2**

(d) Given $u > 0$, solve $\tan^{-1}\left(\dfrac{1}{u-1}\right) + \tan^{-1}\left(\dfrac{1}{u+1}\right) = \tan^{-1}\left(\dfrac{4}{7}\right)$. **4**

End of Question 12

Question 13 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) A new medicine is trialled to test its ability to control a virus. At first the trial is successful and the population P of the virus can be modelled by the equation $P = 50000e^{\frac{t}{2}\log_e 0.7}$ where t is time measured in hours.

(i) When its population reaches 24 500, the virus becomes resistant to the medicine and the population begins to increase. Show that this resistance occurs 4 hours after the trial started. **1**

(ii) The increasing population of the virus can be modelled by the equation $P = 500 + 12000e^{kt}$ where k is a constant and t is the time since the trial started. Show that $k = 0.25\ln 2$. **1**

(iii) Find the value of t when the population returns to its initial value. Give your answer to the nearest 10 minutes. **2**

(iv) Sketch the graph of the population over time, clearly indicating the initial population, the population when the resistance begins and the point where the population returns to its initial value. **2**

(b) A particle is moving with simple harmonic motion. Initially it is at the centre of its motion and its period is 2π. The velocity of the particle is given by $v^2 = 20 + 8x - x^2$.

(i) Find the endpoints of the motion of the particle. **2**

(ii) Find the centre of motion of the particle. **1**

(iii) Express the displacement x of the particle as a function of time t. **3**

Question 13 continues

Question 13 (continued) **Marks**

(c) (i) Expand $(1-x)^{2n}$ using the binomial theorem. **1**

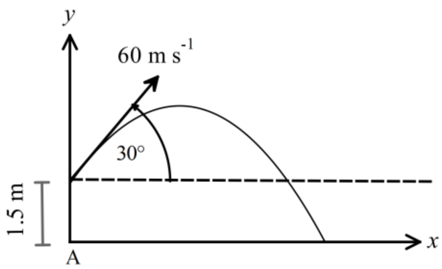
(ii) Hence prove the following identity. **2**

$$^{2n}C_1 + 3\,^{2n}C_3 + ... + (2n-1)\,^{2n}C_{2n-1} = 2\,^{2n}C_2 + 4\,^{2n}C_4 + ... + 2n\,^{2n}C_{2n}$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) An archer shoots an arrow from a bow at an initial velocity of 60 ms^{-1}, while standing at point A. The bow is 1.5 metres above ground level at the time of firing and the angle of projection is 30°.



Allowing gravity to be 9.8 $m\,s^{-2}$.

(i) The archer is aiming for a tree that is 300 metres away and 3.4 metres in height. The ground is horizontal. Show calculations which prove that the arrow will not hit the tree. **2**

(ii) The archer has painted a target on the tree at a point 1 metre above ground. What angle will the archer need to shoot the arrow at in order to hit the target if the initial velocity is 60 ms^{-1}? **3**

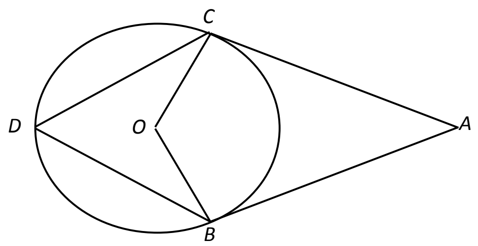
(b) Two towers T_1 and T_2 have heights h metres and $2h$ metres respectively. The second tower is due south of the first tower. The bearing of tower T_1 from a surveyor is 292°. The bearing of the tower T_2 from the surveyor is 232°. The angle of elevation from the surveyor to the top of tower T_1 is 30° while the angle of elevation from the surveyor to the top of tower T_2 is 60°. **4**

Show that the distance d between the two towers is given by $d = \dfrac{\sqrt{21}h}{3}$ metres.

Question 14 continues

Question 14 (continued) **Marks**

(c) The diagram shows a circle with centre *O*. Tangents from the point *A* touch the circle at the points *B* and *C*. Chords *BD* and *CD* are also shown.



(i) Show that $\angle BAC$ and $\angle BOC$ are supplementary. **1**

(ii) Show that $\angle BDC = 90° - \frac{1}{2}\angle BAC$. **2**

(iii) $0° < \angle BDC < 90°$. (Do NOT show this.) **3**

Let $\cos\angle BAC = a$.

Using part (ii), or otherwise, show that $\cos\angle BDC = \sqrt{\frac{1-a}{2}}$.

End of Question 14
End of Section II
End of Examination

2015

Trinity Grammar School
HSC Trial Examination

Mathematics Extension 1

WORKED SOLUTIONS

SECTION 1

1 $\angle OTP = 90°$ (tangent perpendicular to radius) **B**

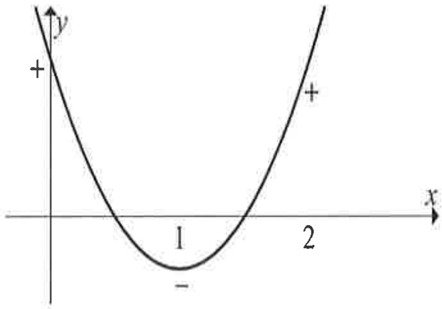
$4x + x + 90 = 180$ (angle sum of triangle)

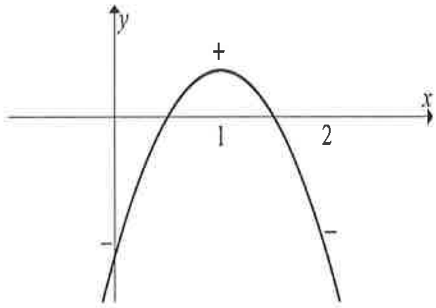
$x = 18$

2 $AP : AB = 3 : 3 + 7$ **A**

$= 3 : 10$

3 The y values of the three points must be +, –, + or –, +, – so that the zeroes lie between the three points. **D**





4 $T = A - Be^{-kt}$ **C**

As $t \to \infty, e^{-kt} \to 0$

$T = A - Be^{-kt} \to A$

5 There are two zeroes, 0 and a negative number. **D**

As $x \to \infty, y \to -\infty$, the leading coefficient $a < 0$.

$f(x) = -x^2(x+2)$ satisfies these conditions

6 **B**

$$\tan\theta - \tan\frac{\theta}{2} = \frac{2t}{1-t^2} - t$$

$$= \frac{2t - t(1-t^2)}{1-t^2}$$

$$= \frac{2t - t + t^3}{1-t^2}$$

$$= \frac{t + t^3}{1-t^2}$$

$$= \frac{t(1+t^2)}{1-t^2}$$

7 Given $\left(x^2 - \frac{4}{x}\right)^9$ C

$$T_{r+1} = {}^9C_r\left(x^2\right)^r\left(-4x^{-1}\right)^{9-r}$$
$$= \left[{}^9C_r\left(-4\right)^{9-r}\right]x^{2r}x^{-9+r}$$
$$2r - 9 + r = 3$$
$$r = 4$$

Coefficient $= {}^9C_4\left(-4\right)^{9-4} = -{}^9C_4 4^5$

8 DISREGARD THIS QUESTION

9 Time of flight $= \dfrac{V\sin\theta}{g}$ B

$$\therefore t = \frac{V\sin\theta}{g}$$
$$= \frac{40 \times \sin 45°}{g}$$
$$= \frac{40}{g\sqrt{2}}$$
$$= \frac{20\sqrt{2}}{g} \text{ s}$$

10

Solution	Answer	Mark
$p = x - 1$ $\therefore y = (x-1)^2 - 1$ $= x^2 - 2x$	B	1

SECTION II

Question 11

(a) $\dfrac{x^2-4}{x}<0$ **3**

By the "Critical value Method"

denominator $x \neq 0$

numerator $x^2-4 \neq 0$

$x \neq \pm 2$

test $x<-2$:

$\dfrac{(-3)^2-4}{-3}<0$ true

test $-2<x<0$:

$\dfrac{(-1)^2-4}{-1}<0$ false

test $0<x<2$:

$\dfrac{(1)^2-4}{1}<0$ true

test $x>2$:

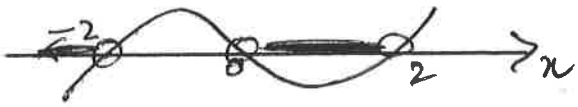
$\dfrac{(3)^2-4}{3}<0$ false

$\therefore x<-2, 0<x<2$

By "squaring" method

$(x^2-4)x < 0 \cdot x^2$

$x(x-2)(x+2)<0$



$\therefore \{x: x<-2 \text{ or } 0<x<2\}$.

✓ — appropriate method that may lead to a correct answer at least $x(x^2-4)<0$ if "square method used"

✓✓ — a correct method plus either $x<-2 \cup 0<x<2 \cup x=0, \pm 2$

✓✓✓ — correct solution, must have an inequality solution..

(b) $\dfrac{d}{dx}\left[\tan^{-1}\left(\dfrac{x^2}{9}\right)\right]=\dfrac{1}{1+\left(\dfrac{x^2}{9}\right)^2}\times\dfrac{2x}{9}$ ← Line 1. **2**

$=\dfrac{81}{81+x^4}\times\dfrac{2x}{9}$

$=\dfrac{18x}{81+x^4}$

[using $\dfrac{d}{dx}\tan^{-1}f(x) = \dfrac{f'(x)}{1+[f(x)]^2}$]

✓ — derivative of $x^2/9$ seen anywhere $\left(\frac{2x}{9}\right)$ or $\left(\frac{1}{1+x^4/81}\right)$ or $\frac{9}{81+x^4}$, $\frac{1}{81+x^4}$ etc...

✓✓ — correct answer as seen in line 1 above or equivalent.

Question 11 (continued)

(c) $f(x)=2\pi\cos^{-1}\left(\frac{x}{\pi}\right)$ **2**

domain: $-1\le\frac{x}{\pi}\le 1$

$-\pi\le x\le\pi$ ✓ correct answer only

range: $0\times 2\pi\le y\le\pi\times 2\pi$

$0\le y\le 2\pi^2$ ✓ correct answer only

(d) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(2\sin^2 x+\cos^2 x\right)\,dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(2\sin^2 x+\cos^2 x\right)dx$ **3**

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2 x+\underbrace{\left(\sin^2 x+\cos^2 x\right)}_{=1}\right)\,dx$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2 x+1\right)\,dx$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1-\cos 2x}{2}+1\right)\,dx$$

$$=\left[\frac{1}{2}x-\frac{1}{4}\sin 2x+x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\left[\frac{3x}{2}-\frac{1}{4}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\left(\frac{3\pi}{4}-\frac{1}{4}\sin\pi\right)-\left(\frac{3\pi}{8}-\frac{1}{4}\sin\frac{\pi}{2}\right)$$

$$=\frac{3\pi}{8}+\frac{1}{4}$$

fact: $\sin^2 x=\frac{1}{2}(1-\cos 2x)$

$\therefore 2\sin^2 x = 1-\cos 2x$

✓ a correct re-arrangement of either $\sin^2 x$ or $\cos^2 x$ or both to "standard" form. If $\int\sin^2 x+\cos^2 x\,dx$ a max. of 1 mark is awarded.

✓✓ correct primitive of at least one expression in $\sin^2 x$ or $\cos^2 x$ [or $\cos 2x$ or $\sin 2x$] [Allow FT only in $\int\sin^2 x$ or $\int\cos^2 x$ used if incorrect.

✓✓✓ correct solution

Alternatively:

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\left[\tfrac{1}{2}(1-\cos 2x)\right]+\tfrac{1}{2}(1+\cos 2x)\,dx$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1-\cos 2x+\tfrac{1}{2}+\tfrac{1}{2}\cos 2x\,dx$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \tfrac{3}{2}-\tfrac{1}{2}\cos 2x\,dx$$

$$=\tfrac{1}{2}\int_{\pi/4}^{\pi/2} 3-\cos 2x\,dx$$

(e)

$$\ddot{x}=\frac{1}{16\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{1}{16\sqrt{x}}$$

$$\frac{1}{2}v^2=\frac{1}{16}\int x^{-\frac{1}{2}}\,dx$$

$$v^2=\frac{1}{8}\int x^{-\frac{1}{2}}\,dx$$

$$v^2=\frac{1}{4}x^{\frac{1}{2}}+C$$

Sub $x=0, v=-4$

$C=\ +16$

$v^2=\frac{1}{4}x^{\frac{1}{2}}\ +16$

$\therefore (-4)^2=\frac{1}{4}(0)^{\frac{1}{2}}+C$

[$C=8$ if $\frac{1}{2}v^2=\frac{\sqrt{x}}{4}+c$ is used]

$C=16$

$\therefore\ v^2=\frac{\sqrt{x}}{4}+16$ ✓

Particle at rest when $v=0$

& $v=\pm\sqrt{\frac{\sqrt{x}}{4}+16}$

ie $0^2=\frac{1}{4}x^{\frac{1}{2}}+16$ ie. $v\ne 0$

No solution.

Hence particle is never at rest.

There are alternative responses e.g $v^2>0$, $x\ge 0$.

✓ correctly identifying/replacing $\ddot{x}$ with $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ or finding their C even if incorrect

✓✓ Correct integration (with/without c) in terms of x.

✓✓✓ correct solution (with reasoning). must use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Alternatively: Let $u = \ln x$ & by Quad · Formula
$\therefore \ln x = \dfrac{e^2(1+e) \pm \sqrt{[e^2(1+e)]^2 - 4e^5}}{2}$

Question 11 (continued)

$\therefore \quad x = e^{\left[\frac{e^2(1+e) \pm \sqrt{e^4(1+e)^2 - 4e^5}}{2}\right]}$

(f) **2**

$$(\log_e x)^2 - e^2(1+e)\log_e x + e^5 = 0$$

$$(\log_e x)^2 - (e^2 + e^3)\log_e x + (e^2 \times e^3) = 0$$

$$(\log_e x - e^2)(\log_e x - e^3) = 0$$

$$\log_e x = e^2, e^3$$

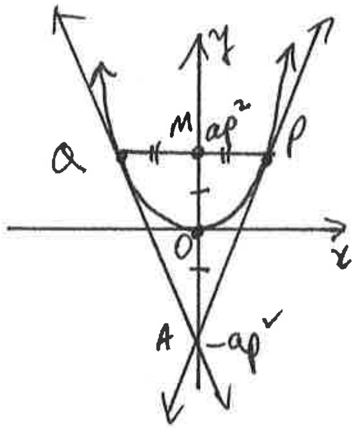
$\underline{\underline{x = e^{e^2}}}$ ... or $\underline{\underline{x = e^{e^3}}}$

✓ attempt to use logarithmic law(s) or correct subst into Quadratic formula.

✓✓ correct answers (with or without working) (exact form only).

Question 12

(a) (i) Tangent at $(2ap, ap^2)$: **3**



$(x_1, y_1) = (2ap, ap^2)$

$x^2 = 4ay$

$y = \dfrac{x^2}{4a}$

$y' = \dfrac{x}{2a}$

$m = \dfrac{2ap}{2a} = p$

$y - ap^2 = p(x - 2ap)$

$y = px - ap^2$

✓ Attempt to find (by valid method) equation of tangent at P or Q or $x = 0$ or $x = -ap^2$ or bald answer of $[0, -ap^2]$.

✓✓ equation of tangent at P and Q and recognising same y-intercept at $-ap^2$ (with reason). [Stating coordinates alone is insufficient]

Tangent at $(-2ap, ap^2)$:

$y - ap^2 = -p(x + 2ap)$

$y = -px - ap^2$

✓✓✓ correct solution and the correct coordinate must be stated.

Both tangents have the same y-intercept, $-ap^2$.
Point of intersection of tangents is $(0, -ap^2)$.

[There a various approaches to this question e.g using symmetric properties].

(ii) Midpoint $(x, y) = \left(\dfrac{2ap - 2ap}{2}, \dfrac{ap^2 + ap^2}{2}\right)$ **1**

$= (0, ap^2)$

✓ correct answer.

(iii) Midpoint $(x, y) = \left(\dfrac{0+0}{2}, \dfrac{ap^2 - ap^2}{2}\right)$ **1**

$= (0, 0)$

✓ obtaining correctly (0,0) and no FT since the question said O is the origin.

✓ S(1) is established

✓✓ S(k) and S(k+1) are correctly stated

✓✓✓ correct solution with all elements of induction must contain incorporation of assumption.

Question 12 (continued)

(b) Let $S(n) = (n+1)^2 + n - 1 \quad n \in \mathbb{Z}^+$ 3

<u>Prove S(1) is true</u>

$$S(1) = (1+1)^2 + 1 - 1$$
$$= 4 + 1 - 1$$
$$= 4$$

Since 4 is divisible by 2 the the statement is true for n=1.

<u>Assume S(k) is true</u> $1 \le k \le n$, n, k are positive integers

<u>Prove S(k+1) is true.</u>

$S(k)$: $(k+1)^2 + k - 1 = 2M$ (M is an integer) ①

$S(k+1)$: $(k+2)^2 + k = 2N$ (N is an integer) ②

in ② LHS $= (k+2)^2 + k$

$= (k+2)^2 + 2M + 1 - (k+1)^2$ from ①

$= k^2 + 4k + 4 + 2M + 1 - k^2 - 2k - 1$

$= 2k + 4 + 2M$

$= 2(k + 2 + M)$

$= 2N$

The statement S(n) is true for n=1. Whenever it is true for n=k, it is also true for n=k+1 and for all positive integers $n \geq 1$.

(C) (i) 1

$$P(x) = -4x^5 + 5x^4 - 0.7$$
$$P(0) = -0.7$$
$$P(1) = 0.3$$
$\therefore$ zero between x = 0 and x = 1

✓ must <u>evaluate</u> p(0) and p(1)

(ii) 3

$$P(x) = -4x^5 + 5x^4 - 0.7$$
$$\frac{dP(x)}{dx} = -20x^4 + 20x^3$$
$$0 = -20x^4 + 20x^3$$
$$0 = -20x^3(x-1)$$

✓ finding p'(x).

✓✓ indicating p'(x) is <u>undefined</u> at x=1.

$\therefore$ $x_n = x_{n-1} - \dfrac{p(x_{n-1})}{p'(x_{n-1})}$ will not be applicable.

$\therefore$ st ps at x = 0 and x = 1

Tangent at turning point will fail to cut the x axis

$\therefore$ Newton's method will fail

[Alternatively establishing equation of tangent at x=1 is horizontal and not x=0 $\therefore$ will not cut the x-axis].

Q12 Continued

Q12 (d) 4

Let $\alpha = \tan^{-1}\left(\dfrac{1}{u-1}\right)$ and $\beta = \tan^{-1}\left(\dfrac{1}{u+1}\right)$

$\tan\alpha = \dfrac{1}{u-1}$ and $\tan\beta = \dfrac{1}{u+1}$

then,

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \quad *$$

$$= \frac{\dfrac{1}{u-1}+\dfrac{1}{u+1}}{1-\dfrac{1}{u-1}\times\dfrac{1}{u-1}}$$

$$= \frac{\dfrac{2u}{u^2-1}}{1-\dfrac{1}{u^2-1}}$$

$$= \frac{\dfrac{2u}{u^2-1}}{\dfrac{u^2-2}{u^2-1}}$$

$$= \frac{2u}{u^2-2}$$

$$\alpha+\beta = \tan^{-1}\left(\frac{2u}{u^2-2}\right) \Rightarrow \tan^{-1}\left(\frac{1}{u-1}\right)+\tan^{-1}\left(\frac{1}{u+1}\right) = \tan^{-1}\left(\frac{2u}{u^2-2}\right)$$

$$\therefore \frac{2u}{u^2-2} = \frac{4}{7} \quad **$$

$14u = 4u^2 - 8$

$2u^2 - 7u - 4 = 0$

$(2u+1)(u-4) = 0 \quad ***$

$\therefore u = 4$ (given $u > 0$) ****

bald answer $u=4$

can also use a diagram to show $\tan(\tan^{-1}x)$.

✓ Attempts to use a correct identity to expand. If expression is incorrect and is quadratic then give max 1 mark.

✓✓ use of $\tan(\alpha+\beta)$ expansion correctly and giving an expression in terms of u only.

✓✓✓ correct solution but fails to give the correct answer. must do so from an equation

✓✓✓✓ correct solution with $u=4$ only concluded.

Question 13

(a) (i) $P = 50000e^{\frac{t}{2}\log_e 0.7}$ **1**

$24500 = 50000e^{\frac{t}{2}\log_e 0.7}$

$\frac{t}{2}\log_e 0.7 = \log_e 0.49$

$t = \frac{2\log_e 0.49}{\log_e 0.7}$

$t = 4$ h

(ii) $P = 500 + 12000e^{kt}$ **1**

$24500 = 500 + 12000e^{4k}$

$4k = \ln 2$

$k = 0.25\ln 2$

Q13 Continues over the page.

Question 13 (continued)

(a) (iii) When $t=0,\ P=50000e^{0}=50000$

$50000=500+12000e^{(0.25\ln 2)t}$

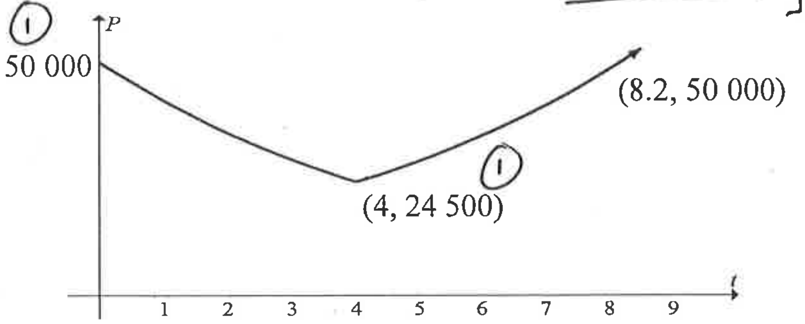
$(0.25\ln 2)t=\ln\left(\dfrac{50000-500}{12000}\right)$ (1)

$t=\dfrac{\ln 4.125}{0.25\ln 2}$

$=8.177576477$

$=8.2\text{ h (1dp)}$ (1) $\left[8.2\times 60\text{ min}=492\text{ min} \doteqdot 490\text{ min.}\right]$

(iv) (1)



(b) (i) At the endpoints: **2**

$v^2=20+8x-x^2=0$

$x^2-8x-20=0$

$(x-10)(x+2)=0$

$x=-2,10$

(ii) centre of motion $=\dfrac{-2+10}{2}=4$ **1**

(b) (iii) amplitude $=a=6$

period $=2\pi$

$n=1$

When $t=0,\ x=4$

$\therefore\ x=6.\sin(t+\alpha)+4$

$4=6.\sin(0+\alpha)+4$

$\sin\alpha=0$

$\alpha=0$

$\boxed{x=6\sin t+4}$ (3)

(b)(ii) (Alternative solution)

$\frac{1}{2}v^2=10+4x-\frac{1}{2}x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{d}{dx}\left(10+4x-\frac{1}{2}x^2\right)$

$\ddot{x}=4-x$

$\ddot{x}=-x+4$

$\ddot{x}=-1.(x-4)$

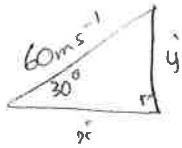
$\ddot{x}=-n^2(x-b)$

$\therefore$ centre of motion $=4$. (1)

Question 13 (continued)

c) (i)	$(1-x)^{2n} = 1 - {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \ldots + (-1)^r\, {}^{2n}C_r x^r \ldots + {}^{2n}C_{2n} x^{2n}$	1 Mark: Correct answer.
c) (ii)	Differentiating both sides. $-2n(1-x)^{2n-1} = -{}^{2n}C_1 + 2\,{}^{2n}C_2 x + .. + (-1)^r r\,{}^{2n}C_r x^{r-1} .. + 2n\,{}^{2n}C_{2n} x^{2n-1}$ Substitute $x=1$ $0 = -{}^{2n}C_1 + 2\,{}^{2n}C_2 + \ldots + (-1)^r r\,{}^{2n}C_r \ldots + 2n\,{}^{2n}C_{2n}$ $\therefore\ {}^{2n}C_1 + 3\,{}^{2n}C_3 + \ldots + (2n-1)\,{}^{2n}C_{2n-1} = 2\,{}^{2n}C_2 + 4\,{}^{2n}C_4 + \ldots + 2n\,{}^{2n}C_{2n}$	2 Marks: Correct answer. 1 Mark: Differentiates both sides.

Q14 Equations of Motion



a) $\boxed{\ddot{x} = 0}$

$\dot{x} = c$ at $t=0$ $\dot{x} = V\cos\theta$
$= 60\cos 30$
$= 30\sqrt{3}$

$\therefore \boxed{\dot{x} = 30\sqrt{3}}$

$x = 30\sqrt{3}t + C_1$ at $t=0$ $x=0$
$\therefore C_1 = 0$

$\boxed{x = 30\sqrt{3}t}$

$t = \dfrac{x}{30\sqrt{3}}$

$\boxed{\ddot{y} = -9.8}$

$\dot{y} = -9.8t + C_2$ at $t=0$
$\dot{y} = V\sin\theta$
$= \cancel{8}0 \times \sin 30$

$\therefore C_2 = 30$

$\boxed{\dot{y} = -9.8t + 30}$

$y = \dfrac{-9.8t^2}{2} + 30t + C_3$

at $t=0$ $y=1.5$ $\therefore C_3 = 1.5$

$\boxed{y = -4.9t^2 + 30t + 1.5}$

---

a) (i) A $x = 300m$

$t = \dfrac{300}{30\sqrt{3}}$ ✓ 1

$t = \dfrac{10}{\sqrt{3}}$

y is the height above ground.

$y = -4.9\left(\dfrac{10}{\sqrt{3}}\right)^2 + 30\left(\dfrac{10}{\sqrt{3}}\right) + 1.5$

$y = 11.37m$ ✓

$\therefore$ The arrow will be at 11.37m above the ground when $t = \dfrac{10}{\sqrt{3}}$ sec and $x = 300m$. This is 8m above the top of the tree at that point in time and so will not hit the tree. 1

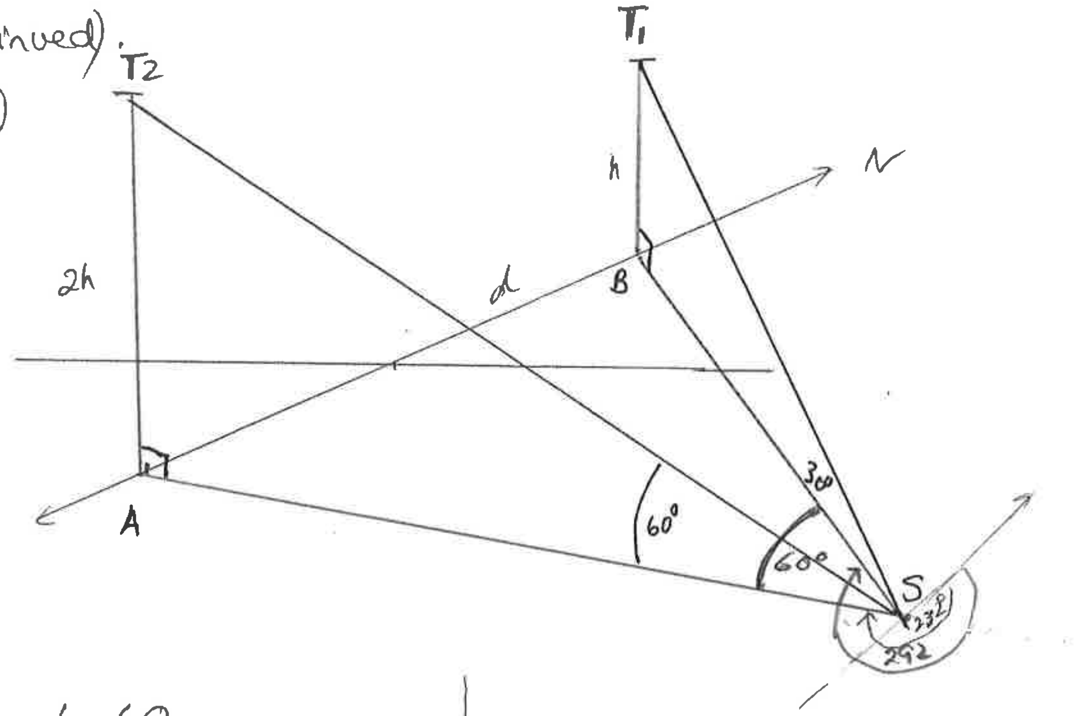
Q14 Continued,

Q14 a) (ii)

(ii) Using the derivations above, but with $\theta°$ instead of $30°$. Horizontal displacement is given by $x = 60t\cos\theta$ Vertical displacement is given by $y = -4.9t^2 + 60t\sin\theta + 1.5$	1	Modifying equation motions correctly
When horizontal displacement is 300 metres, we want vertical displacement to be + 1 metre. $300 = 60t\cos\theta$ $t = \frac{300}{60\cos\theta}$ $= \frac{5}{\cos\theta}$ When $t = \frac{5}{\cos\theta}$ vertical displacement is: $y = -4.9\left(\frac{25}{\cos^2\theta}\right) + 60\left(\frac{5}{\cos\theta}\right)\sin\theta + 1.5$ $= -122.5\sec^2\theta + 300\tan\theta + 1.5$ $= -122.5(\tan^2\theta + 1) + 300\tan\theta + 1.5$ $= -122.5\tan^2\theta - 122.5 + 300\tan\theta + 1.5$ $= -122.5\tan^2\theta + 300\tan\theta - 121$	1	Correct equation for vertical motion in terms of $\tan\theta$.
We want $y = 1$, so $1 = -122.5\tan^2\theta + 300\tan\theta - 121$ $0 = -122.5\tan^2\theta + 300\tan\theta - 122$ Using quadratic formula: $\tan\theta = \frac{-300 \pm \sqrt{(-300)^2 - 4 \times -122.5 \times -122}}{-245}$ $= 0.5149428327$ or 1.934036759 $\theta = 27°15'$ or $62°40'$ Therefore, the arrow must be fired at an angle of 27°15' or 62° 40' in order to hit the mark on the tree.	1	Correct angles. or one correct angle.

(14. continued)

Q14 b)



Tower 2

In ΔT_2AS

$$\frac{2h}{AS} = \tan 60$$

$$AS = \frac{2h}{\tan 60}$$

$$AS = \frac{2h}{\sqrt{3}}$$ ✓ (1)

Tower 1

In ΔT_1BS

$$\frac{h}{BS} = \tan 30$$

$$BS = \frac{h}{\tan 30}$$

$$= \frac{h}{1/\sqrt{3}}$$

$$BS = \sqrt{3}h$$ ✓ (1)

Using cosine rule

$$d^2 = AS^2 + BS^2 - 2 \times AS \times BS \times \cos 60°$$

$$= \frac{4h^2}{3} + 3h^2 - 2 \times \frac{2h}{\sqrt{3}} \times \sqrt{3}h \times \frac{1}{2}$$ ✓ (1)

$$= \frac{4h^2}{3} + 3h^2 - 2h^2$$

$$= \frac{4h^2}{3} + h^2$$

$$= \frac{7h^2}{3}$$ ✓ (1)

$$\therefore d = \sqrt{\frac{7h^2}{3}}$$

$$= \frac{\sqrt{7}h}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{21}\,h}{3}$$ as required.

Question 14 (continued)

(c) (i) $\angle OBA = OCA = 90°$ (tangent perpendicular to radius) **1**

$ABOC$ is cyclic (opposite angles are supplementary)

$\therefore \angle BAC + \angle BOC = 180°$ (opposite angles of a cyclic quadrilateral)

(ii) $\angle BOC = 180° - \angle BAC$ (supplementary angles) **2**

$\angle BDC = \frac{1}{2}\angle BOC$ (angle at centre twice angle at circumference)

$$= \frac{1}{2}(180° - \angle BAC)$$

$$= 90° - \frac{1}{2}\angle BAC$$

(iii) $a = \cos\angle BAC$ **3**

$\therefore a = 2\cos^2\left(\frac{1}{2}\angle BAC\right) - 1$ $\left(\cos\theta = 2\cos^2\frac{\theta}{2} - 1\right)$

$$\cos^2\left(\frac{1}{2}\angle BAC\right) = \frac{a+1}{2}$$

From (ii), $\frac{1}{2}\angle BAC = 90° - \angle BDC$

$$\cos^2(90° - \angle BDC) = \frac{a+1}{2}$$

$\sin^2(\angle BDC) = \frac{a+1}{2}$ $(\sin\theta = \cos(90° - \theta))$

$1 - \cos^2(\angle BDC) = \frac{a+1}{2}$ $(\sin^2\theta + \cos^2\theta = 1)$

$$\cos^2(\angle BDC) = 1 - \frac{a+1}{2} = \frac{1-a}{2}$$

$$\cos(\angle BDC) = \pm\sqrt{\frac{1-a}{2}}$$

$\therefore \cos(\angle BDC) = \sqrt{\frac{1-a}{2}}$ $(0° < \angle BDC < 90,\ \cos > 0 \text{ in the 1st quad})$

Q14 (ii)

Alternative Solution

$a = \cos \angle BAC$

$\cos^{-1}(a) = \angle BAC$

Now $\angle BDC = 90^\circ - \frac{1}{2}\angle BAC$ from (ii)

$2\angle BDC = 180^\circ - \angle BAC$

$2\angle BDC = 180^\circ - \cos^{-1}(a)$

$\cos^{-1}(a) = 180^\circ - 2\angle BDC$

$a = \cos(180 - 2\angle BDC)$ (1)

$= -\cos(2\angle BDC)$ (Now $\cos 2\theta = 2\cos^2\theta - 1$)

$-a = \cos(2\angle BDC)$

$= 2\cos^2(\angle BDC) - 1$ (1)

$1 - a = 2\cos^2(\angle BDC)$

$\therefore \cos^2 \angle BDC = \frac{1-a}{2}$

$\cos \angle BDC = \pm\sqrt{\frac{1-a}{2}}$ but $0 < (\angle BDC) < 90^\circ$ (1)

So $\cos \angle BDC = +\sqrt{\frac{1-a}{2}}$ as required.