TRINITY GRAMMAR SCHOOL

Mathematics Department

(NESA Student Number)

2023

Year 12

Mathematics Extension 1

TRIAL EXAMINATION

ASSESSMENT TASK 4

Date of Assessment Task: Wednesday, 23 August 2023

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Erasable pens and all electronic devices (or timekeepers), unless otherwise advised, are not permitted in this Assessment
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- *Write your NESA Student Number at the top of this page*

Total marks: 70

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 12)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

- **Year 12 Assessment Weighting:** 30%

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 $x-2$ is a linear factor of x^3+3x^2+x-22. Which of the following is the other factor?

A. x^2+x-1

B. x^2+x-11

C. x^2+5x-9

D. $x^2+5x+11$

2 What is the gradient of the tangent to the curve $y=\sin^{-1}x$ at the point where $x=\frac{1}{5}$?

A. $\frac{5\sqrt{6}}{12}$

B. $\frac{2\sqrt{6}}{5}$

C. $\frac{4\sqrt{3}}{25}$

D. $\frac{25}{24}$

3 Which of the following identities is false?

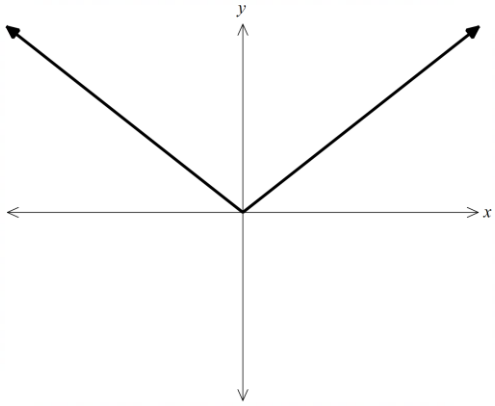
A. $\sec^2(4\theta)-1\equiv\tan^2(4\theta)$

B. $\cot^2(4\theta)-\operatorname{cosec}^2(4\theta)\equiv 1$

C. $\cos(4\theta)\equiv\cos^2(2\theta)-\sin^2(2\theta)$

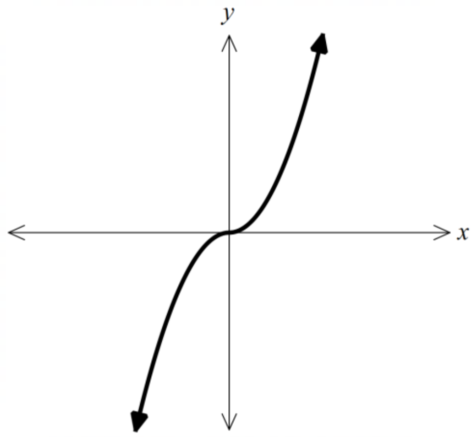
D. $\cos(4\theta)+4\sin^2(\theta)\cos^2(\theta)\equiv\cos^2(2\theta)$

4 The diagram shows the graph of $y = |x|$

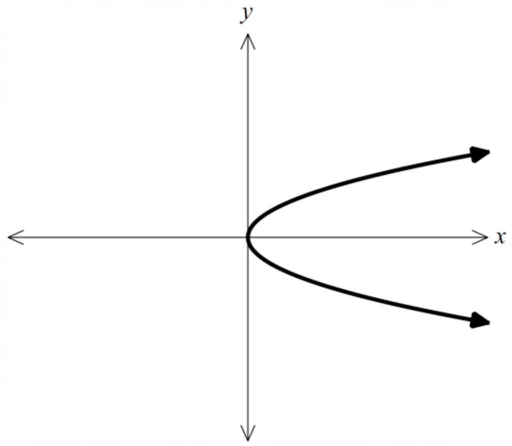


Which of the following could be a sketch of the graph of $y = x|x|$?

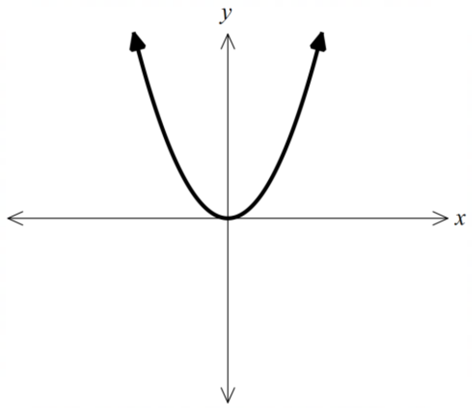
A.



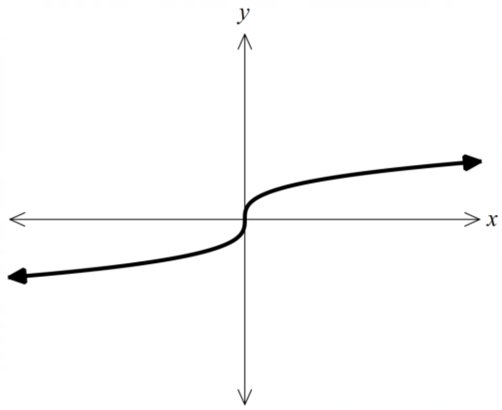
B.



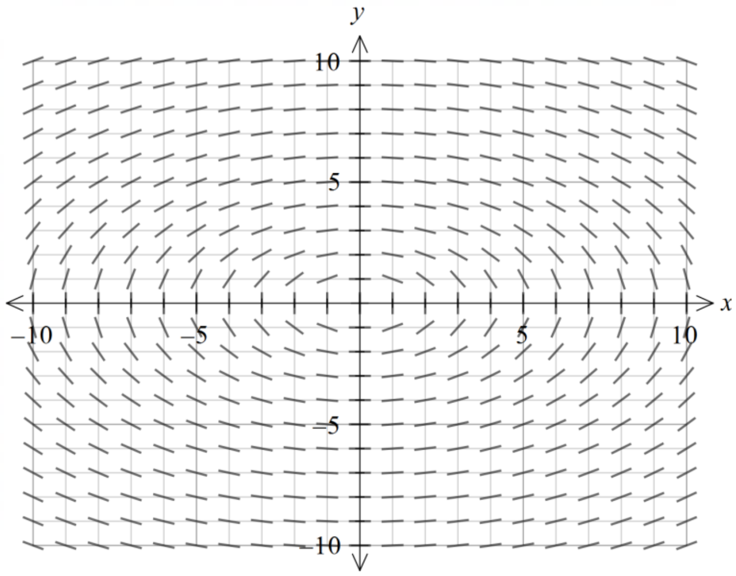
C.



D.



5 Which of the following differential equations could be represented by the slope field diagram below?



A. $\dfrac{dy}{dx} = \dfrac{x^2}{2} + y^2$

B. $\dfrac{dy}{dx} = x^2 + \dfrac{y^2}{2}$

C. $\dfrac{dy}{dx} = -\dfrac{x}{2y}$

D. $\dfrac{dy}{dx} = -\dfrac{y}{2x}$

6 Consider the function f with rule $f(x) = a\cos^{-1}(x-b)$. Given that f has domain [2, 4] and range $[0,\ 6\pi]$, which of the following is true?

A. $a = 6,\ b = -3$

B. $a = 3,\ b = 6$

C. $a = -3,\ b = 6$

D. $a = 6,\ b = 3$

7 A committee of 6 people is to be chosen from 8 students and 6 teachers so that it contains at least 3 students and at least 2 teachers. What is the number of ways this can be done?

A. 1050

B. 1120

C. 2170

D. 7560

8 An ingot of aluminium is placed in a room with a surrounding air temperature of 25ºC and allowed to cool. It loses heat according to Newton's law of cooling, $\frac{dT}{dt} = -k(T - A)$ where T is the temperature of the metal in degrees Celsius at time t minutes, A is the surrounding air temperature and k is a positive constant. The initial temperature of the aluminium is 1350ºC. After 10 minutes the temperature of the aluminium drops to 720ºC.

What total time elapses for the aluminium to cool to a temperature of 50ºC?

A. 60.35 minutes

B. 61.53 minutes

C. 64.52 minutes

D. 65.28 minutes

9 Given that $x = a(t - \sin t)$ and $y = a(1 - \cos t)$. Using the chain rule or otherwise, which of the following is $\frac{dy}{dx}$?

A. $\frac{\sin t}{1 - \cos t}$

B. $\frac{\cos t}{1 - \sin t}$

C. $\frac{\sin t}{1 + \cos t}$

D. $\frac{\cos t}{1 + \sin t}$

10 It is given that $\vec{P} = \begin{pmatrix} 8 \\ a \end{pmatrix}$, $\vec{Q} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}$. If the angle θ between vectors $\vec{P}$ and $\vec{Q}$ is an obtuse angle, what is the value(s) of a?

A. $a \neq 5$ and $a > \frac{64}{5}$

B. $a < -5$

C. $a > -5$ and $a \neq \frac{64}{5}$

D. $a > -5$

Section II

60 marks

Attempt Questions 11-14
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Allow about 1 hour and 45 minutes for this section

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

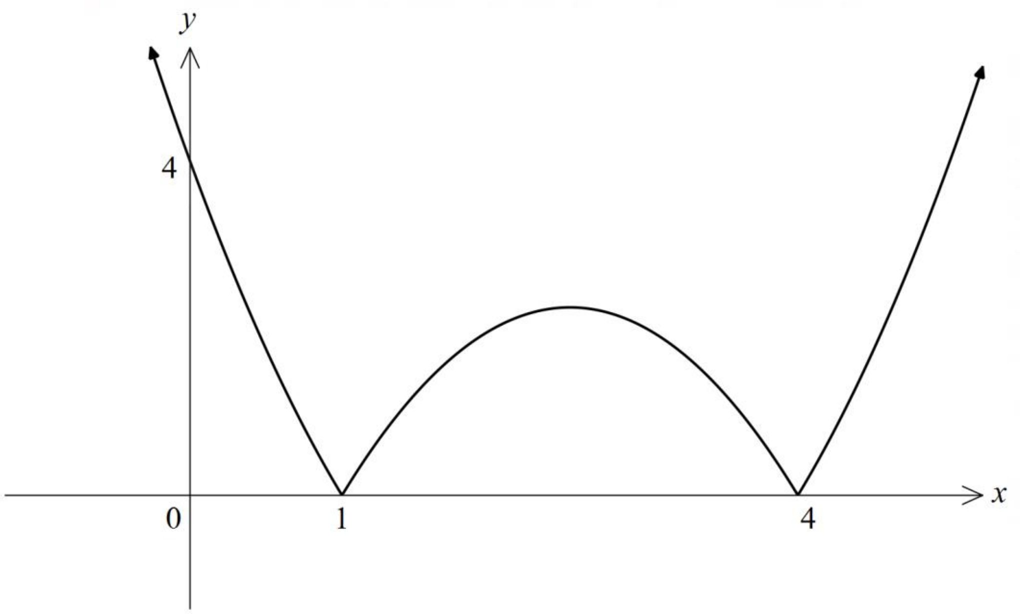
Question 11 (15 marks) Use a SEPARATE Writing Booklet.

(a) Vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are such that $\underset{\sim}{a} = 5\underset{\sim}{i} - 6\underset{\sim}{j}$, $\underset{\sim}{b} = 11\underset{\sim}{i} - 15\underset{\sim}{j}$ and $3\underset{\sim}{a} + \underset{\sim}{c} = \underset{\sim}{b}$.

(i) Find $\underset{\sim}{c}$. **2**

(ii) Find the unit vector in the direction of $\underset{\sim}{b}$. **2**

(b) The diagram shows the graph of $y = |f(x)|$, where $f(x)$ is a quadratic function in the form $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{Z}$. Write down the two possible equations for $f(x)$. **2**



(c) Solve $\dfrac{x}{2+x} \leq 5$. **3**

Question 11 continues on page 7

Question 11 (continued)

(d) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$. **2**

(ii) Hence or otherwise, solve $\sqrt{3}\cos x - \sin x = -\sqrt{3}$, in the domain $[0, 2\pi]$. **2**

(e) A 6-character password is to be chosen from the following characters. **2**

Digits	4	5	7
Letters	T	G	S
Symbols	#	&	%

No character may be used more than once in any password. Find the number of different passwords that may be chosen if the password starts with two letters and ends with two digits.

End of Question 11

Question 12 (15 marks) Use a SEPARATE Writing Booklet.

(a) Prove by mathematical induction that $2^{4n}+31^{n}-2$ is divisible by 15 for all integers $n>0$. **3**

(b) The variables x and y are related by the differential equation $\dfrac{dy}{dx}=e^{2y}\cos 3x.$ When $x=0,\ y=0.$ **2**

Solve the differential equation, obtaining an expression for y in terms of x.

(c) The three roots of $p(x)=0,$ where $p(x)=5x^{3}+ax^{2}+bx-2$ are $x=\dfrac{1}{5},\ x=n$ and $x=n+1,$ where a and b are positive integers and n is a negative integer. **4**

Find $p(x)$, simplifying your coefficients.

(d) Find the value of the independent term in the expansion of $\left(\dfrac{x^{4}}{2}+\dfrac{2}{x}\right)^{15}$. **3**

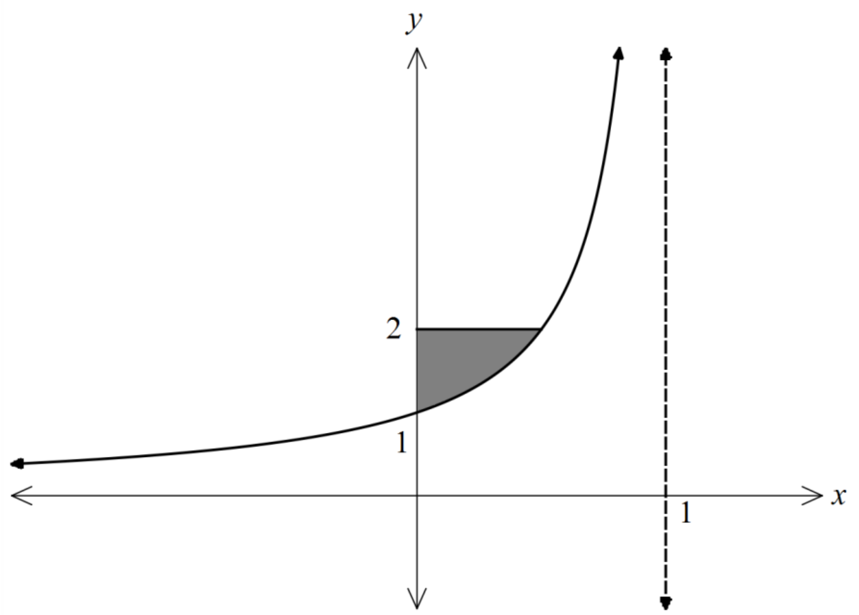
(e) Given that $\cos 2x=a+b$ and $\sin 2x=a-b.$ Show that $a^{2}+b^{2}=\dfrac{1}{2}.$ **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE Writing Booklet.

(a) The shaded region bounded by the graph $y=\dfrac{1}{1-x}$ for $x<1$, the line $y=2$ and the $y-$axis is rotated about the $y-$axis to form a solid of revolution. **4**

Find the volume V of the solid formed, correct to 3 significant figures.



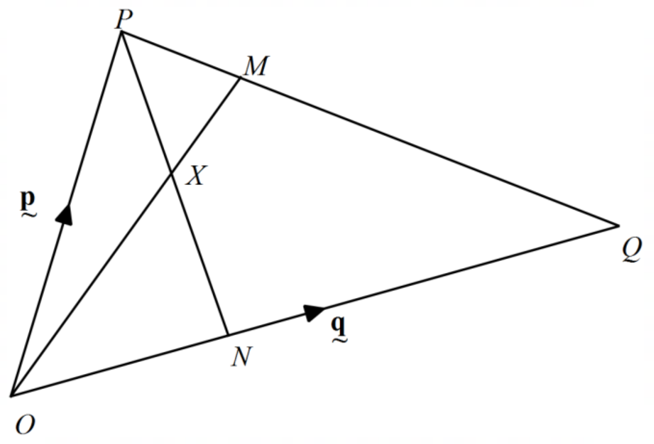
(b) The angles A and B (in degrees) are such that $\sin(A+45°)=2\sqrt{2}\cos A$ and $4\sec^2 B+5=12\tan B$. **3**

Find the exact value of $\tan(A-B)$.

Question 13 continues on page 10

Question 13 (continued)

(c) In the diagram, let $\overrightarrow{OP} = \underset{\sim}{p}, \overrightarrow{OQ} = \underset{\sim}{q}, \overrightarrow{PM} = \frac{1}{3}\overrightarrow{PQ}$ and $\overrightarrow{ON} = \frac{2}{5}\overrightarrow{OQ}$.

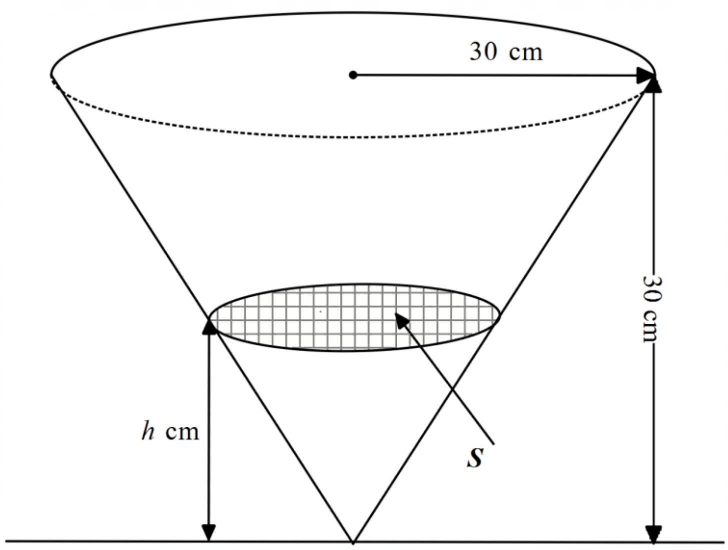


(i) Given that $\overrightarrow{OX} = m\overrightarrow{OM}$ and $0 < m < 1$, show that $\overrightarrow{OX} = \frac{1}{3}m\left(2\underset{\sim}{p} + \underset{\sim}{q}\right)$. **1**

(ii) Given that $\overrightarrow{PX} = n\overrightarrow{PN}$ and $0 < n < 1$, show that $\overrightarrow{OX} = (1-n)\underset{\sim}{p} + \left(\frac{2}{5}n\right)\underset{\sim}{q}$. **2**

(iii) Hence, evaluate m and n. **2**

(d) Water is poured into a conical vessel of radius 30 cm and perpendicular height of 30 cm at a constant rate of 24 cm^3s^{-1}. The depth of water is h cm at any time t seconds as shown in the diagram. **3**

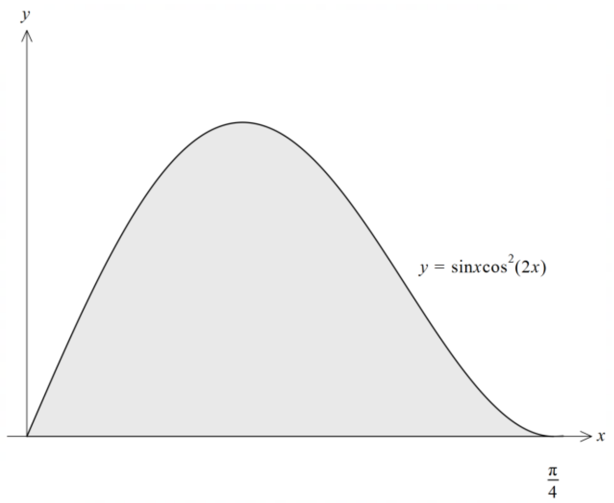


Find the rate of increase of the area of surface S of the water when the depth is 16 cm.

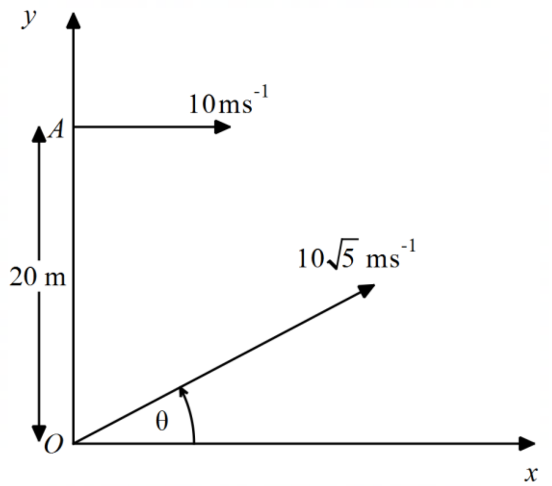
End of Question 13

Question 14 (15 marks) Use a SEPARATE Writing Booklet.

(a) The diagram shows the curve $y = \sin x \cos^2 2x$ for $0 \le x \le \dfrac{\pi}{4}$. Using the substitution $u = \cos x$, find the **exact area** of the shaded region bounded by the curve and the $x-$axis. **4**



(b) OA is a vertical building of height 20 metres. A particle is projected horizontally from A with speed of 10 ms^{-1}. At the same instant, a second particle is projected from O with speed $10\sqrt{5}$ ms^{-1} at the angle θ above the horizonal. The two particles travel in the same plane of motion. Take $g = 10$ ms^{-1}.



(i) Write down expressions for the horizontal (x) and vertical (y) displacements relative to O of each particle after time t seconds. **2**

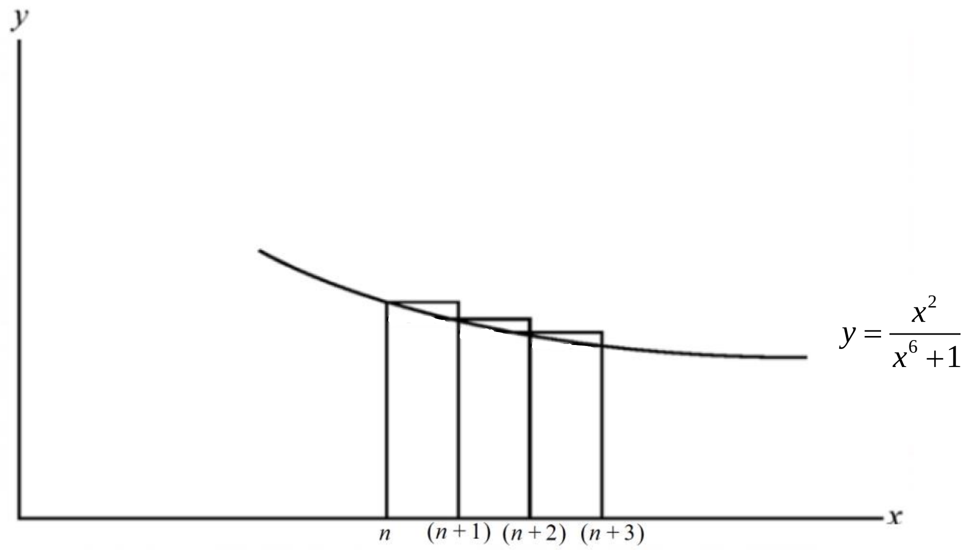
(ii) Show that when the two particles collide, then they do so after 1 second. **2**

(iii) Show that when the two particles collide, they do so when their paths of motion are **perpendicular** to each other. **2**

Question 14 continues on page 12

Question 14 (continued)

(c) The diagram below shows the curve $y = \dfrac{x^2}{x^6+1}$ for $x > 0$, together with line segments parallel to the coordinate axes. The vertical line segments are at $x = n,\ x = n+1,\ x = n+2,\ x = n+3$ and so on, where $n \in \mathbb{Z}^+$.



(i) By considering the sum of the areas of these rectangles, show that **2**

$$\int_n^\infty \frac{x^2\,dx}{x^6+1} < \sum_{r=n}^{\infty} \frac{r^2}{r^6+1}.$$

(ii) Let $S = \displaystyle\sum_{r=1}^{\infty} \frac{r^2}{r^6+1}$.

Show that an upper bound for S is given by **3**

$$\sum_{r=1}^{n} \frac{r^2}{r^6+1} + \frac{\pi}{6} - \frac{1}{3}\tan^{-1}\left(n^3\right).$$

End of paper

Section I: [D, A, B, A, C, D, C, B, A, C/D]

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Q1: $x^3 + 3x^2 + x - 22 = (x-2)(x^2+5x+11)$ by inspection.

[otherwise:

$$\begin{array}{r|l} & \quad x^2 + 5x + 11 \\ x-2 & x^3 + 3x^2 + x - 22 \\ & x^3 - 2x^2 \\ \hline & \quad 5x^2 + x - 22 \\ & \quad 5x^2 - 10x \\ \hline & \qquad 11x - 22 \\ & \qquad 11x - 22 \\ \hline & \qquad\quad 0 \end{array}$$

(other methods may exist)].

$\therefore$ D

Q2:

$$y = \sin^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$\therefore$ when $x = \frac{1}{5}$, $y' = \dfrac{1}{\sqrt{1 - \frac{1}{25}}}$

$$= \frac{1}{\sqrt{\frac{25-1}{25}}}$$

$$= \frac{1}{\frac{\sqrt{24}}{5}}$$

$$= \frac{5}{\sqrt{24}}$$

$$= \frac{5}{2\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{5\sqrt{6}}{12}$$

$\therefore$ A

Q3: B

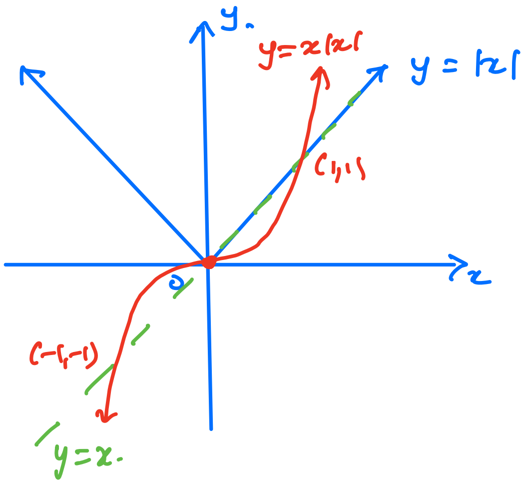
$LHS = \cot^2 4\theta - \operatorname{cosec}^2(4\theta)$

$= (\operatorname{cosec}^2 4\theta - 1) - \operatorname{cosec}^2(4\theta)$

$= -1$

$\neq RHS.$

Q4:



$y = x|x|$ this is an odd function

∴ point symmetric about O.

A

Note: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

∴ $y = x|x|$

$= \begin{cases} x \times x = x^2 & \text{if } x \geq 0 \\ x(-x) = -x^2 & \text{if } x < 0 \end{cases}$

Q5: By examining the slope fields and the available choices

$\frac{dy}{dx} = -\frac{x}{2y} \Rightarrow y \neq 0$ ∴ C

Q6: Let $y = a\cos^{-1}(x-b)$

∴ $-1 \leq (x-b) \leq 1$ and $0 \leq y/a \leq \pi$

$-1+b \leq x \leq 1+b$ $\quad 0 \leq y \leq a\pi$

Given: D: $x \in [2,4]$ then $1+b = 4 \Rightarrow b = 3$

R: $y \in [0, 6\pi]$ then $a\pi = 6\pi \Rightarrow a = 6$

∴ D

Q7: $3S, 3T = {}^8C_3 \times {}^6C_3 = 1120$

$4S, 2T = {}^8C_4 \times {}^6C_2 = 1050$

∴ Total $= 1120 + 1050 = 2170$ C

Q8: $\dfrac{dT}{dt} = -k(T-25)$ given $A = 25$ (surrounding temp)

when $t = 0$, $T = 1350°C$

$t = 10$, $T = 720°C$

Using separation of variables / integration / $\underbrace{T = A + Be^{-kt}}$

intended method in Multiple choice.

$$\frac{dT}{T-25} = -k\,dt$$

$$\int_{T=1350}^{720} \frac{dT}{T-25} = \int_{t=0}^{10} -k\,dt$$

$$\ln|T-25|\Big]_{1350}^{720} = -kt\Big]_0^{10}$$

$$\ln|720-25| - \ln|1350-25| = -10k$$

$$\therefore \quad k = -\frac{1}{10}\ln\left(\frac{695}{1325}\right)$$

$$= -\frac{1}{10}\ln\left(\frac{139}{265}\right)$$

$$\therefore \quad T = 25 + \overbrace{(1350-25)}^{B} e^{-kt}$$

when $T = 50°C$ $\therefore$ $50 = 25 + 1325e^{-kt}$

$$\frac{25}{1325} = e^{-kt}$$

$$\therefore \quad t = -\frac{1}{k}\ln\left(\frac{25}{1325}\right)$$

$$= -\frac{1}{\left[-\frac{1}{10}\ln\left(\frac{139}{265}\right)\right]}\ln\left(\frac{25}{1325}\right)$$

$$\doteqdot 61.53 \text{ mins.}$$

B

Q9:

$$x = a(t - \sin t) \qquad \therefore \frac{dx}{dt} = a(1 - \cos t)$$

$$y = a(1 - \cos t) \qquad \frac{dy}{dt} = a \sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{a \sin t}{a(1 - \cos t)}$$

$$= \frac{\sin t}{1 - \cos t}$$

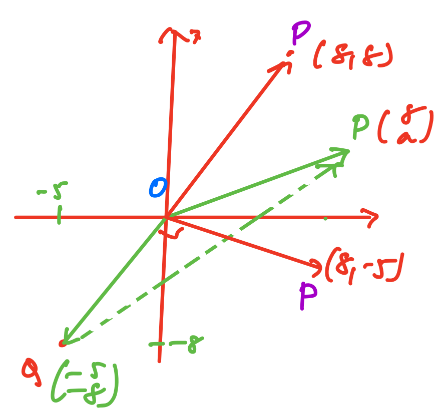
A

Q10:

$$\cos\theta = \frac{\vec{P} \cdot \vec{Q}}{|\vec{P}|\,|\vec{Q}|}$$

$$= \frac{\begin{pmatrix} 8 \\ a \end{pmatrix} \begin{pmatrix} -5 \\ -8 \end{pmatrix}}{\sqrt{8^2 + a^2}\,\sqrt{25 + 64}}$$

$$= \frac{-40 - 8a}{\sqrt{89}\,\sqrt{64 + a^2}}$$



if $a = -5$, $\cos\theta = 0$

$\theta = \pi/2$.

$\therefore a > -5$

if $\frac{\pi}{2} < \theta < \pi$ (obtuse-angled) $\Rightarrow \cos\theta < 0$

then $-40 - 8a < 0$

$-8a < 40$

$a > -5$

D

told Δ is obtuse $\therefore \theta \neq \pi$ at all.

but we need to exclude the case Q, O, P are collinear i.e. $\begin{pmatrix} 8 \\ a \end{pmatrix} \neq t \begin{pmatrix} -5 \\ -8 \end{pmatrix}$

$\therefore$ C was also accepted (but this case should not arise)

when $t = -8/5$ $\therefore a = \frac{-8}{5} \times -8$
$= \frac{64}{5}$.

Section II:

11

a) i) $\underset{\sim}{c} = \underset{\sim}{b} - 3\underset{\sim}{a}$

$= \begin{pmatrix} 11 \\ -15 \end{pmatrix} - 3\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ ✓ valid method.

$= \begin{pmatrix} 11 - 15 \\ -15 + 18 \end{pmatrix}$

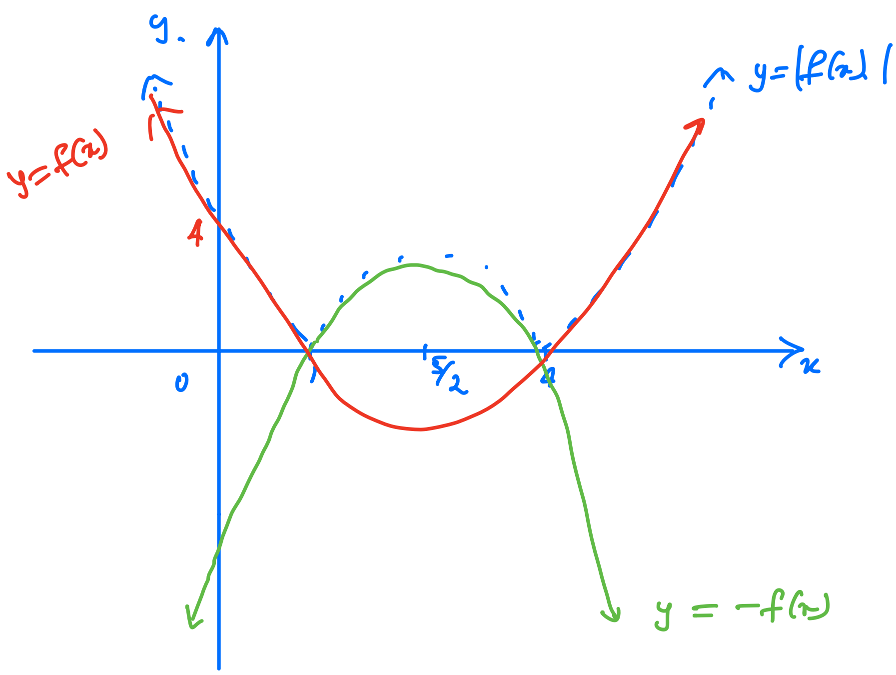
$= \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ ✓

$= -4\underset{\sim}{i} + 3\underset{\sim}{j}$

ii) $\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \frac{1}{\sqrt{121 + 225}}\begin{pmatrix} 11 \\ -15 \end{pmatrix}$ ✓

$= \frac{11}{\sqrt{346}}\underset{\sim}{i} - \frac{15}{\sqrt{346}}\underset{\sim}{j}$

b)



This question is seeking "possible" equations for $y=f(x)$

Consider $\quad f(x) = (x-1)(x-4)$

when $x=0$, $\quad f(x)=4 \quad \therefore \quad f(x)=(x-1)(x-4)$

and $\quad f(x) = -(x-1)(x-4)$

are two possible expressions for $f(x)$

$\left[\ f(x) = \begin{cases} x^2-5x+4 \checkmark \\ -4+5x-x^2 \checkmark \end{cases}\right.$ is acceptable].

Alternate method: $y=ax^2+bx+c$

subs $(0,4)$, $(1,0)$, $(4,0)$

$\Rightarrow \quad 4=c \quad \therefore \ y=ax^2+bx+4$

& $\left.\begin{cases} a+b+4=0 \\ 16a+4b+4=0 \end{cases}\right\} \Rightarrow \left.\begin{matrix} a=1 \\ b=-5 \end{matrix}\right\}$

etc...

c) $\quad \dfrac{x}{2+x} \leq 5 \quad , (x \neq -2)$

$$\left(\frac{x}{2+x}\right)(2+x)^2 \leq 5(2+x)^2$$

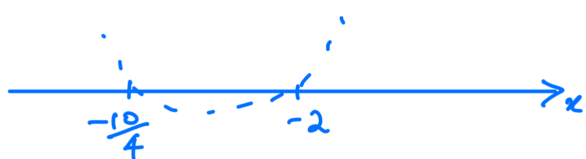
$$x(2+x) - 5(2+x)^2 \leq 0$$

$$(2+x)\left[x-5(2+x)\right] \leq 0$$

$$(2+x)(-4x-10) \leq 0$$

$$(2+x)(4x+10) \geq 0$$

$\checkmark$ valid method.



$\therefore \ \{x : \ x \leq -{}^{10}/_{4} \checkmark \ \cup \ x > -2\} \checkmark$. or equivalent methods e.g critical values.

d) i) $\quad \sqrt{3}\cos x - \sin x = R\cos x\cos\alpha - R\sin\alpha\sin x$

$\Rightarrow \quad R\cos\alpha = \sqrt{3} \quad ①$

$\qquad R\sin\alpha = 1 \quad ②$

$$③ \div ① \Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6} \quad \checkmark \qquad \left(0 < \alpha < \frac{\pi}{2}\right)$$

$$①^2 + ②^2 \Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) = (\sqrt{3})^2 + 1^2$$

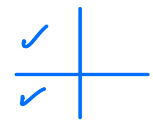
$$= 4$$

$$\therefore R = 2 \quad (R > 0). \quad \checkmark$$

$$\therefore \sqrt{3}\cos x - \sin x = 2\cos\left(x + \frac{\pi}{6}\right)$$

(ii) Solving $2\cos\left(x + \frac{\pi}{6}\right) = -\sqrt{3}$

$$\cos\left(x + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$



$$x + \frac{\pi}{6} = \pi - \cos^{-1}\frac{\sqrt{3}}{2}, \quad \pi + \cos^{-1}\frac{\sqrt{3}}{2}$$

$$= \pi - \frac{\pi}{6}, \quad \pi + \frac{\pi}{6}$$

$$\therefore x = \pi - \frac{\pi}{3}, \quad \pi$$

$$\therefore x = \frac{2\pi}{3}, \ \pi \quad \checkmark \checkmark \quad \text{for } x \in [0, 2\pi].$$

[Note: t- method can be used]

if both correct answers not seen then award [1] max. for valid working

(e) $\boxed{L}\ \boxed{L}\ \_\_\ \_\_\ \boxed{D}\ \boxed{D}$

$3 \times 2 \quad 5 \times 4 \quad 3 \times 2$ ✓ valid method

$$\therefore \text{No. of arrangement} = (3\times2)^2 \times (5\times4)$$

$$\left[= ({}^3P_2)^2 \times {}^5P_2\right].$$

$$= 720. \quad \checkmark$$

Q12:

a) Let $S(n): \left(2^{4n} + 31^n - 2\right) \div 15$

Prove $S(1)$ is true.

$$S(1) = 2^4 + 31 - 2$$
$$= 16 + 31 - 2$$
$$= 45$$
$$(= 3 \times 15)$$

Since 45 is $\div$ by 15 then $S(1)$ is true.

Assume $S(k)$ is true for $1 \le k \le n$, $k, n \in \mathbb{Z}^+$

ie $2^{4k} + 31^k - 2 = 15M$ ① (M is an integer)

RTP: $S(k+1)$ is true

ie $2^{4k+4} + 31^{k+1} - 2 = 15N$ ② (N is an integer)

Must see evidence claim is a multiple of 15.

in ②

$$LHS = 2^{4k} \cdot 2^4 + 31^{k+1} - 2$$
$$= 2^4\left[15M - 31^k + 2\right] + 31^{k+1} - 2 \quad \text{from ①}$$
$$= 2^4(15M) - 2^4(31^k) + 2^5 + 31^{k+1} - 2$$
$$= 2^4(15M) + 30 + 31^k(31 - 2^4)$$
$$= 15(16M + 2) + 15(31)^k$$
$$= 15(16M + 2 + 31^k)$$
$$= 15N$$

Must see evidence that clearly leads to a multiple of 15.

The statement is true for $n = 1$. Whenever it is true for $n = k$, it is also true for $n = k+1$ and for all positive integer values of n.

b) $$\frac{dy}{dx} = e^{2y}(\cos 3x)$$

separation of variables $\Rightarrow$ $$\frac{dy}{e^{2y}} = (\cos 3x)\,dx$$

$$e^{-2y}\,dy = (\cos 3x)\,dx$$

$$\therefore \int_{y=0}^{y} e^{-2y}\,dy = \int_{x=0}^{x} \cos 3x\,dx$$

$$-\tfrac{1}{2}\left[e^{-2y}\right]_0^y = \tfrac{1}{3}\left[\sin 3x\right]_0^x$$

✓ valid method.

$$-\tfrac{1}{2}\left[e^{-2y} - e^{-0}\right] = \tfrac{1}{3}(\sin 3x - \sin 0)$$

$$-\tfrac{1}{2}\left[e^{-2y} - 1\right] = \tfrac{1}{3}\sin 3x$$

$$e^{-2y} - 1 = \tfrac{2}{3}\sin 3x$$

$$e^{-2y} = \tfrac{2}{3}\sin 3x + 1$$

$$-2y = \ln\left|1 + \tfrac{2}{3}\sin 3x\right|$$

$$y = -\tfrac{1}{2}\ln\left|1 + \tfrac{2}{3}\sin 3x\right|$$ ✓

[acceptable: $y = -\frac{1}{2}\ln\left|1 - \frac{2}{3}\sin 3x\right|$ if the method of find a constant is used].

[There is/are alternate methods]

c) $$p(x) = 5x^3 + ax^2 + bx - 2$$

let $\alpha = \frac{1}{5}$, $\beta = n$, $\gamma = n+1$

$$\sum \alpha = -\frac{a}{5}$$

ie $\frac{1}{5} + n + n + 1 = -\frac{a}{5}$ $(-b/a)$

$2n + \frac{6}{5} = -\frac{a}{5}$ ①

also $\prod \alpha = \frac{1}{5}(n)(n+1) = \frac{2}{5}$ $(-c/a)$

ie $n(n+1) = 2$ ②

✓ valid method

$n^2 + n - 2 = 0$

$(n+2)(n-1) = 0$

$\therefore n = -2$ or $n = 1$

but $n < 0$ $\therefore n = -2$ ✓

hence in ① $-4 + \frac{6}{5} = -\frac{a}{5}$

$-\frac{14}{5} \times (5) = a$

$\therefore a = 14.$

if $n = -2$, $\alpha = \frac{1}{5}$, $\beta = -2$, $\gamma = -1$

$\therefore \alpha\beta + \alpha\gamma + \beta\gamma = d/a$

$-\frac{2}{5} - \frac{1}{5} + 2 = \frac{b}{5}$

$\therefore b = 5\left(-\frac{2}{5} - \frac{1}{5} + 2\right)$

$= 7$

✓ for $\underline{a}$ or $\underline{b}$

$\therefore p(x) = 5x^3 + 14x^2 + 7x - 2.$ ✓ must state the full expression as required.

(d) $T_{r+1} = {}^{15}C_r \left(\frac{x^4}{2}\right)^{15-r} \left(\frac{2}{x}\right)^r$ $0 \le r \le 15.$

$$= {}^{15}C_r \frac{1}{2^{15-r}} x^{60-4r} 2^r \frac{1}{x^r}$$

$$= {}^{15}C_r \; 2^{2r-15} \; x^{60-5r}$$ ✓ or equivalent

we want $60 - 5r = 0$ (independent term)

$\Rightarrow r = 12.$ ✓ (implied)

$\therefore$ we want ${}^{15}C_{12} \; 2^{24-15}$

$$= {}^{15}C_{12} (2^9)$$

$$= 232\,960.$$ ✓

e) $\cos 2x = a + b$ ①

$\sin 2x = a - b$ ②

evidence of squaring results seen.

$①^2 + ②^2 \Rightarrow \cos^2 2x + \sin^2 2x = (a+b)^2 + (a-b)^2$ ✓

(must see evidence of where (1) came from

$\rightarrow 1 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2$ ✓

$1 = 2a^2 + 2b^2$ ✓

$\therefore a^2 + b^2 = \frac{1}{2}.$ (already given)

13. a) $$y = \frac{1}{1-x}$$

$$\Rightarrow \quad 1-x = \frac{1}{y}$$ ✓ seen everywhere. or equivalent.

$$\therefore \quad x = 1 - \frac{1}{y}$$

$$x^2 = 1 - \frac{2}{y} + \frac{1}{y^2}$$ ✓ either or equivalent

$$\therefore \quad V = \pi \int_1^2 1 - \frac{2}{y} + \frac{1}{y^2} \, dy$$

$$= \pi \int_1^2 1 - \frac{2}{y} + y^{-2} \, dy$$

$$= \pi \left[y - 2\ln y - \frac{1}{y} \right]_1^2$$ ✓

$$= \pi \left[\left(2 - 2\ln 2 - \frac{1}{2}\right) - \left(1 - 2\ln 1 - 1\right) \right]$$

$$= \pi \left[\tfrac{3}{2} - 2\ln 2 - 0 \right]$$

$$= \pi \left[\tfrac{3}{2} - 2\ln 2 \right] \text{ (exact)}.$$

$$= 0.357 \text{ (3 s.f.)}$$ ✓

b) $$\begin{cases} \sin(A+45) = 2\sqrt{2}\cos A & (1) \\ 4\sec^2 B + 5 = 12\tan B & (2) \end{cases}$$

from ①

$$\sin A \cos 45 + \sin 45 \cos A = 2\sqrt{2}\cos A$$

$$\frac{1}{\sqrt{2}} \sin A + \frac{1}{\sqrt{2}} \cos A = 2\sqrt{2}\cos A.$$

$\div\ 2\sqrt{2}\cos A]$

$$\frac{1}{\sqrt{2}} \frac{\sin A}{2\sqrt{2}\cos A} + \frac{1}{\sqrt{2}} \cos A \left(\frac{1}{2\sqrt{2}\cos A}\right) = 1$$

ie

$$\frac{1}{4} \tan A + \frac{1}{4} (1) = 1$$

$$\therefore \tan A = (3/4)(4) = 3$$

from ②

$$4 \sec^2 B + 5 = 12 \tan B$$

$$4 (1 + \tan^2 B) + 5 = 12 \tan B$$

$$4 \tan^2 B + (5+4) - 12 \tan B = 0$$

$$4 \tan^2 B - 12 \tan B + 9 = 0$$

$$(2\tan B)^2 - 2(2)(3)\tan B + 3^2 = 0$$

$$(2\tan B - 3)^2 = 0$$

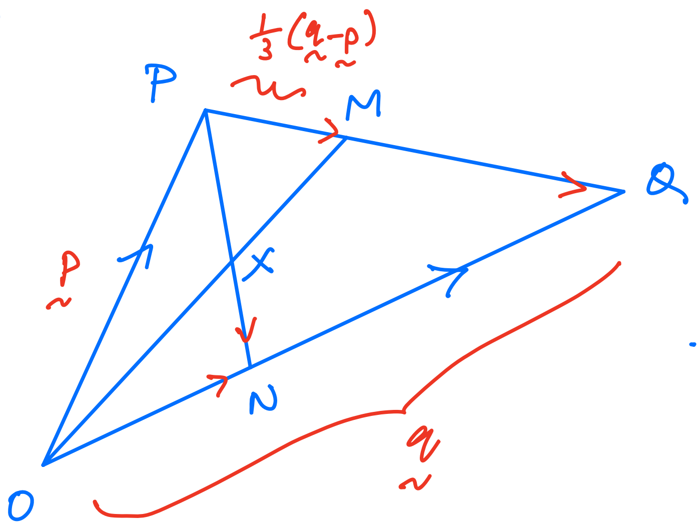
$$\therefore \tan B = 3/2.$$

$$\therefore \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{3 - 3/2}{1 + 3(3/2)}$$

$$= 3/11.$$

c)



$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$

$\therefore \overrightarrow{PM} = \frac{1}{3}(\mathbf{q} - \mathbf{p})$

$\overrightarrow{OQ} = \mathbf{q}$

$\therefore \overrightarrow{ON} = \frac{2}{5}(\mathbf{q})$

i) $\overrightarrow{OX} = m\,\overrightarrow{OM} \qquad 0 < m < 1$

$= m\left(\overrightarrow{OP} + \overrightarrow{PM}\right)$

$= m\left(\mathbf{p} + \frac{1}{3}(\mathbf{q} - \mathbf{p})\right)$

$= m\left(\mathbf{p} + \frac{1}{3}\mathbf{q} - \frac{1}{3}\mathbf{p}\right)$

$= m\left(\frac{2}{3}\mathbf{p} + \frac{1}{3}\mathbf{q}\right)$

$= \frac{m}{3}\left(2\mathbf{p} + \mathbf{q}\right)$ as required.

ii) $\overrightarrow{PX} = n\,\overrightarrow{PN} \qquad 0 < n < 1.$

$= n\left(\overrightarrow{ON} - \overrightarrow{OP}\right)$

$= n\left(\frac{2}{5}\mathbf{q} - \mathbf{p}\right)$

$\overrightarrow{OX} = \overrightarrow{OP} + \overrightarrow{PX}$

$= \mathbf{p} + n\left(\frac{2}{5}\mathbf{q} - \mathbf{p}\right)$

$$= \underset{\sim}{p} + \frac{2n}{5}\underset{\sim}{q} - n\underset{\sim}{p}$$

$$= \underset{\sim}{p}(1-n) + \frac{2}{5}n\underset{\sim}{q}$$ as required

iii) From (i) and (ii)

$$\underset{\sim}{p}(1-n) + \frac{2}{5}n\underset{\sim}{q} = \frac{1}{3}m(2\underset{\sim}{p}+\underset{\sim}{q})$$

equating "matching" parts

$$(1-n) = \frac{2}{3}m \quad ① \quad \text{and}$$

$$\frac{2}{5}n = \frac{1}{3}m \quad ②$$

from ② $m = \frac{6}{5}n$ ③

subs ③ into ① $\Rightarrow$ $1-n = \frac{2}{3}\left(\frac{6}{5}n\right)$

$$1-n = \frac{12}{15}n$$

$$1 = \frac{9}{5}n$$

$$n = 5/9$$ ✓ correct answer only (no FT if intermediate steps incorrect ∵ equations given)

$$\therefore \quad m = \frac{6}{5}\times\frac{5}{9}$$

$$= 2/3$$ ✓

$$\therefore (m,n) = \left(\frac{2}{3}, \frac{5}{9}\right).$$

(d) Cone: $V = \frac{1}{3}\pi r^2 h$

but $r = h$ $\therefore$ $V = \frac{1}{3}\pi h^3$

we know: $\frac{dV}{dt} = 24$ cm^3/s.

$$\frac{dV}{dh} = \pi h^2$$

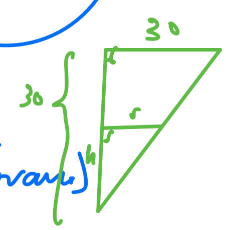
$$\therefore \frac{dh}{dV} = \frac{1}{\pi h^2}$$

we want $\frac{dS}{dt} = \frac{dV}{dt} \times \frac{dS}{dV}$

$S = \pi r^2$
$= \pi h^2$

$$= \frac{dV}{dt} \times \left[\frac{dS}{dh} \times \frac{dh}{dV}\right]$$

now $\frac{r}{30} = \frac{h}{30}$ (similar Δ's - see diagram)



$\therefore$ $r = h$.

hence: $\frac{dS}{dt} = 24 \times 2\pi h \times \frac{1}{\pi h^2}$

with $h = 16$

$$\frac{dS}{dt} = 24 \times 2\pi \times 16 \times \frac{1}{16^2\pi}$$

$$= 3 \text{ cm}^2/\text{s}.$$

Q11f.

a) $y = \sin x (\cos^2 2x)$

$A = \int_0^{\frac{\pi}{4}} \sin x (\cos^2 2x) \, dx$

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

when $x = 0$, $u = 1$

$x = \pi/4$ $u = \frac{1}{\sqrt{2}}$

✓ demonstrated use of subst. method.

$\cos^2 2x = (\cos 2x)^2$

$= (2\cos^2 x - 1)^2$

$= (2u^2 - 1)^2$

$\therefore \int_0^{\frac{\pi}{4}} \sin x \cos^2 2x \frac{dx}{du} \, du$

$= \int_1^{\frac{1}{\sqrt{2}}} -(2u^2-1)^2 \, du$

✓ must be seen in terms of u.

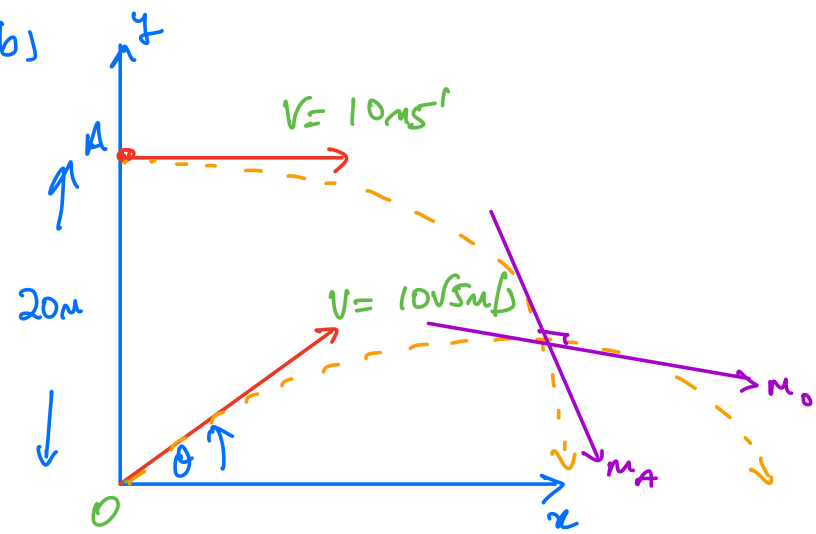
$= \int_{\frac{1}{\sqrt{2}}}^1 4u^4 - 4u^2 + 1 \, du$

$$= \left[\frac{4u^5}{5} - \frac{4u^3}{3} + u\right]_{1/\sqrt{2}}^{1}$$ ✓

$$= \left(\frac{4(1)^5}{5} - \frac{4(1)^3}{3} + 1\right) - \left(\frac{4\left(\frac{1}{\sqrt{2}}\right)^5}{5} - \frac{4\left(\frac{1}{\sqrt{2}}\right)^3}{3} + \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{15}\left(7 - 4\sqrt{2}\right)$$ ✓ from calculator. (exact form)

(b)



(other methods are valid)

i) **At A:** $\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}0\\ -10\end{pmatrix}$

$\therefore \quad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}c_1\\ -10t + c_2\end{pmatrix}$

when $t = 0$, $\dot{x} = 10$, $\dot{y} = 0$

$\therefore \quad c_1 = 10, \quad c_2 = 0$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}10\\ -10t\end{pmatrix}$$

$$\therefore \quad \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}10t + c_3\\ -\frac{10t^2}{2} + c_4\end{pmatrix}$$

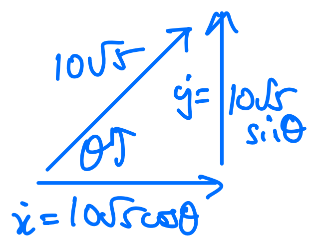
when $t=0$, $x=0$, $y=20$.

$$\therefore \quad c_3 = 0, \quad c_4 = 20.$$

$$\therefore \left.\begin{cases} x(t) = 10t\\ y(t) = -5t^2 + 20\end{cases}\right\} \checkmark$$

If no working, If any incorrect award [7] max if valid working shown.

At O:



$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}0\\ -10\end{pmatrix}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}c_5\\ -10t + c_6\end{pmatrix}$$

when $t=0$, $\dot{x} = 10\sqrt{5}\cos\theta$ $\quad \dot{y} = 10\sqrt{5}\sin\theta$

$$\therefore \quad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}10\sqrt{5}\cos\theta\\ 10\sqrt{5}\sin\theta - 10t\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}10\sqrt{5}\cos\theta \, t\\ 10\sqrt{5}\sin\theta \, t - 5t^2\end{pmatrix}$$

$$\therefore \left.\begin{cases} x(t) = (10\sqrt{5}\cos\theta)t\\ y(t) = -5t^2 + 10\sqrt{5}\sin\theta\end{cases}\right\} \checkmark$$

as above if no working shown.

ii) Particles collide when $x_A = x_O$ & $y_A = y_O$

ie: $10t = (10\sqrt{5}\cos\theta)\,t$

$\therefore\ 1 = \sqrt{5}\cos\theta$

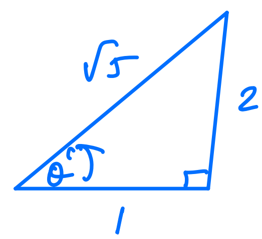
ie $\cos\theta = \frac{1}{\sqrt{5}}$ ✓

& $-5t^2 + 20 = -5t^2 + (10\sqrt{5}\sin\theta)\,t$

$20 = (10\sqrt{5}\sin\theta)\,t$

$\sin\theta = \frac{2}{\sqrt{5}}$ $\quad\therefore\ t = \dfrac{20}{10\sqrt{5}\sin\theta}$

for $0 \leq \theta < \frac{\pi}{2}$,



$$t = \frac{20}{10\sqrt{5}\left(\frac{2}{\sqrt{5}}\right)}$$ ✓ a correct valid method.

$\therefore\ t = 1$ (as required) → already given.

iii) we want to show $m_A \times m_O = -1$ (see diagram above)

From A: $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}10\\ -10t\end{pmatrix}$

From O: $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}10\sqrt{5}\cos\theta\\ 10\sqrt{5}\sin\theta - 10t\end{pmatrix}$

When $t=1$, $\dot{x}_A = 10$ $\quad \dot{y}_A = -10$.

$\dot{x}_0 = 10\sqrt{5}\left(\frac{1}{\sqrt{5}}\right)$ $\quad \dot{y}_0 = 10\sqrt{5}\left(\frac{2}{\sqrt{5}}\right) - 10$

$= 10$ $\quad = 10$.

$\therefore \quad m_A = \frac{-10}{10} \qquad m_0 = \frac{10}{10}$

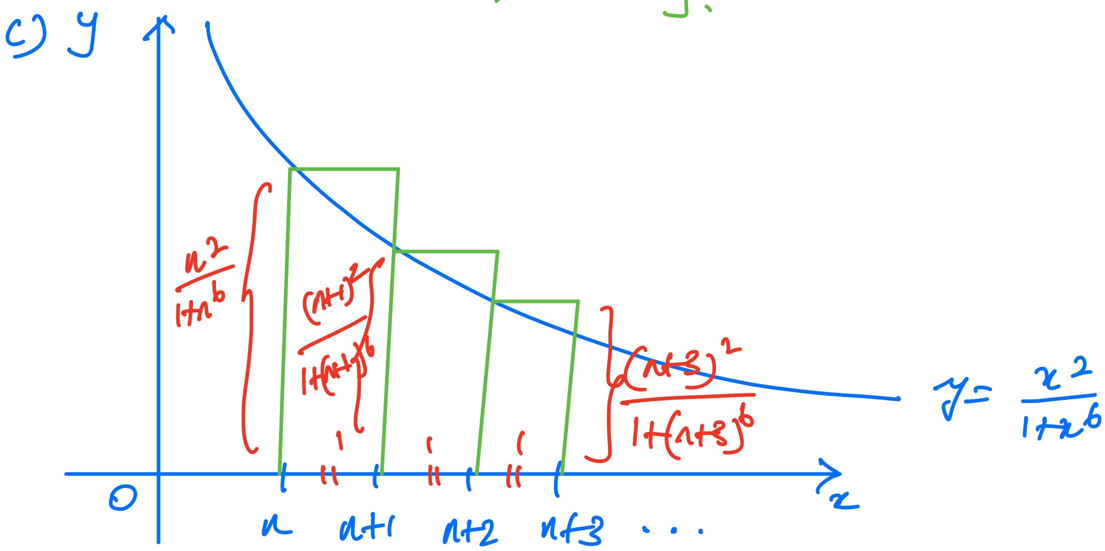
Valid method ✓

$-\frac{10}{10} \times \frac{10}{10} = -1$ ✓

$\therefore$ Particle's path from O is $\perp$ to particle from O path when $t=1$.

[Can also show the dot product between the two tangent (vectors) is zero].

c)



i) Using left end points of each rectangle

$A \doteq (1)\left(\frac{n^2}{1+n^6}\right) + (1)\left(\frac{(n+1)^2}{1+(n+1)^6}\right) + \cdots$ ✓

but the exact area $= \int_n^\infty \frac{x^2}{x^6+1}\,dx$

Reason

Since each rectangle has some part above $y = \frac{x^2}{1+x^6}$ and $\frac{x^2}{1+x^6}$ is a decreasing function $\left(\frac{dy}{dx} < 0\right)$ — see diagram then

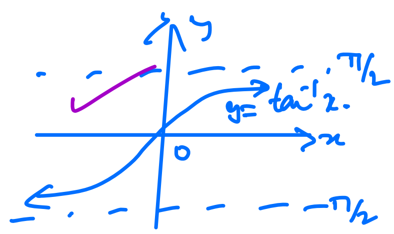
$$\int_n^{\infty} \frac{x^2}{1+x^6}\,dx < \sum_{r=n}^{\infty} \frac{r^2}{r^6+1}$$

ii) $$\int_n^{\infty} \frac{x^2}{1+(x^3)^2}\,dx = \frac{1}{3}\tan^{-1}(x^3)\Big]_n^{\infty}$$

$$= \frac{1}{3}\lim_{b\to\infty}\left[\tan^{-1}(x^3)\Big|_n^b\right]$$

$$= \frac{1}{3}\left[\lim_{b\to\infty}\tan^{-1}(b^3) - \tan^{-1}(n^3)\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{2} - \tan^{-1}(n^3)\right]$$



$$= \frac{\pi}{6} - \frac{1}{3}\tan^{-1}(n^3)$$

as $x\to\infty$, $\tan^{-1}x \to \frac{\pi}{2}$

now:

$$S = \sum_{r=1}^{\infty} \frac{r^2}{1+r^6}$$

$$= \sum_{r=1}^{n} \frac{r^2}{1+r^6} + \sum_{r=n+1}^{\infty} \frac{r^2}{r^6+1}$$

if $\displaystyle\int_n^\infty \frac{x^2}{1+x^6}\,dx < \sum_{r=n}^{\infty} \frac{r^2}{1+r^6}$

$$\therefore \int_{n+1}^{\infty} \frac{x^2}{1+x^6}\,dx < \int_n^\infty \frac{x^2}{1+x^6}\,dx < \sum_{r=n}^{\infty} \frac{r^2}{1+r^6}$$

$$\Rightarrow \int_{n+1}^{\infty} \frac{x^2}{1+x^6}\,dx < \int_n^\infty \frac{x^2}{1+x^6}\,dx < \sum_{r=n+1}^{\infty} \frac{r^2}{1+r^6} < \sum_{r=n}^{\infty} \frac{r^2}{1+r^6}$$

$$\therefore S < \sum_{r=1}^{n} \frac{r^2}{1+r^6} + \left[\frac{\pi}{6} - \frac{1}{3}\tan^{-1}(n^3)\right]$$

This is quite a "complex" question with some assumption to be made. Since $y = \frac{x^2}{1+x^6}$ is a decreasing function, $\forall x \in \mathbb{R}$, and continuous/ differentiable the conclusion reached are acceptable.