TRINITY GRAMMAR SCHOOL

Mathematics Department

(NESA Student Number | Year 12 only)

2024

Year 12

Mathematics Extension 1

TRIAL EXAMINATION

ASSESSMENT TASK 4

Date of Assessment Task: Friday, 30 August 2024

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen.
- Erasable pens and all electronic devices (or timekeepers), unless otherwise advised, are not permitted in this Assessment.
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.
- *Write your NESA Student Number (Year 12) at the top of this page*

Total marks: 70

Section I – 10 marks (pages 2 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II – 60 marks (pages 7 – 13)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section.

- **Year 12 HSC Assessment Weighting:** 30%

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following vectors are perpendicular?

A. $\underset{\sim}{u} = 3\underset{\sim}{i} - 7\underset{\sim}{j},\ \underset{\sim}{v} = \frac{7}{3}\underset{\sim}{i} - \underset{\sim}{j}$

B. $\underset{\sim}{u} = \frac{2}{5}\underset{\sim}{i} - 2\underset{\sim}{j},\ \underset{\sim}{v} = \frac{7}{3}\underset{\sim}{i} - \underset{\sim}{j}$

C. $\underset{\sim}{u} = \frac{9}{2}\underset{\sim}{i} + \frac{7}{2}\underset{\sim}{j},\ \underset{\sim}{v} = -\frac{2}{9}\underset{\sim}{i} - \frac{1}{7}\underset{\sim}{j}$

D. $\underset{\sim}{u} = 14\underset{\sim}{i} + \underset{\sim}{j},\ \underset{\sim}{v} = \frac{1}{2}\underset{\sim}{i} - 7\underset{\sim}{j}$

2 What is the solution to the inequality $|2 - 5x| < 7$?

A. $x < \text{-}1$ or $x > \frac{9}{5}$

B. $\text{-}1 < x < \frac{9}{5}$

C. $x < -\frac{9}{5}$ or $x > 1$

D. $-\frac{9}{5} < x < 1$

3 What is the value of $\sin 2x$ given that $\sin x = \frac{\sqrt{3}}{4}$ given x is obtuse angled?

A. $-\frac{\sqrt{39}}{8}$

B. $-\frac{\sqrt{3}}{2}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{\sqrt{39}}{8}$

4 The radius of a spherical balloon is expanding at a constant rate of 1.3 cms^{-1}. What is the rate of change of the surface area of the balloon when its radius is 6.3 cm?

A. 68.61 cm^2s^{-1}

B. 158.34 cm^2s^{-1}

C. 205.84 cm^2s^{-1}

D. 498.76 cm^2s^{-1}

5 Using the digits 0, 1, 2, 3, 4 and 5, how many odd numbers between 2000 and 5000 can be formed if no digits are repeated?

A. 48

B. 96

C. 120

D. 360

6 Consider the integral:

$$\int_0^k \frac{2}{8+(2x)^2}\,dx = \frac{\pi}{8\sqrt{2}}$$

What is the value of k?

A. $\dfrac{1}{\sqrt{3}}$

B. 1

C. $\sqrt{2}$

D. $\sqrt{3}$

7 The polynomial $x^2 + x + 2$ is a factor of the polynomial $x^4 + 4x^2 + x + a$.
What is the value of a?

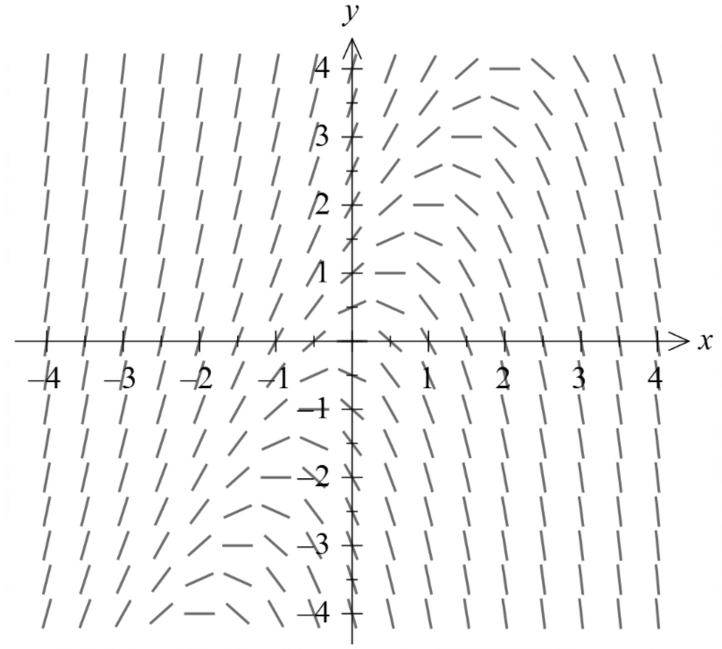
A. -6

B. -3

C. 3

D. 6

8 The direction field for a differential equation is shown below.



Which of the following equations could give this direction field?

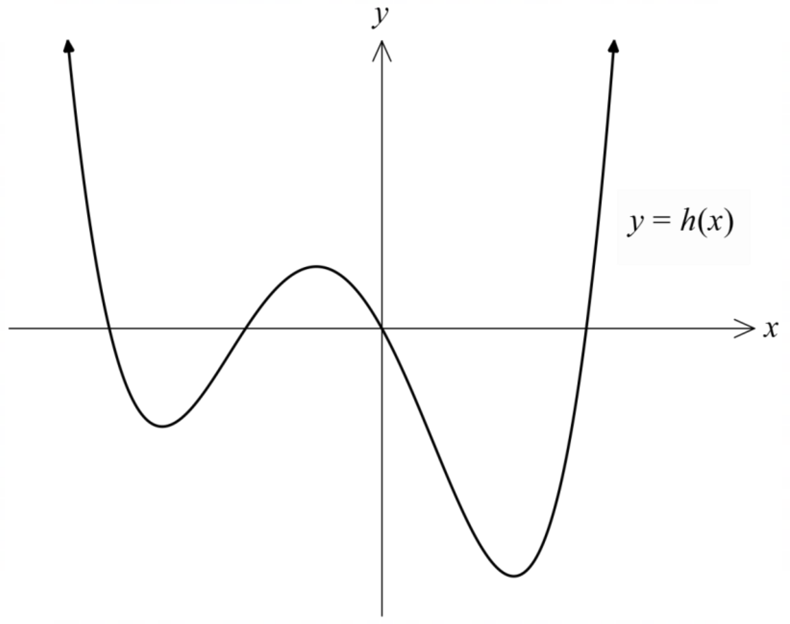
A. $y' = y - 2x$

B. $y' = y + 2x$

C. $y' = y + x$

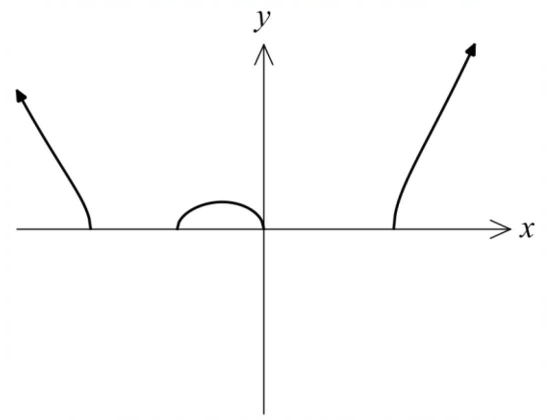
D. $y' = x - y$

9 The graph of $y = h(x)$ is shown below.

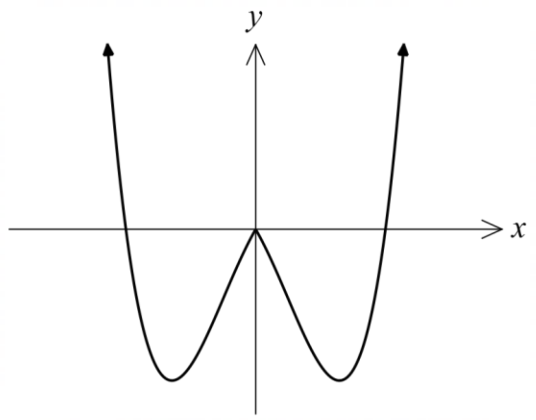


Which graph below shows $y = |h(x)|$?

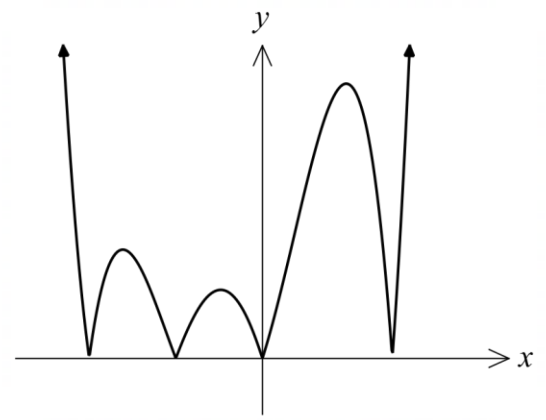
A.



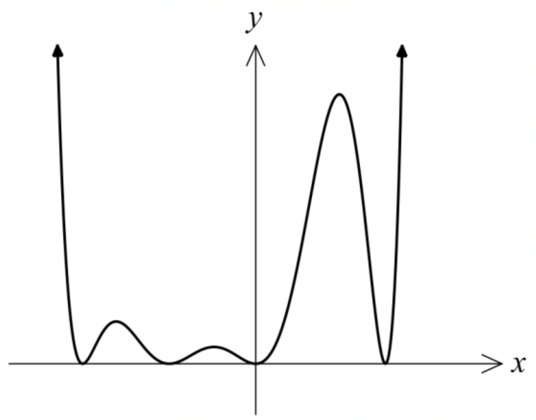
B.



C.



D.



10 The graph of the function $y = \sin^{-1}\left(\frac{1}{2}(x-2)\right)$ is to be transformed by a translation by 1 unit to the left, then a horizontal dilation with a scale factor of 2.

What is the domain of the resultant function?

A. $-1 \le x \le 7$

B. $-2 \le x \le 6$

C. $1 \le x \le 3$

D. $-\frac{1}{2} \le x \le \frac{3}{2}$

Section II

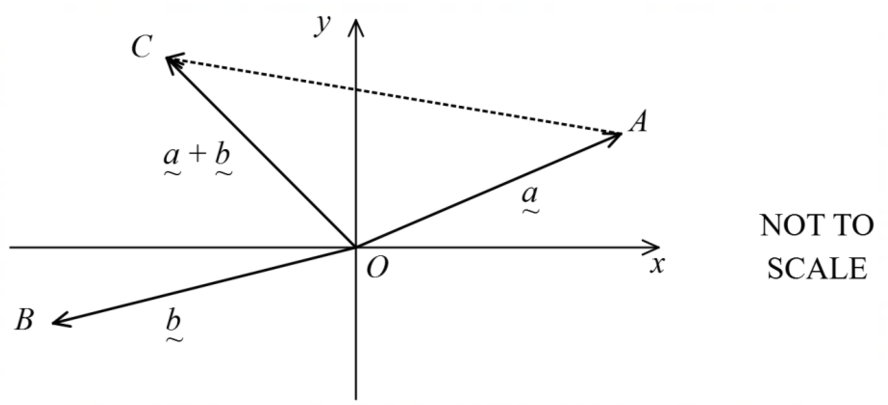
60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra Writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the vectors $\underset{\sim}{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} -9 \\ -1 \end{pmatrix}$ and the resultant vector $\underset{\sim}{a} + \underset{\sim}{b}$.



(i) Show that ΔAOC is an isosceles triangle. **2**

(ii) Find the size of $\angle OAC$. **1**

Question 11 continues on page 8

Question 11 (continued)

(b) (i) Rewrite $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x + \alpha)$, where $0 \le \alpha \le \frac{\pi}{2}$, $R > 0$. **2**

(ii) Hence, or otherwise, solve $\cos x - \sqrt{3}\sin x = 1$ for $0 \le x \le 2\pi$. **2**

(c) Evaluate $\int_1^{\sqrt{2}} \frac{x}{\sqrt{4 - x^2}}\, dx$ using the substitution $u = 4 - x^2$. **3**

(d) The coefficient of x in the expansion of $\left(x + \frac{1}{ax^2}\right)^7$ is $\frac{7}{3}$. **3**

Find the value(s) of a.

(e) Solve the inequality $\frac{1}{x} \le \frac{1}{x + 1}$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to show that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n. **4**

(b) The polynomial $P(x) = x^4 - 3x^3 + ax^2 + bx - 6$ leaves a remainder of 8 when divided by $(x+1)$. If $(x-3)$ is a factor of $P(x)$, find the values of a and b. **4**

(c) Consider the function $f(x) = \sqrt{4 - x^2}$. A new function is created such that $g(x) = \dfrac{1}{f(x)}$.

(i) Sketch the graph of $y = g(x)$, showing all important features. **2**
(You do NOT need to use calculus to verify any points on the graph.)

(ii) Find the exact area bounded by $y = g(x)$, $x = \sqrt{2}$, $x = \sqrt{3}$ and the x-axis. **3**

(d) Given that ${}^{n}\mathbf{C}_4 = p$ and ${}^{n}\mathbf{C}_5 = q$, use the identities of Pascal's Triangle, or otherwise, to find a simplified expression for ${}^{n+1}\mathbf{C}_{n-5} + {}^{n}\mathbf{C}_{n-4}$ in terms p and q. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The letters of the word PERSEVERE are arranged in a row.

(i) How many different arrangements of the letters are possible? **1**

(ii) What is the probability that an arrangement will have all E's together in one group and all the R's together in one group? **2**

(b) If $2\sin\left(\theta - \frac{\pi}{3}\right) = \cos\left(\theta - \frac{\pi}{3}\right)$ express $\tan\theta$ in the form $a + b\sqrt{c}$ where a, b and c are integers. **3**

(c) Consider $f(x) = 4\sin^2\left[2\left(x + \frac{\pi}{8}\right)\right] - 2$ for $0 < x < \frac{\pi}{4}$.

(i) Verify that $f(x)$ could be a continuous probability density function. **3**

(ii) Find the mode of the distribution. **2**

(d) The curve $y = -\log_e(x + 1)$, the y-axis and the line $y = 1$ enclose a region in the number plane.

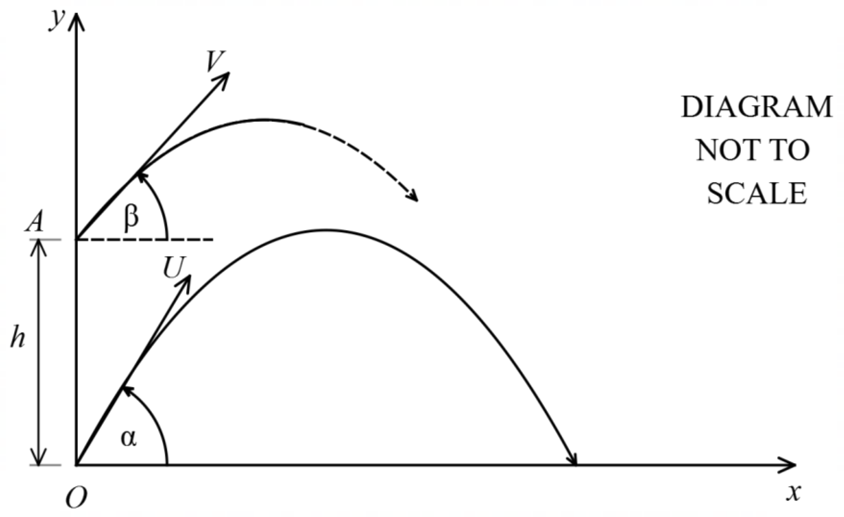
(i) Sketch the region, clearly showing any asymptotes and intercepts with axes. **1**

(ii) The region is rotated about the y-axis to form a solid of revolution. Find the volume, giving your answer correct to 3 significant figures. **3**

End of Question 13

Question 14 (14 marks) Use a SEPARATE writing booklet.

(a) An object is projected on level ground from the origin, O, with initial speed U m/s at an angle of elevation of α. At the same instant another object is projected from a point A which is h m above the origin. The second object is projected with initial speed V m/s at an angle of elevation of β, where $\beta < \alpha$.



(i) The parametric equations of displacement for the object projected from the origin, O, are **1**

$$x = Ut\cos\alpha \text{ and } y = Ut\sin\alpha - \frac{1}{2}gt^2$$

(DO NOT PROVE THESE)

Write down the two parametric equations of displacement for the object projected from A.

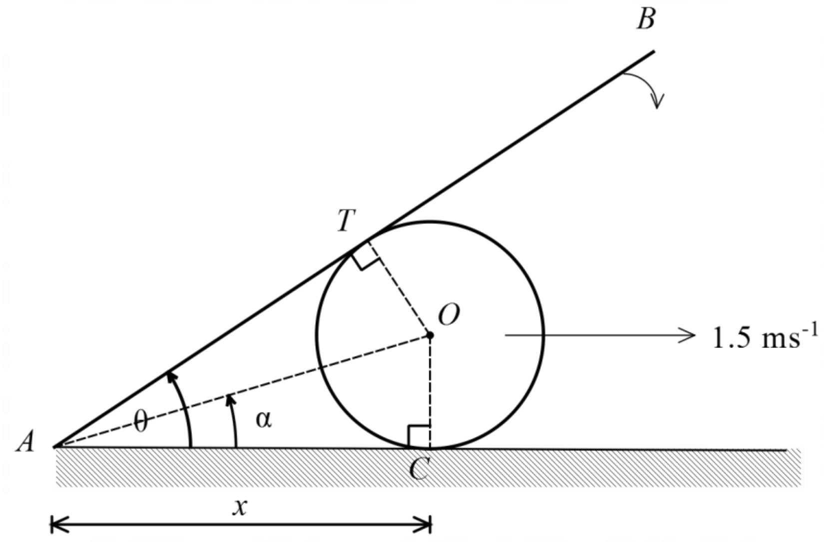
(ii) If the two objects collide T seconds after they are projected, show that $T = \dfrac{h\cos\beta}{U\sin(\alpha - \beta)}$. **4**

Question 14 continues on page 12

Question 14 (continued)

(b) A rod AB is leaning on a metal disc of radius 1 m, as shown in the diagram below. The centre of the disc is at O.

One end of the rod, A, is fixed to the ground, while the disc is tangential to the ground at C, such that $\angle ACO = \angle ATO = 90°$.



The disc rolls horizontally to the right with a constant speed of 1.5 ms^{-1}. As the disc rolls the end of the rod in the air, B, descends to the ground.

Let $\angle OAC = \alpha$, $\angle BAC = \theta$ and $AC = x$.

(i) Write an expression for $\tan\alpha$ in terms of x. **1**

(ii) Hence, or otherwise, find the rate of change of θ in radians per second when $x = 4\text{ m}$, giving your answer correct 3 decimal places. **4**

Question 14 continues on page 13

(c) A tiny pond is initially unpolluted and holds 450 litres of water. There is an input stream with a concentration of 5 g/L of pollutant flowing into the pond at a rate of 30 L/day. Moreover, there is an output stream with a volumetric flow rate of 30 L/day.

Let m be the mass (in grams) of pollutant in the pond and t be the number of days after the input stream first begins to flow into the pond.

(i) Show that a differential equation to model the mass of pollutant after t days is given by $\frac{dm}{dt} = 150 - \frac{m}{15}$. **1**

(ii) What is the concentration of the pollutant, m, in the pond after 10 days? **3**

End of Question 14

End of Paper

TRINITY GRAMMAR SCHOOL

Mathematics Department

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2024

SOLUTIONS

Year 12

Mathematics Extension 1

TRIAL EXAMINATION

ASSESSMENT TASK 4

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General Instructions

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Total marks: 40

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II – 60 marks (pages 6 – 12)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section.

- **Year 12 HSC Assessment Weighting:** 30%

- ***Shade completely only ONE answer for each question.***

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D
6	A	B	C	D
7	A	B	C	D
8	A	B	C	D
9	A	B	C	D
10	A	B	C	D

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

---

1 Which of the following vectors are perpendicular? $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

A. $\underset{\sim}{u} = 3\underset{\sim}{i} - 7\underset{\sim}{j},\ \underset{\sim}{v} = \frac{7}{3}\underset{\sim}{i} - \underset{\sim}{j}$

B. $\underset{\sim}{u} = \frac{2}{5}\underset{\sim}{i} - 2\underset{\sim}{j},\ \underset{\sim}{v} = \frac{7}{3}\underset{\sim}{i} - \underset{\sim}{j}$

C. $\underset{\sim}{u} = \frac{9}{2}\underset{\sim}{i} + \frac{7}{2}\underset{\sim}{j},\ \underset{\sim}{v} = -\frac{2}{9}\underset{\sim}{i} - \frac{1}{7}\underset{\sim}{j}$

D. $\underset{\sim}{u} = 14\underset{\sim}{i} + \underset{\sim}{j},\ \underset{\sim}{v} = \frac{1}{2}\underset{\sim}{i} - 7\underset{\sim}{j}$

2 What is the solution to the inequality $|2-5x| < 7$?

A. $x < -1$ or $x > \frac{9}{5}$

B. $-1 < x < \frac{9}{5}$

C. $x < -\frac{9}{5}$ or $x > 1$

D. $-\frac{9}{5} < x < 1$

$2 - 5x > -7$ or $2 - 5x < 7$

$-5x > -9$ $\quad$ $-5x < 5$

$x < \frac{9}{5}$ $\quad$ $x > -1$

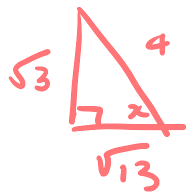
3 What is the value of $\sin 2x$ given that $\sin x = \frac{\sqrt{3}}{4}$ given x is obtuse angled?

A. $-\frac{\sqrt{39}}{8}$

B. $-\frac{\sqrt{3}}{2}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{\sqrt{39}}{8}$



$\sin 2x = 2\sin x \cos x$

$= 2 \cdot \frac{\sqrt{3}}{4} \cdot -\frac{\sqrt{13}}{4}$

4 The radius of a spherical balloon is expanding at a constant rate of 1.3 cms^{-1}. What is the rate of change of the surface area of the balloon when its radius is 6.3 cm?

A. 68.61 cm^2s^{-1}

B. 158.34 cm^2s^{-1}

C. 205.84 cm^2s^{-1}

D. 498.76 cm^2s^{-1}

$\frac{dr}{dt} = 1.3$

$S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = 1.3 \times 8 \times \pi \times 6.3$

5 Using the digits 0, 1, 2, 3, 4 and 5, how many odd numbers between 2000 and 5000 can be formed if no digits are repeated?

A. 48

B. 96

C. 120

D. 360

$2: 1 \times 4 \times 3 \times 3$

$3: 1 \times 4 \times 3 \times 2$

$4: 1 \times 4 \times 3 \times 3$

6 Consider the integral:

$$\int_0^k \frac{2}{8 + (2x)^2}\, dx = \frac{\pi}{8\sqrt{2}}$$

What is the value of k?

A. $\frac{1}{\sqrt{3}}$

B. 1

C. $\sqrt{2}$

D. $\sqrt{3}$

$$\left[\frac{1}{2\sqrt{2}} \cdot \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^k = \frac{\pi}{8\sqrt{2}}$$

$$\tan^{-1}\left(\frac{k}{\sqrt{2}}\right) - \tan^{-1} 0 = \frac{\pi}{4}$$

$$\tan\frac{\pi}{4} = \frac{k}{\sqrt{2}}$$

$$1 = \frac{k}{\sqrt{2}}$$

7 The polynomial $x^2 + x + 2$ is a factor of the polynomial $x^4 + 4x^2 + x + a$.
What is the value of a?

A. -6

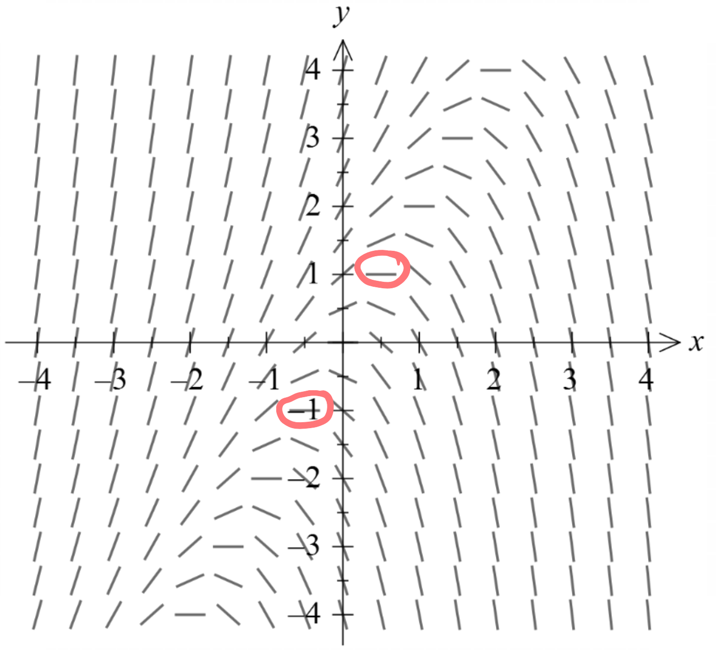
B. -3

C. 3

D. 6

$$
\begin{array}{r}
x^2 - x + 3 \\
x^2 + x + 2 \overline{)\, x^4 + \quad + 4x^2 + x + a} \\
x^4 + x^3 + 2x^2 - \\
\hline
-x^3 + 2x^2 + x \\
-x^3 - x^2 - 2x - \\
\hline
3x^2 + 3x + a \\
3x^2 + 3x + 6
\end{array}
$$

8 The direction field for a differential equation is shown below.



Which of the following equations could give this direction field?

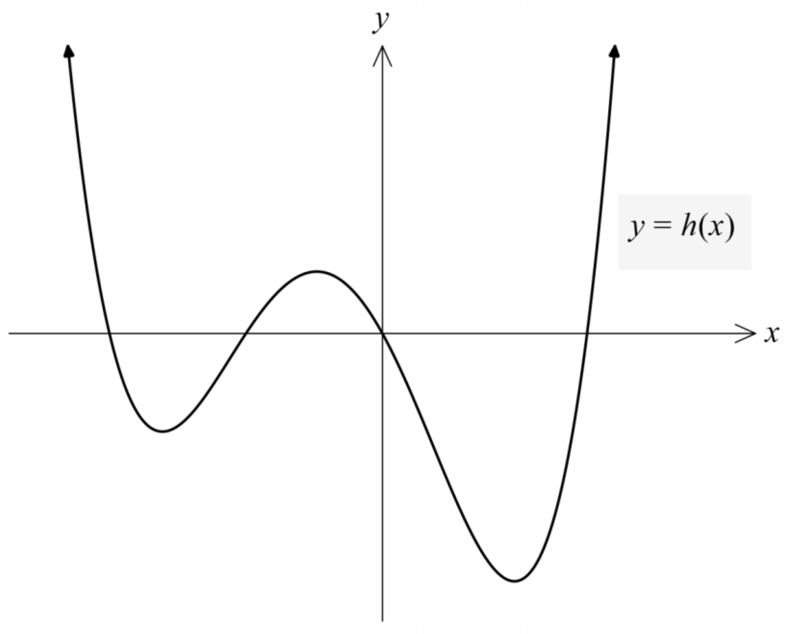
A. $y' = y - 2x$

B. $y' = y + 2x$

C. $y' = y + x$

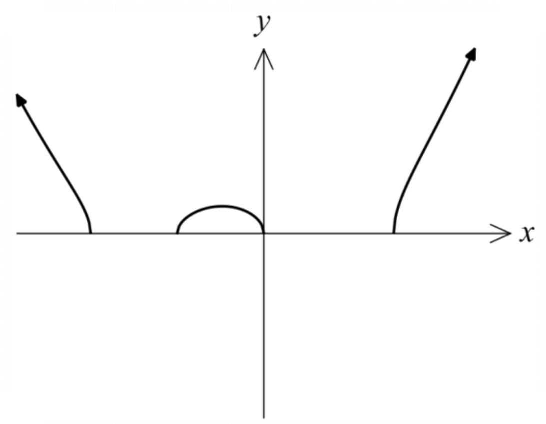
D. $y' = x - y$

9 The graph of $y = h(x)$ is shown below.

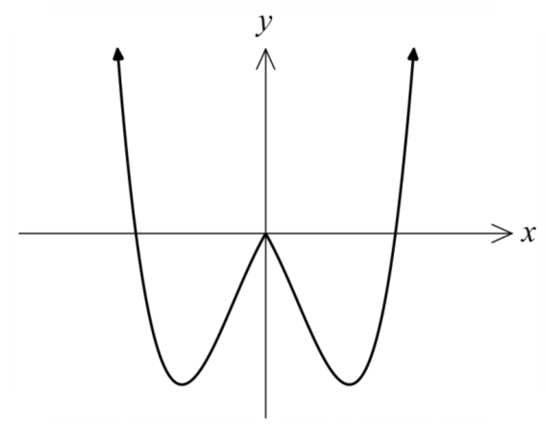


Which graph below shows $y = |h(x)|$?

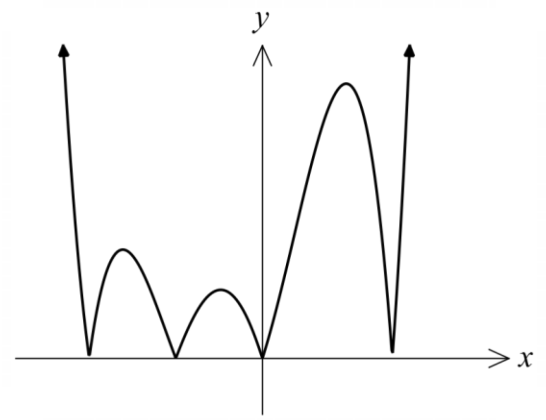
A.



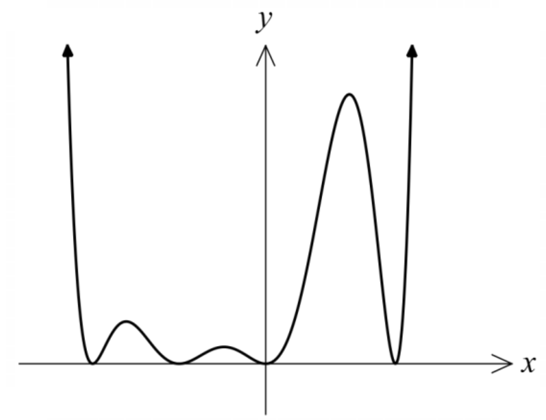
B.



C.



D.



10 The graph of the function $y = \sin^{-1}\left(\frac{1}{2}(x-2)\right)$ is to be transformed by a translation by 1 unit to the left, then a horizontal dilation with a scale factor of 2.

What is the domain of the resultant function?

A. $-1 \le x \le 7$

B. $-2 \le x \le 6$

C. $1 \le x \le 3$

D. $-\frac{1}{2} \le x \le \frac{3}{2}$

$\sin^{-1} x$ -1 to 1

stretch -2 to 2

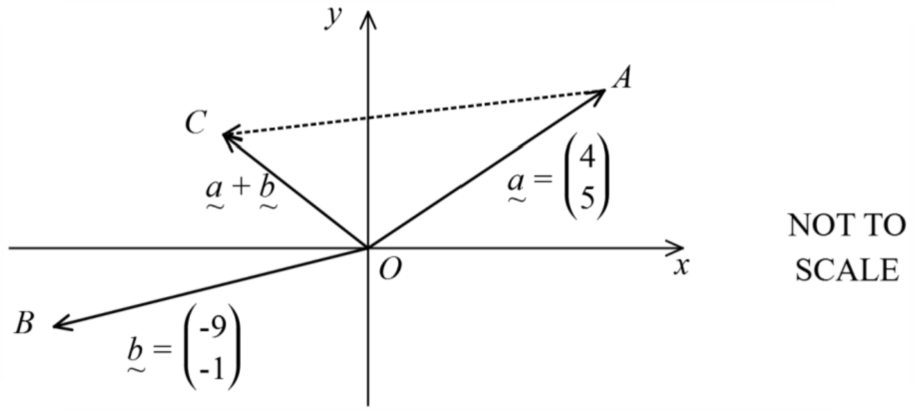
$\sin^{-1}\left(\frac{1}{2}(x-2)\right)$ shift 0 to 4

then: -1 to 3

-2 to 6

Question 11 (16 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the vectors $\underset{\sim}{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} -9 \\ -1 \end{pmatrix}$ and the resultant vector $\underset{\sim}{a} + \underset{\sim}{b}$.



(i) Show that ΔAOC, formed by the vectors, is an isosceles triangle **2**

$|\underset{\sim}{a}| = \sqrt{4^2+5^2}$
$= \sqrt{41}$

1 mark: one correct magnitude OR $\underset{\sim}{a}+\underset{\sim}{b}$ correct.

$\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

2 marks: all correct including final statement.

$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{(-5)^2+4^2}$
$= \sqrt{41}$

$\therefore \Delta AOC$ is isosceles.

(ii) Find the size of $\angle OAC$. **1**

$\underset{\sim}{a} \cdot (\underset{\sim}{a}+\underset{\sim}{b}) = |\underset{\sim}{a}||\underset{\sim}{a}+\underset{\sim}{b}| \cos(A\hat{O}C)$
$-20 + 20 = \sqrt{41}\sqrt{41} \cos A\hat{O}C$
$0 = \cos A\hat{O}C$
$\therefore A\hat{O}C = 90°$
$\therefore$ since isosceles $O\hat{A}C = 45°$ (1)

(b) (i) Rewrite $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$, where $0 \le \alpha \le \frac{\pi}{2}$, $R > 0$. **2**

$R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

$\therefore R = \sqrt{1^2 + (\sqrt{3})^2}$ $\quad\quad \tan\alpha = \sqrt{3}$

$= 2$ $\quad\quad \alpha = \frac{\pi}{3}$

$\therefore \cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$

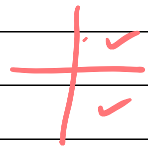
① correct R $\quad\quad$ ① correct angle

(ii) Hence, or otherwise, solve $\cos x - \sqrt{3}\sin x = 1$ for $0 \le x \le 2\pi$. **2**

$\cos x - \sqrt{3}\sin x = 1$ $\quad\quad 0 \le x \le 2\pi$

$2\cos\left(x + \frac{\pi}{3}\right) = 1$ $\quad\quad \frac{\pi}{3} \le x + \frac{\pi}{3} \le \frac{7\pi}{3}$

$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$



$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$x = 0, \frac{4\pi}{3}, 2\pi$

① for $\frac{\pi}{3}$

① for all three angles

(c) Evaluate $\int_1^{\sqrt{2}} \frac{x}{\sqrt{4-x^2}}\,dx$ using the substitution $u = 4 - x^2$. **3**

$u = 4 - x^2$ $\quad$ $x = \sqrt{2}$ $\quad$ $u = 2$

$du = -2x\,dx$ $\quad$ $x = 1$ $\quad$ $u = 3$

① change bounds

$\therefore I = -\frac{1}{2}\int_1^{\sqrt{2}} \frac{-2x}{\sqrt{4-x^2}}\,dx$

$= -\frac{1}{2}\int_3^2 \frac{1}{\sqrt{u}}\,du$

① correct subst. must not mix variables

$= -\frac{1}{2}\left[2u^{\frac{1}{2}}\right]_3^2$

$= -\frac{1}{2}\left(2\sqrt{2} - 2\sqrt{3}\right)$

$= \sqrt{3} - \sqrt{2}$

① answer.

(d) The coefficient of x in the expansion of $\left(x + \frac{1}{ax^2}\right)^7$ is $\frac{7}{3}$. **3**

Find the value(s) of a.

$T_{n+1} = \binom{7}{n}(x)^{7-n}\left(\frac{1}{ax^2}\right)^n$

$= \binom{7}{n}\frac{1}{a^n}\cdot x^{7-n}\cdot x^{-2n}$

① general term

For x term

$x^{7-n}\cdot x^{-2n} = x^1$

$\therefore 7 - 3n = 1$

$3n = 6$

$n = 2$

① correct power

For co-efficient

$\binom{7}{2}\frac{1}{a^2} = \frac{7}{3}$

$\frac{21}{a^2} = \frac{7}{3}$

$a^2 = 9$

$\therefore a = \pm 3$

① correct values of a

(e) Solve the inequality $\dfrac{1}{x} \le \dfrac{1}{x+1}$. **3**

$$\frac{1}{x} \le \frac{1}{x+1} \qquad x \ne 1, 0$$

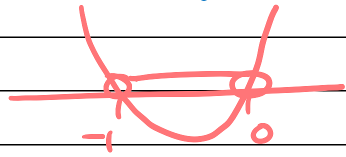
$$x(x+1)^2 \le x^2(x+1)$$

① for attempt at to solve - critical - square - graph.

$$x(x+1)^2 - x^2(x+1) \le 0$$

$$x(x+1)(x+1-x) \le 0$$

$$x(x+1) \le 0$$



$$\therefore \quad -1 < x < 0$$

① for correct inequality signs

① for correct values.

Question 12 (~~14~~ 15 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to show that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n. **4**

For $n=1$

$9^{1+2} - 4^1$ ①

$= 9^3 - 4$

$= 729 - 4$

$= 725$ which is divisible by 5

$\therefore$ The statement is true for $n=1$

Assume true for $n=k$, $k \in \mathbb{Z}^+$

$9^{k+2} - 4^k = 5M \quad M \in \mathbb{Z}$ ①

$4^k = 9^{k+2} - 5M$

Prove true for $n=k+1$

$9^{(k+1)+2} - 4^{k+1}$

$= 9^{k+3} - 4.4^k$

$= 9^{k+3} - 4(9^{k+2} - 5M)$ from assumption

$= 9.9^{k+2} - 4.9^{k+2} + 20M$ ① use of assumption

$= 5.9^{k+2} + 20M$

$= 5(9^{k+2} + 4M)$ ① correct solution

which is divisible by 5 as $(9^{k+2} + 4M)$ is an integer.

$\therefore$ When the statement is true for $n=k$ it is true for $n=k+1$

$\therefore$ since the statement is true for $n=1$ it is true for $n=2$ & so on for all positive integers.

(b) The polynomial $P(x) = x^4 - 3x^3 + ax^2 + bx - 6$ leaves a remainder of 8 when divided by $(x+1)$. If $(x-3)$ is a factor of $P(x)$, find the values of a and b. **4**

$P(-1) = 8 = (-1)^4 - 3(-1)^3 + a(-1)^2 + b(-1) - 6$

$8 = -2 + a - b$

$10 = a - b$ 1 use of remainder theorem

$b = a - 10$ (1)

$P(3) = 0 = (3)^4 - 3(3)^3 + a(3)^2 + 3b - 6$

$0 = -6 + 9a + 3b$

$6 = 9a + 3b$

$2 = 3a + b$ 1 use of factor theorem

subst (1) into (2)

$2 = 3a + a - 10$

$4a = 12$

$a = 3$ 1 answer

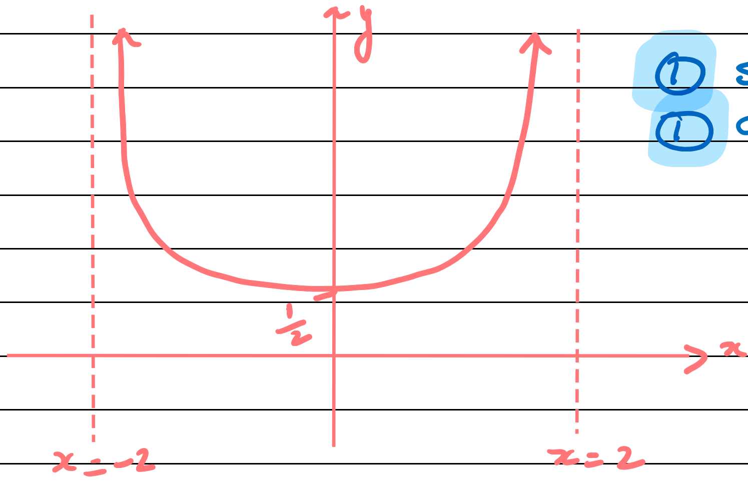
$b = 3 - 10$

$b = -7$ 1 answer.

(c) Consider the function $f(x)=\sqrt{4-x^2}$. A new function is created such that $g(x)=\dfrac{1}{f(x)}$.

(i) Sketch $g(x)$, showing all important features. $\cancel{3}$ 2
(You do NOT need to use calculus to verify any points on the graph.)

$$g(x) = \frac{1}{\sqrt{4-x^2}}$$



① shape
① asymptotes and y-intercept

(ii) Find the exact area bound by $g(x)$, $x=\sqrt{2}$, $x=\sqrt{3}$ and the x-axis. **3**

$$A = \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}} \quad ①$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}\ u^2$$

① ①

(d) Given that ${}^{n}\mathbf{C}_4 = p$ and ${}^{n}\mathbf{C}_5 = q$, use the identities of Pascal's Triangle, or otherwise, to find a simplified expression for ${}^{n+1}\mathbf{C}_{n-5} + {}^{n}\mathbf{C}_{n-4}$ in terms p and q. **2**

$${}^{n+1}C_5 = {}^{n}C_4 + {}^{n}C_5$$
$$= p+q$$

$${}^{n+1}C_{(n+1-5)} = {}^{n+1}C_5 = {}^{n+1}C_{n-4} = p+q$$

$${}^{n}C_{n-4} = {}^{n}C_4 = p$$ (1) for one identity

$$\therefore {}^{n+1}C_{n-5} + {}^{n}C_{n-4} = p+q+p$$
$$= 2p+q$$ (1) correct

ERROR:

$${}^{n}C_{n-4} = {}^{n}C_4 = p$$ (1) still possible.

$${}^{n}C_5 = \frac{n!}{5!(n-5)!} = q$$

$${}^{n+1}C_{n-5} = \frac{(n+1)!}{(n-5)!(n+1-(n-5))!}$$ (1)

$$= \frac{(n+1)!}{6!(n-5)!}$$

$$= \frac{(n+1)\cdot n!}{6\times 5!\cdot(n-5)!}$$

$$= \left(\frac{n+1}{6}\right)q.$$

$$\therefore {}^{n+1}C_{n-5} + {}^{n}C_{n-4} = \left(\frac{n+1}{6}\right)q + p.$$

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The letters of the word PERSEVERE are arranged in a row.

(i) How many different arrangements of the letters are possible? **2**

$\text{No.} = \dfrac{9!}{4!\,2!}$ ① for 9! and a division

$= 7560$ ① correct answer

(ii) What is the probability that an arrangement will have all E's together in one group and all the R's together in one group? **2**

No. with E + R = 5! P E R S V

$= 120$

$\therefore P = \dfrac{120}{7560}$

$= \dfrac{1}{63}$ ① for answer

(b) If $2\sin\left(\theta - \dfrac{\pi}{3}\right) = \cos\left(\theta - \dfrac{\pi}{3}\right)$ express $\tan\theta$ in the form $a + b\sqrt{c}$, where a, b, and c are integers. **3**

$2\sin\left(\theta - \frac{\pi}{3}\right) = \cos\left(\theta - \frac{\pi}{3}\right)$

$\therefore 2\tan\left(\theta - \frac{\pi}{3}\right) = 1$ ① use of one trig identity

$2 \cdot \dfrac{\tan\theta - \tan\frac{\pi}{3}}{1 + \tan\theta\tan\frac{\pi}{3}} = 1$ ① use of sum

$2\tan\theta - 2\sqrt{3} = 1 + \sqrt{3}\tan\theta$

$2\tan\theta - \sqrt{3}\tan\theta = 1 + 2\sqrt{3}$

$\tan\theta = \dfrac{1 + 2\sqrt{3}}{2 - \sqrt{3}}$

$= 8 + 5\sqrt{3}$ ① correct.

(c) Consider $f(x) = 4\sin^2\left[2\left(x+\frac{\pi}{8}\right)\right] - 2,\ 0 < x < \frac{\pi}{4}$.

(i) Verify that $f(x)$ could be a probability density function. **3**

For a pdf $A = 1$

$4\sin^2\left(2\left(x+\frac{\pi}{8}\right)\right) = 4\left(\frac{1}{2}\left(1-\cos\left(4\left(x+\frac{\pi}{8}\right)\right)\right)\right)$

$= 2 - 2\cos\left(4x+\frac{\pi}{2}\right)$ 1

$\therefore A = \int_0^{\pi/4} 2 - 2\cos\left(4x+\frac{\pi}{2}\right) - 2\, dx$

$= \int_0^{\frac{\pi}{4}} -2\cos\left(4x+\frac{\pi}{2}\right) dx$

$= \left[-\frac{1}{2}\sin\left(4x+\frac{\pi}{2}\right)\right]_0^{\frac{\pi}{4}}$ 1

$= -\frac{1}{2}\sin\left(\frac{3\pi}{2}\right) - \left(-\frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right)$

$= \left(-\frac{1}{2}\cdot -1\right) - \left(-\frac{1}{2}\cdot 1\right)$

$= \frac{1}{2} + \frac{1}{2} = 1$ 1 $\therefore$ $f(x)$ is a pdf.

(ii) Find the mode of the distribution. **2** ~~1~~

max occurs when

$\sin^2\left(2x+\frac{\pi}{4}\right) = 1$

$\sin\left(2x+\frac{\pi}{4}\right) = 1$ or $\sin\left(2x+\frac{\pi}{4}\right) = -1$ 1

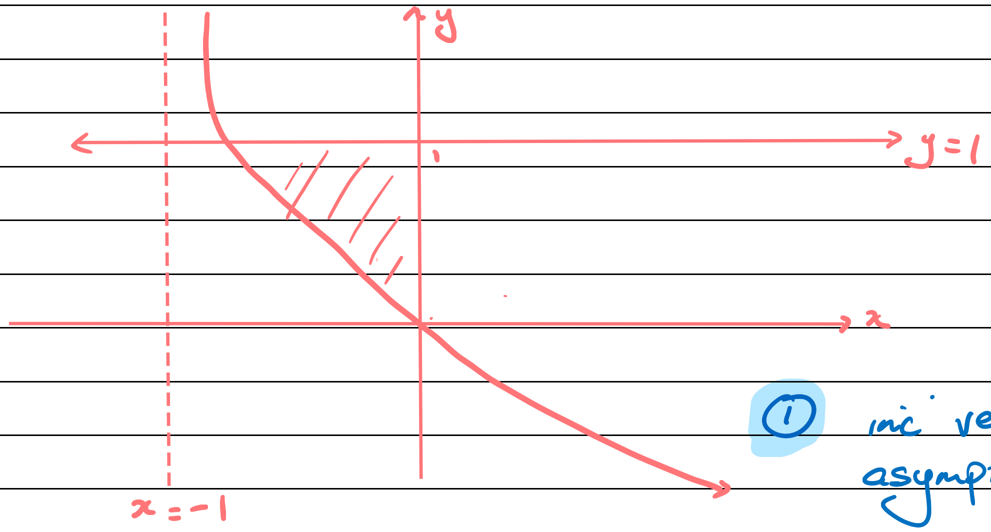
$2x + \frac{\pi}{4} = \frac{\pi}{2}$ | $2x + \frac{\pi}{4} = \frac{3\pi}{2}$

$2x = \frac{\pi}{4}$ | $2x = \frac{5\pi}{4}$

$x = \frac{\pi}{8}$ 1 | $x = \frac{5\pi}{8}$ outside domain

(d) The curve $y = -\log_e(x+1)$, the y-axis and the line $y = 1$ enclose a region in the number plane.

(i) Sketch the region, clearly showing any asymptotes and intercepts with axes. **1**



1 inc vertical asymptote.

(ii) The region is rotated about the y-axis to form a solid of revolution. Find the volume, giving your answer correct to 3 significant figures. **3**

$y = -\ln(x+1)$

$-y = \ln(x+1)$

$x+1 = e^{-y}$

$x = e^{-y} - 1$ 1

$x^2 = (e^{-y}-1)^2$

$= e^{-2y} - 2e^{-y} + 1$

$V = \pi\int_0^1 x^2\,dy$

$= \pi\int_0^1 e^{-2y} - 2e^{-y} + 1\,dy$

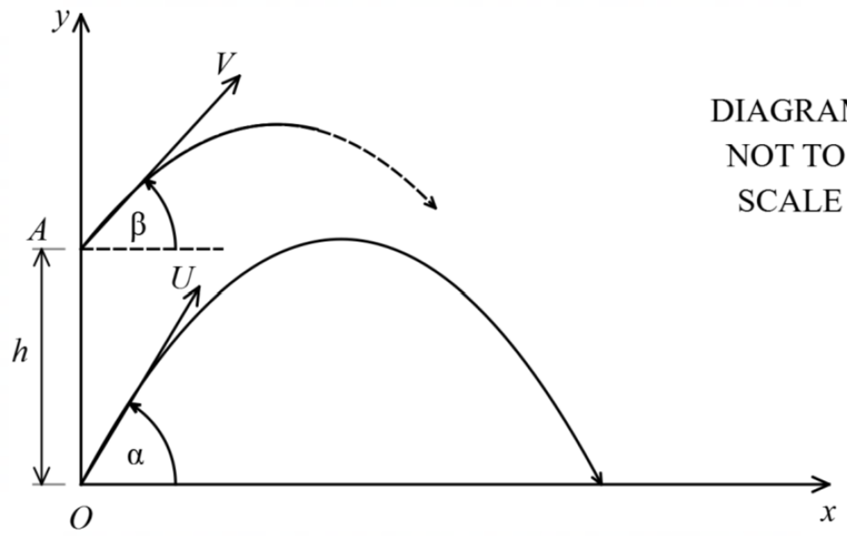
$= \pi\left[-\frac{1}{2}e^{-2y} + 2e^{-y} + y\right]_0^1$ 1

$= \pi\left(-\frac{1}{2}e^{-2} + 2e^{-1} + 1 - \left(-\frac{1}{2} + 2\right)\right)$

$= 0.5280742207 = 0.528$ to 3s.f. 1

Question 14 (14 marks) Use a SEPARATE writing booklet.

(a) An object is projected on level ground from the origin, O, with initial speed U m/s at an angle of elevation of α. At the same instant another object is projected from a point A which is h m above the origin. The second object is projected with initial speed V m/s at an angle of elevation of β, where $\beta < \alpha$.



(i) The parametric equations of displacement for the object projected from the origin, O, are **1**

$$x = Ut\cos\alpha \text{ and } y = Ut\sin\alpha - \frac{1}{2}gt^2$$

(DO NOT PROVE THESE)

Write down the two parametric equations of displacement for the object projected from A.

From A:

$x = Vt\cos\beta$ $\quad$ $y = Vt\sin\beta - \frac{1}{2}gt^2 + h$

1

(ii) If the objects collide T seconds after they are projected, show that $T = \dfrac{h\cos\beta}{U\sin(\alpha-\beta)}$. **4**

When collide at $t = T$, both x & y are equal.

For x:

$VT\cos\beta = UT\cos\alpha$ ① for equating x or y

$\therefore \quad V = \dfrac{U\cos\alpha}{\cos\beta}$ ①

For y:

$VT\sin\beta - \frac{1}{2}gT^2 + h = UT\sin\alpha - \frac{1}{2}gT^2$

$h = UT\sin\alpha - VT\sin\beta$ ①

$h = T(U\sin\alpha - V\sin\beta)$

$= T\left(U\sin\alpha - \dfrac{U\cos\alpha\sin\beta}{\cos\beta}\right)$

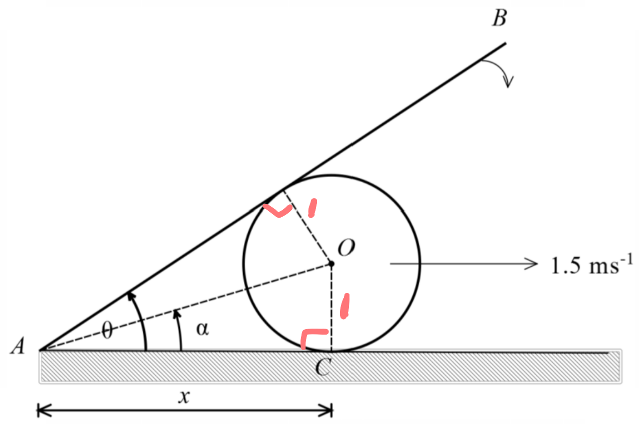
$h = UT\left(\dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\beta}\right)$ ①

$h = UT\dfrac{\sin(\alpha-\beta)}{\cos\beta}$

$\dfrac{h\cos\beta}{U\sin(\alpha-\beta)} = T$

QED

Q 14(b)



(i) Write a relationship between $\angle OAC$, OC and AC. **1**

$\tan\alpha = \frac{1}{x}$ or $\tan\alpha = \frac{OC}{AC}$ ①

(ii) Hence, or otherwise, find the rate of change of θ in radians per second when $x = 4$ m, giving your answer correct 3 decimal places. **4**

$\theta = 2\alpha$ (bisected angle in a kite)

$\frac{d\theta}{dt} = \frac{dx}{dt} \times \frac{d\theta}{dx}$ $\qquad \frac{dx}{dt} = 1.5$ m/s

$\tan\alpha = \frac{1}{x}$

$\alpha = \tan^{-1}\left(\frac{1}{x}\right)$

$2\alpha = 2\tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \theta = 2\tan^{-1}(x^{-1})$ ① expression for θ

$\frac{d\theta}{dx} = \frac{-2x^{-2}}{1+(x^{-1})^2}$

$= \frac{-2}{x^2+1}$ ① $\frac{dx}{d\theta}$

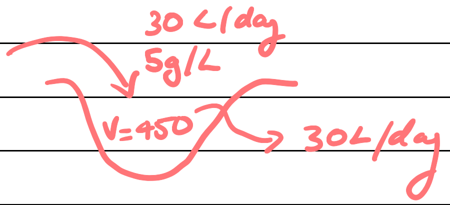
$\therefore \frac{d\theta}{dt} = 1.5 \times \frac{-2}{x^2+1}$ ① attempt at related rate

at $x = 4$ ① for answer

$\frac{d\theta}{dt} = 1.5 \times \frac{-2}{4^2+1} = -0.176$ rads/sec

Q14(c)

(i) Show that a differential equation to model the mass of pollutant after t days is $\frac{dm}{dt} = 150 - \frac{m}{15}$. **1**



$$\frac{dm}{dt} = \text{rate in} - \text{rate out}$$

$$= \frac{5g}{L} \times \frac{30L}{day} - \frac{m}{450L} \times \frac{30L}{day} \quad ①$$

$$\frac{dm}{dt} = 150 - \frac{m}{15}$$

(ii) What is the concentration of the pollutant in the pond after 10 days? **3**

$$\frac{dm}{dt} = 150 - \frac{1}{15}m$$

$$\frac{dm}{dt} = \frac{2250 - m}{15}$$

$$\frac{1}{2250 - m}\, dm = \frac{1}{15}\, dt$$

$$\int \frac{1}{2250 - m}\, dm = \int \frac{1}{15}\, dt$$

$$-\ln|2250 - m| = \frac{t}{15} + c$$ ① correct integral

at $t = 0$ $m = 0$

$$-\ln|2250| = 0 + c$$

$$c = -\ln|2250|$$

$$\therefore \quad -\ln|2250 - m| = \frac{t}{15} - \ln|2250|$$

at $t=10$

$$-\ln|2250-m| = \frac{10}{15} - \ln|2250|$$

$$\ln|2250-m| = \ln 2250 - \frac{2}{3}$$

$$2250-m = e^{\ln 2250 - 2/3}$$

$$2250-m = 2250 \cdot e^{-2/3}$$

$$m = 2250 - 2250 \cdot e^{-2/3}$$
$$= 1094.811 \text{ g.}$$

1 correct m.

$$\therefore \text{Concentration} = \frac{1094.811}{450}$$
$$= 2.4329\ldots$$
$$= 2.43 \text{ g/L.}$$

1 correct C.