# TRINITY GRAMMAR SCHOOL

## Mathematics Department

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**(NESA Student Number | Year 12 only)**

# 2022

# Year 12

# Mathematics Extension 1

## TRIAL EXAMINATION

## HSC ASSESSMENT TASK 4

**Date of Assessment Task:** Friday, 26 August 2022

**General Instructions**

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Erasable pens and all electronic devices (or timekeepers), unless otherwise advised, are not permitted in this Assessment
- NESA approved calculators may be used
- A formula and data sheet is provided
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations
- Write your NESA Student Number and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating N/A and your NESA Student Number or Name

**Total marks: 70**

**Section I – 10 marks** (pages 2 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

**Section II – 60 marks** (pages 8 – 15)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

- **HSC Assessment Weighting:** 30%

**Section I**

**10 marks**
**Attempt Questions 1 – 10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1 – 10.

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**1** Vectors **a**, **b** and **c** are shown below.

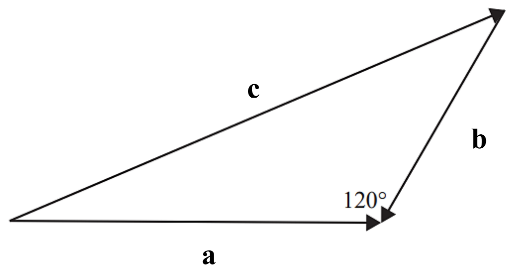



*diagram not to scale*

Which statement is correct?

A. $|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$

B. $|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{ab}|$

C. $|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}||\mathbf{b}|$

D. $|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{a}||\mathbf{b}|$

**2** What is the gradient of the tangent at $(7, 2)$ on the **inverse** function of $f(x) = x^3 - 1$ ?

A. $\dfrac{1}{147}$

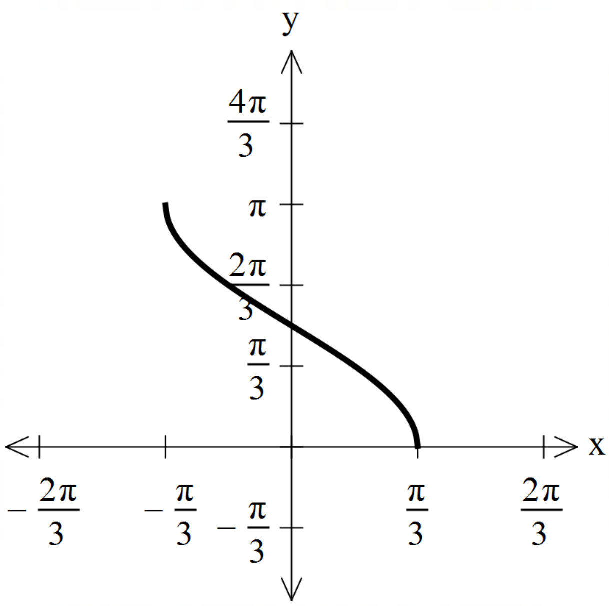
B. $\dfrac{1}{12}$

C. $12$

D. $147$

**3** The equation of the graph drawn below is in the form $y = A\cos^{-1}(Bx)$ where $A, B \in \mathbb{R}$ .

Which of the following represents the equation of the graph drawn below?




A. $y = \cos^{-1}\left(\frac{3x}{\pi}\right)$

B. $y = \cos^{-1}\left(\frac{\pi x}{3}\right)$

C. $y = \frac{\pi}{3}\cos^{-1}(x)$

D. $y = \frac{3}{\pi}\cos^{-1}(x)$

**4** Consider the vectors $\boldsymbol{a} = x\boldsymbol{i} + \boldsymbol{j}$ , $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$ , $\boldsymbol{c} = \boldsymbol{i} + x\boldsymbol{j}$ where $x$ is a constant.

$\theta$ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\varphi$ is the angle between vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ .

What is the value of $\cos\theta\cos\varphi$ ?

A. $-\frac{(x+1)^2}{2(1+x^2)}$

B. $-\frac{(x-1)^2}{2(1+x^2)}$

C. $\frac{\sqrt{2}(1-x^2)}{(1+x^2)}$

D. $\frac{\sqrt{2}(1+x^2)}{(1-x^2)}$

**5** A box contains 28 red balls, 17 yellow balls and 21 green balls.

What is the smallest number $n$ (where $n$ is a positive integer) such that any $n$ balls drawn from the box will contain at least 15 balls of the same colour?

A. 43

B. 45

C. 46

D. 66

**6**

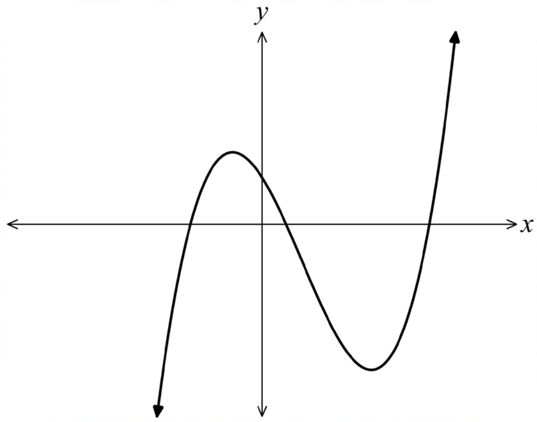
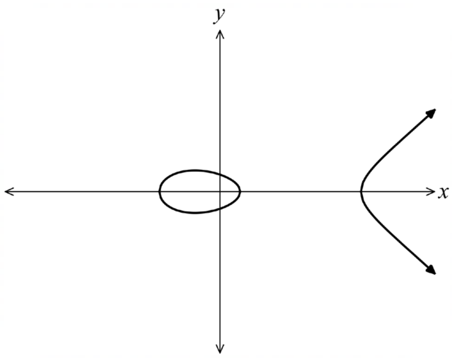



*diagram not to scale*

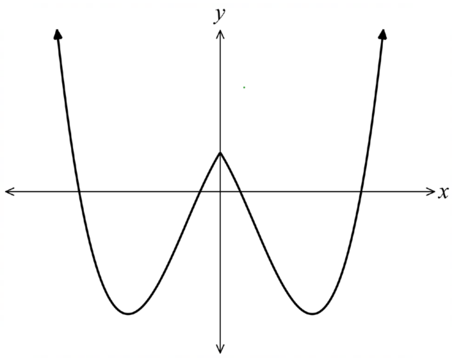
The graph of $y = f(x)$ is shown above.

Which of the following is a possible graph for $y^2 = f(x)$ ?

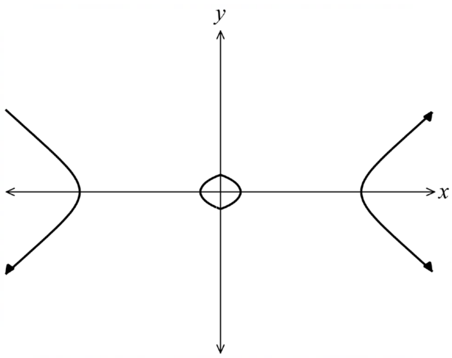
A.



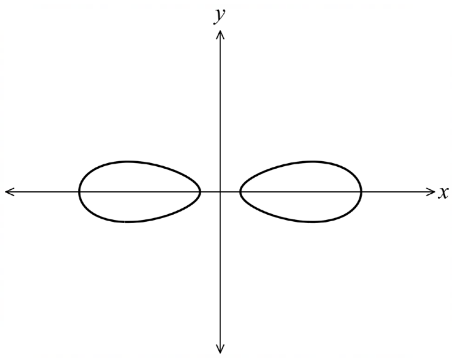

B.




C.




D.



**7** When $x^4 + 2x^3 - 5x^2 + x + 3$ is divided by $x^2 + x - 2$ , the remainder is in the form $Ax + B$ . What is the sum of $A$ and $B$ ?

A. $-5$

B. $-3$

C. 1

D. 2

**8** Which of the following represents the volume of the solid of revolution formed by rotating the graph of $y = \sqrt{9 - (x - 1)^2}$ about the $x$-axis?

A. $4\pi(3)^2$

B. $\pi\int_{-2}^{4}\left(\sqrt{9 - (x - 1)^2}\right)dx$

C. $\pi\int_{-2}^{4}(9 - (x - 1)^2)^2\,dx$

D. $\pi\int_{-4}^{2}(9 - (x - 1)^2)\,dx$

**9** A curve is defined parametrically by $x = \sec(t) + 1$ and $y = \tan(t)$ where $t \in [0, \frac{\pi}{2})$ .
Which of the following statements is correct for the Cartesian equation **and** domain of the curve?

A. $y = \sqrt{x^2 - 2x}\,,\ x \in [2,\ \infty)$

B. $y = \sqrt{x^2 - 2x}\,,\ x \in (-\infty, 0] \cup [2,\ \infty)$

C. $y^2 = x^2 - 2x\,,\ x \in [2,\ \infty)$

D. $y^2 = x^2 - 2x\,,\ x \in (-\infty, 0] \cup [2,\ \infty)$

**10** Sand falls from a chute to form a pile in the shape of a right circular cone of radius $r$, with a semi-vertex angle 60° as shown below.
Sand is added to the pile at a rate of 1.5 $m^3$ per minute.

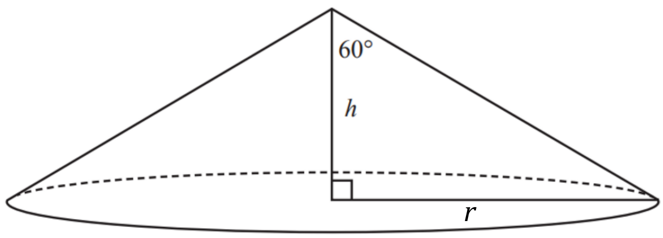



*diagram not to scale*

When the height of the pile, $h$, is 0.5 metres, what is the approximate instantaneous rate of change in height, in metres per minute?

A. 0.21

B. 0.31

C. 0.64

D. 3.82

**End of Section I**

**Section II**

**60 marks**
**Attempt Questions 11 – 14**
**Allow about 1 hours and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a SEPARATE writing booklet.

(a) The polynomial $P(x) = 4x^5 + 9x + 2$ is divided by $(x + 3)$ . Find the remainder. **1**

(b) Solve the inequality $\dfrac{2}{2x+3} \leq 1$ . **3**

(c) Find $\displaystyle\int \frac{1}{\sqrt{9 - 4x^2}}\, dx$ . **2**

(d) The polynomial $P(x) = x^4 + ax^3 - 3x^2 + bx - 2$ has a triple zero at $x = -1$ . **3**
Find the values of $a$ and $b$.

**Question 11 continues on page 9**

**Question 11 (continued)**

(e) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2(2\theta)\, d\theta$ . **2**

(f) Ten people sit around a round table. How many arrangements are possible if three particular people want to sit together? **1**

(g) Find the term independent of $x$ in the expansion of $\left(2x + \frac{1}{x^2}\right)^6$ . **3**

**End of Question 11**

**Question 12** (16 marks) Use a SEPARATE writing booklet.

(a) A household iron is cooling in a room of constant temperature of 23°C. The initial temperature of the iron is 90°C and it cools to 70°C after 10 minutes.
You may assume that the rate of cooling can be expressed by the differential equation

$$\frac{dT}{dt} = -k(T - 23) \text{ where } k \text{ is a positive constant.}$$

(i) Show, by differentiation, that the temperature $T$ can be expressed in the form **1**

$$T = 23 + Ae^{-kt} \text{ where } A \text{ is a constant.}$$

(ii) Find the **exact** values of $A$ and $k$ . **2**

(iii) How long will it take for the iron to cool down to 40°C ? Give your answer to the nearest minute. **2**

(b) (i) Differentiate $\tan^{-1}\left(\frac{1}{x}\right)$ , $x \neq 0$ , **and** hence show that **2**

$$\frac{d}{dx}\left(\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right) = 0 \ .$$

(ii) Hence explain why $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$ for $x > 0$ . **2**

**Question 12 continues on page 11**

**Question 12 (continued)**

(c) Functions $f$, $g$ and $h$ are defined such that

$$f(x) = \frac{1}{x-1} \quad , \quad g(x) = x^2 \quad , \quad h(x) = \sqrt{x} \quad .$$

(i) State the equation of $f(h(x))$. **1**

(ii) Determine the domain **and** range for $f(h(x))$ . **4**

(iii) Is it true that $f\left(h\big(g(x)\big)\right) = f(x)$ ? Justify your answer. **2**

**End of Question 12**

**Question 13** (14 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that for all integers $n \geq 1$, **3**

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2} \quad .$$

(b) Find the **exact** volume of the solid of revolution formed when the graph of **2**

$x = \sqrt{\dfrac{1+2y}{1+y^2}}$ is rotated $360°$ about the $y$-axis for $y \in [0, 1]$ .

(c) Four runners have a race. All the runners complete the race. **3**

Determine the total number of ways they could finish the race, including the cases where two or more runners finish at exactly the same time.

**Question 13 continues on page 13**

**Question 13 (continued)**

(d)

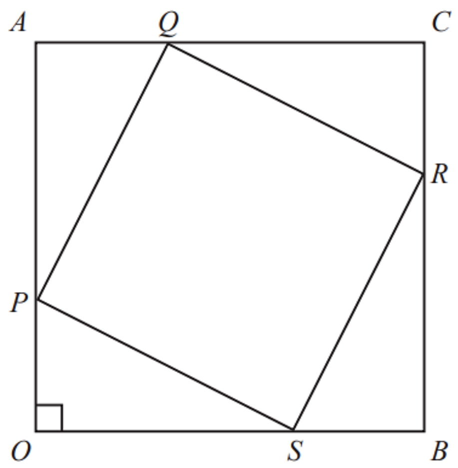



*diagram not to scale*

The diagram above shows the square $OACB$ where point $O$ is the origin. Let the position vector for points $A$, $B$ be defined as $\boldsymbol{a}$, $\boldsymbol{b}$ respectively, i.e. $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$ .

Let points $P$, $Q$, $R$ and $S$ be defined so that $\overrightarrow{OP} = \overrightarrow{RC} = k\boldsymbol{a}$ and $\overrightarrow{AQ} = \overrightarrow{SB} = k\boldsymbol{b}$ where $0 \leq k \leq 1$ . This means that points $P$, $Q$, $R$ and $S$ are positioned along their respective sides in equal proportion.

(i) Explain why $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ . **1**

(ii) Using vectors, prove that $\angle PQR = 90°$ . **2**

Now suppose that in square $OACB$, it is known that $OA = 10$ cm and that point $P$ is moving away from the origin at a constant speed of 0.2 cm per second.
This means that point $P$, $Q$, $R$ and $S$ are moving at the same speed along their respective sides.

Let $x$ = the distance $OP$.

(iii) Determine the rate, in $\text{cm}^2$ per second, at which the area of square $PQRS$ is changing when $x = 3$ cm. **3**

**End of Question 13**

**Question 14** (15 marks) Use a SEPARATE writing booklet.

(a) Solve $\sin x - 3\cos x = 3$ for $0° \leq x \leq 360°$ . **3**

(b) An object is fired from a point $O$ with initial speed of $V$ ms$^{-1}$ at an angle of elevation $\theta$ to the horizontal plane.

The position vector $\boldsymbol{s}(t)$ of at any time, $t$ seconds, for $t \geq 0$, is

$$\boldsymbol{s}(t) = (Vt\cos\theta)\,\boldsymbol{i} + \left(Vt\sin\theta - \frac{1}{2}gt^2\right)\boldsymbol{j}\ ,$$

where $g$ ms$^{-2}$ is the acceleration due to gravity, $\boldsymbol{i}$ is a unit vector in the horizontal direction of motion and $\boldsymbol{j}$ is a unit vector vertically up.

The object falls to a point $P$ below the level of $O$ such that $PM = OM$.

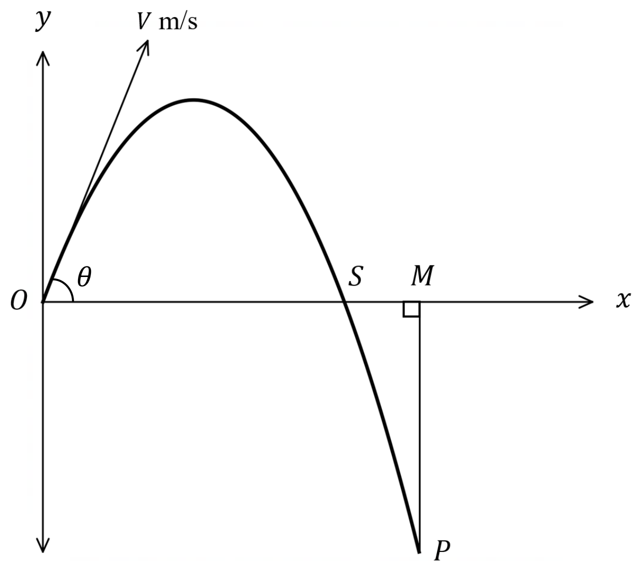



*diagram not to scale*

(i) Show that the distance $OM$ is $\dfrac{V^2}{g}(\sin 2\theta + \cos 2\theta + 1)$ metres. **3**

(ii) Given that $\dfrac{OM}{OS} = \dfrac{4}{3}$, prove that $\sin 2\theta - 3\cos 2\theta = 3$ . **3**

(iii) Find the value of $\theta$ . **2**

**Question 14 continues on page 15**

**Question 14 (continued)**

(c) The exact value of ${}^{100}\mathbf{C}_3 + {}^{100}\mathbf{C}_5 + \ldots + {}^{100}\mathbf{C}_{99}$ can be written in the form $a^b + c$ where $a,\ b,\ c\ \in\ \mathbb{Z}$. **4**

Find the values of $a, b$ and $c$ .

**End of Question 14**

**End of Task**

# 2022 HSC 12MAX Task 4 MC Solutions

1. $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos 120°$

$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{a}||\underset{\sim}{b}|$

$\therefore$ D

2. $y = x^3 - 1$

inverse function : $x = y^3 - 1$

$$\frac{dx}{dy} = 3y^2$$

$$\frac{dy}{dx} = \frac{1}{3y^2}$$

at $(7, 2)$, $\frac{dy}{dx} = \frac{1}{3 \times 2^2} = \frac{1}{12}$

$\therefore$ B

3. domain $[-1, 1] \rightarrow [-\frac{\pi}{3}, \frac{\pi}{3}]$

$\cos^{-1}(x) \rightarrow \cos^{-1}\left(\frac{3x}{\pi}\right)$

$\therefore$ A

4. $\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$= \frac{x - 1}{\sqrt{x^2+1} \times \sqrt{1+1}}$

$= \frac{x-1}{\sqrt{2}\sqrt{x^2+1}}$

$\cos\varphi = \frac{\underset{\sim}{b} \cdot \underset{\sim}{c}}{|\underset{\sim}{b}||\underset{\sim}{c}|}$

$= \frac{1 - x}{\sqrt{1+1} \times \sqrt{1+x^2}}$

$= \frac{1-x}{\sqrt{2}\sqrt{x^2+1}}$

$\cos\theta\cos\varphi = -\frac{(x-1)^2}{2(x^2+1)}$

$\therefore$ B

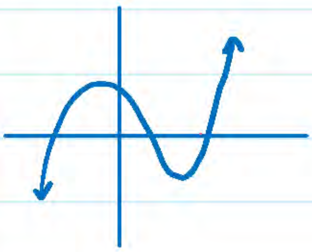
5. red    blue    green

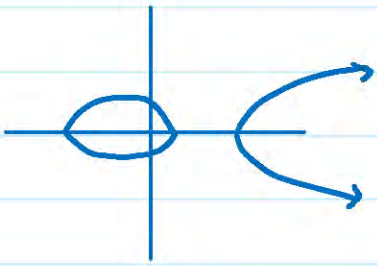
$14 + 14 + 14 = 42$

at least 15 : $42 + 1 = 43$

$\therefore$ A

6. $y = f(x)$      $y^2 = f(x)$





$\therefore$ A

7. Method 1

$$\begin{array}{r} x^2 + x - 4 \\ x^2 + x - 2 \,\big)\, x^4 + 2x^3 - 5x^2 + x + 3 \\ x^4 + x^3 - 2x^2 \\ \hline x^3 - 3x^2 + x \\ x^3 + x^2 - 2x \\ \hline -4x^2 + 3x + 3 \\ -4x^2 - 4x + 8 \\ \hline \end{array}$$

Method 2

$P(x) = x^4 + 2x^3 - 5x^2 + x + 3$

$P(x) = (x^2 + x - 2)\,Q(x) + (Ax + B)$

$P(x) = (x+2)(x-1)\,Q(x) + (Ax + B)$

$P(1) = A + B$

$A + B = 1 + 2 - 5 + 1 + 3$

$$\begin{array}{r} -4x \quad + 3x + 3 \\ -4x^2 - 4x + 8 \\ \hline 7x \quad - 5 \end{array}$$

$P(1) = A + B$

$A + B = 1 + 2 - 5 + 1 + 3$

$= 2$

$Ax + B = 7x - 5 \Rightarrow A + B = 2$

$\therefore$ D

8. $y = \sqrt{9 - (x-1)^2}$ semicircle with centre $(1, 0)$ and radius 3

↳ domain : $[-2, 4]$

$V = \int_{-2}^{4} 9 - (x-1)^2 \, dx$ or $V = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi \times 3^3$

$= 4\pi \times 3^2$

$\therefore$ A

9. $x = \sec t + 1$ $\sec^2 t = (x-1)^2$

$y = \tan t$ $\tan^2 t = y^2$

$1 + \tan^2 t = \sec^2 t$ since $t \in [0, \frac{\pi}{2})$

$1 + y^2 = (x-1)^2$ $x \in [2, \infty)$

$y^2 = x^2 - 2x$ $y \in [0, \infty)$

$\therefore$ A

10. $r = h \tan 60°$ $\frac{dV}{dh} = 9\pi h^2$

$= \sqrt{3} h$

$$\frac{dV}{dh} = 9\pi h^2$$

$= \sqrt{3}\ h$

$V = \pi r^2 h$

$= \pi (\sqrt{3}\ h)^2\ h$

$= 3\pi h^3$

$$\frac{dV}{dt} = 1.5\ m^3/min$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{9\pi \times 0.5^2} \times 1.5$$

$$= 0.212206 \ldots$$

$\therefore$ A

2022 HSC 12MAX Task 4 Question 11 Solutions

## Question 11

(a) $P(-3) = 4 \times (-3)^5 + 9 \times (-3) + 2$

$= -997$ 1

(b) $\dfrac{2}{2x+3} \le 1 \qquad x \ne -\frac{3}{2}$

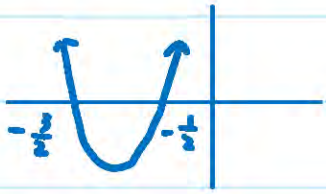
$\times (2x+3)^2 \quad \times (2x+3)^2$

$2(2x+3) \le (2x+3)^2$ 1

$(2x+3)^2 - 2(2x+3) \ge 0$

$(2x+3)(2x+3-2) \ge 0$

$(2x+3)(2x+1) \ge 0$




$\therefore \quad x < -\frac{3}{2}, \quad x \ge -\frac{1}{2}$

1 1

(c) $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$

$\displaystyle = \frac{1}{2}\int \frac{2}{\sqrt{9-4x^2}}\,dx$

$\displaystyle = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$

1 1

(d) 2 is the other root since $(-1)^3 \times 2 = -2$

$a = $ [illegible] 1

(d) 2 is the other root since $(-1)^3 \times 2 = -2$

$-\frac{a}{1} = -1 + -1 + -1 + 2$ 1

$-a = -1$

$a = 1$

$P(x) = x^4 + x^3 - 3x^2 + bx - 2$

$P(-1) = 1 - 1 - 3 - b - 2 = 0$

$b = -5$

$\therefore \; a = 1$ 1 $\quad b = -5$ 1

(e) $\int_0^{\frac{\pi}{2}} \cos^2(2\theta)\, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 4\theta)\, d\theta$ 1

$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{1}{4}\sin 2\pi\right)$

$= \frac{\pi}{4}$ 1

(f) $7! \times 3! = 30240$ 1

(g) ${}^6C_r (2x)^{6-r} (x^{-2})^r = {}^6C_r \times 2^{6-r} \times x^{6-r} \times x^{-2r}$ 1

$= {}^6C_r \times 2^{6-r} \times x^{6-3r}$

$$= {}^{6}C_{r} \times 2^{6-r} \times x^{6-3r}$$

$6 - 3r = 0$ $\quad\therefore$ term $= {}^{6}C_{2} \times 2^{6-2}$

$r = 2$ 1 $\qquad = 240$ 1

## Question 12

(a) (i) $T = 23 + Ae^{-kt}$

$$\frac{dT}{dt} = -k(Ae^{-kt}) \quad 1$$

$$= -k(T - 23)$$

(ii) $t = 0, \quad T = 90$

$t = 10, \quad T = 70$

$$\Rightarrow \begin{cases} 90 = 23 + Ae^{-k\times 0} \\ 70 = 23 + Ae^{-k\times 10} \end{cases}$$

$$23 + A = 90$$

$$A = 67 \quad 1$$

$$67e^{-10k} = 47$$

$$e^{-10k} = \frac{47}{67}$$

$$10k = \ln\frac{67}{47}$$

$$k = \frac{1}{10}\ln\frac{67}{47} \quad 1$$

(iii) $T = 23 + 67e^{-\frac{1}{10}\ln\frac{67}{47}t}$

$$40 = 23 + 67e^{\frac{1}{10}\ln\frac{47}{67}t} \quad 1$$

$$e^{\frac{1}{10}\ln\frac{47}{67}t} = \frac{17}{67}$$

$$\frac{1}{10}\ln\frac{47}{67}t = \ln\frac{17}{67}$$

$$t = 10\ln\frac{17}{67} \div \ln\frac{47}{67}$$

$$t \approx 39 \text{ mins} \quad 1$$

(b) (i) $LHS = \dfrac{1}{1+x^2} + \dfrac{-x^{-2}}{1+(x^{-1})^2}$ 1

$= \dfrac{1}{1+x^2} + \dfrac{-x^{-2}}{1+x^{-2}}$

$= \dfrac{1}{1+x^2} + \dfrac{-1}{x^2+1}$ 1

$= 0$

$= RHS$

(ii) let $f(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$

$f'(x) = 0$ (from part (a))

↳ $f(x)$ is a constant 1

$x = 1$, $f(1) = \tan^{-1} 1 + \tan^{-1} 1$

$= \frac{\pi}{4} + \frac{\pi}{4}$ 1

$= \frac{\pi}{2}$

$\therefore f(x) = \frac{\pi}{2}$ for $x > 0$

(c) (i) $f(h(x)) = f(\sqrt{x})$

$= \dfrac{1}{\sqrt{x} - 1}$ 1

(ii) domain : $x \geq 0$ and $x \neq 1$

$\therefore x \in [0, 1) \cup (1, \infty)$
1 1

range : $\sqrt{x} \geq 0$

$\sqrt{x} - 1 \geq -1$

range : $\sqrt{x} \geq 0$

$\sqrt{x} - 1 \geq -1$

$\swarrow \qquad \searrow$

$-1 \leq \sqrt{x} - 1 < 0 \qquad \sqrt{x} - 1 > 0$

$\frac{1}{\sqrt{x} - 1} \leq -1 \qquad \frac{1}{\sqrt{x} - 1} > 0$

1 1

$\therefore f(h(x)) \in (-\infty, -1] \cup (0, \infty)$

(iii) $h(g(x)) = \sqrt{x^2}$

1

$= |x|$

1

$f(h(g(x))) = f(|x|) \neq f(x)$ since $f(x)$ is not even

$\therefore$ the statement is false

## Question 13

(a) ① Prove true for $n = 1$.

$$LHS = 1 - \frac{1}{(1+1)^2} = \frac{3}{4} \qquad RHS = \frac{1+2}{2\times1+2} = \frac{3}{4}$$

$LHS = RHS \Rightarrow$ the statement is true for $n=1$

② Assume true for $n=k$, $k \in \mathbb{Z}^+$

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$$

③ Prove true for $n=k+1$, that is

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right) = \frac{k+3}{2k+4}$$

$$LHS = \frac{k+2}{2k+2} \times \left(1-\frac{1}{(k+2)^2}\right) \quad \text{from step 2}$$

$$= \frac{k+2}{2k+2} \times \frac{k^2+4k+3}{(k+2)^2}$$

$$= \frac{k+2}{2(k+1)} \times \frac{(k+1)(k+3)}{(k+2)^2}$$

$$= \frac{k+3}{2(k+2)}$$

$$= \frac{k+3}{2k+4}$$

$= RHS$

$\therefore$ if the statement is true for $n=k$, then it is true for $n=k+1$

④ The statement is true for $n=1$. Whenever (or if) the statement is true for $n=k$, it is also true

(a) The statement is true for $n=1$. Whenever (or if) the statement is true for $n=k$, it is also true for $n=k+1$. Hence, by principle of mathematical induction, the statement is true for positive integers $n \geq 1$.

(b) $V = \pi \int_0^1 \frac{1+2y}{1+y^2} \, dy$

$= \pi \int_0^1 \frac{1}{1+y^2} + \frac{2y}{1+y^2} \, dy$

$= \pi \left[ \tan^{-1} y + \ln |1+y^2| \right]_0^1$

$= \pi \left( \tan^{-1} 1 + \ln 2 \right)$

$= \frac{\pi^2}{4} + \pi \ln 2$ units$^3$

(c) four different : $4!$

three different : ${}^4C_2 \times 3!$

two different : ${}^4C_2$

one different : ${}^4C_3 \times 2!$

all same : $1$

total $= 4! + {}^4C_2 \times 3! + {}^4C_2 + {}^4C_3 \times 2! + 1$

$= 75$

(d) (i) $\vec{OA} \perp \vec{OB}$ therefore $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

(ii) $\vec{PQ} = \vec{PO} + \vec{OA} + \vec{AQ}$

$= -k\underset{\sim}{a} + \underset{\sim}{a} + k\underset{\sim}{b}$

$= (1-k)\underset{\sim}{a} + k\underset{\sim}{b}$

$\vec{QR} = \vec{QA} + \vec{AC} + \vec{CR}$

$= -k\underset{\sim}{b} + \underset{\sim}{b} + -k\underset{\sim}{a}$

$= (1-k)\underset{\sim}{b} - k\underset{\sim}{a}$

$\vec{PQ} \cdot \vec{QR} = ((1-k)\underset{\sim}{a} + k\underset{\sim}{b}) \cdot ((1-k)\underset{\sim}{b} - k\underset{\sim}{a})$

$= (1-k)^2 \underset{\sim}{a} \cdot \underset{\sim}{b} - k(1-k)|\underset{\sim}{a}|^2 + k(1-k)|\underset{\sim}{b}|^2 - k^2 \underset{\sim}{a} \cdot \underset{\sim}{b}$

since ABCD is a square,

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ and $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

$\vec{PQ} \cdot \vec{QR} = -k(1-k)|\underset{\sim}{a}|^2 + k(1-k)|\underset{\sim}{b}|^2$

$= -k(1-k)|\underset{\sim}{a}|^2 + k(1-k)|\underset{\sim}{a}|^2$

$= 0$

$\therefore \angle PQR = 90°$

(iii) $x = OP$

$\frac{dx}{dt} = 0.2$ cm/s

$A_{PQRS} = 10^2 - 4 \times \frac{1}{2} \times x(10-x)$

$= 100 - 2x(10-x)$

$= 2x^2 - 20x + 100$

$\frac{dA}{dx} = 4x - 20$

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$= (4x - 20) \times 0.2$

$= (4 \times 3 - 20) \times 0.2$

$= -1.6 \ cm^3 / s$

2022 HSC 12MAX Task 4 Question 14 Solutions

## Question 14

(a) $\sin x - 3\cos x = R\sin(x-\alpha)$

$$\begin{cases} R\cos\alpha = 1 \\ R\sin\alpha = 3 \end{cases} \Rightarrow \begin{cases} R = \sqrt{10} \\ \alpha = \tan^{-1}(3) \end{cases}$$

$$\sqrt{10}\sin(x - \tan^{-1}(3)) = 3$$

$$\sin(x - \tan^{-1}(3)) = \frac{3\sqrt{10}}{10}$$

$$x - \tan^{-1}(3) = 71.56505\ldots^\circ \quad \text{or} \quad 108.4349\ldots^\circ$$

$$x = 143^\circ \quad \text{or} \quad 180^\circ$$

(b) (i) at P, $Vt\cos\theta = \frac{1}{2}gt^2 - Vt\sin\theta$

$$2Vt\cos\theta = gt^2 - 2Vt\sin\theta$$

$$-gt^2 + 2Vt(\cos\theta + \sin\theta) = 0$$

$$t(-gt + 2V(\cos\theta + \sin\theta)) = 0$$

$$t = 0, \quad -gt + 2V(\cos\theta + \sin\theta) = 0$$

$$gt = 2V(\cos\theta + \sin\theta)$$

$$t = \frac{2V(\cos\theta + \sin\theta)}{g}$$

$$t = \frac{2V(\cos\theta + \sin\theta)}{g}, \quad x = V\cos\theta \times \frac{2V(\cos\theta + \sin\theta)}{g}$$

$$= \frac{2V^2(\cos^2\theta + \cos\theta\sin\theta)}{g}$$

$$= \frac{V^2}{g}(2\cos^2\theta + 2\sin\theta\cos\theta)$$

$$= \frac{V^2}{g}(\cos 2\theta + 1 + \sin 2\theta)$$

$$= \frac{V^2}{g}(\sin 2\theta + \cos 2\theta + 1)$$

(ii) OS: $Vt\sin\theta - \frac{1}{2}gt^2 = 0$

$$2Vt\sin\theta - gt^2 = 0$$

$$t(2V\sin\theta - gt) = 0$$

$$t = 0 \quad t = \frac{2V\sin\theta}{\ }$$

$$t(2V\sin\theta - gt) = 0$$

$$t = 0, \quad t = \frac{2V\sin\theta}{g}$$

$$x = V\cos\theta \times \frac{2V\sin\theta}{g}$$

$$= \frac{V^2}{g}\sin 2\theta$$

$$\frac{OM}{OS} = \frac{4}{3} \Rightarrow \frac{\frac{V^2}{g^2}(\sin 2\theta + \cos 2\theta + 1)}{\frac{V^2}{g}\sin 2\theta} = \frac{4}{3}$$

$$\frac{\sin 2\theta + \cos 2\theta + 1}{\sin 2\theta} = \frac{4}{3}$$

$$3\sin 2\theta + 3\cos 2\theta + 3 = 4\sin 2\theta$$

$$\sin 2\theta - 3\cos 2\theta = 3$$

(iii) $2\theta = 143°, \ 180°$

$\theta = 71.5°, \ 90°$

Since $\theta$ is angle of elevation,

$\theta = 71.5°$

(c) $(1+x)^{100} - (1-x)^{100}$

$$= {}^{100}C_0 x^0 + {}^{100}C_1 x^1 + {}^{100}C_2 x^2 + \cdots + {}^{100}C_{99} x^{99} + {}^{100}C_{100} x^{100}$$

$$- ({}^{100}C_0 x^0 - {}^{100}C_1 x^1 + {}^{100}C_2 x^2 - \cdots - {}^{100}C_{99} x^{99} + {}^{100}C_{100} x^{100})$$

$$= 2 \times ({}^{100}C_1 x^1 + {}^{100}C_3 x^3 + \cdots + {}^{100}C_{99} x^{99})$$

Sub $x = 1$,

$LHS = 2^{100} - 0^{100}$ $\qquad$ $RHS = 2 \times ({}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{99})$

$= 2^{100}$ $\qquad$ $= 2 \times (100 + {}^{100}C_3 + \cdots + {}^{100}C_{99})$

$$2^{100} = 2 \times (100 + {}^{100}C_3 + \cdots + {}^{100}C_{99})$$

$$2^{99} = 100 + {}^{100}C_3 + \cdots + {}^{100}C_{99}$$

$${}^{100}C_3 + \cdots + {}^{100}C_{99} = 2^{99} - 100$$

$$2^{99} = 100 + {}^{100}C_3 + \cdots + {}^{100}C_{99}$$

$$ {}^{100}C_3 + \cdots + {}^{100}C_{99} = 2^{99} - 100$$