Year 11 Mathematics Extension Task 2
Thursday 1st August 2019
Time Allowed: 70 minutes plus 5 minutes reading time

Instructions:
a) Write all your answers on the appropriate page in the answer booklet. Extra paper is available
b) Show all necessary working.
c) Marks may be deducted for careless or badly arranged work.
d) Use black or blue pen to write your solutions. Black pen is preferred.

1. Define the following terms that are used when describing polynomials;

(i) coefficients *1*

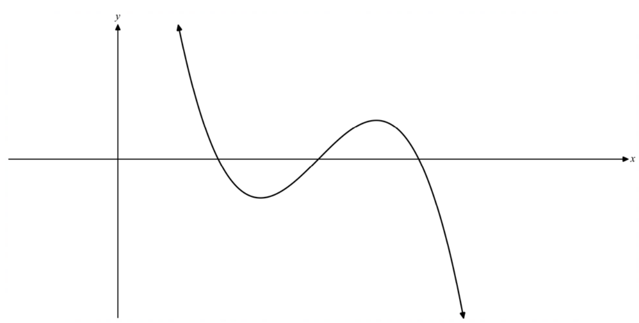
(ii) degree *1*

(iii) leading term *1*

(iv) monic polynomial *1*

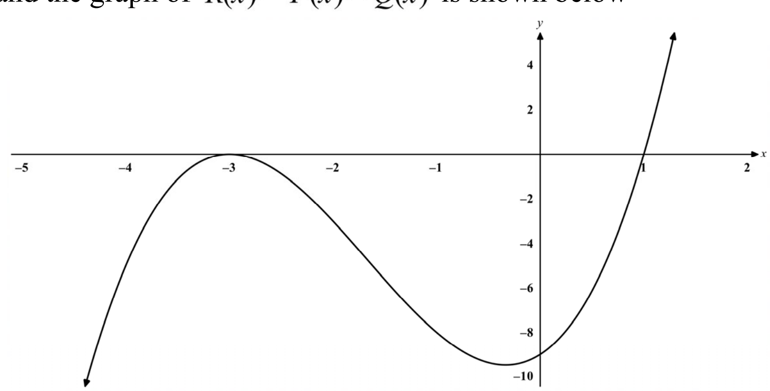
(v) roots *1*

2. Consider the graph shown below *3*



Give at least THREE reasons why the equation of the graph could **NOT** be $y = x^3 + 7x^2 - 11x$

3. The two polynomials $P(x) = 2x^3 + 7x^2 - 3x + 12$ and $Q(x) = x^3 + 2x^2 - 6x + 21$ are solved simultaneously and the graph of $R(x) = P(x) - Q(x)$ is shown below *3*



List at least THREE facts that this graph tells us about the points of intersection of $y = P(x)$ and $y = Q(x)$

4. $P(x) = 5x^3 - 7x^2 + 8x - 3$ and $A(x) = x + 1$

(i) Using the division algorithm, divide $P(x)$ by $A(x)$ and write the result in the form $P(x) = A(x)Q(x) + R(x)$ *3*

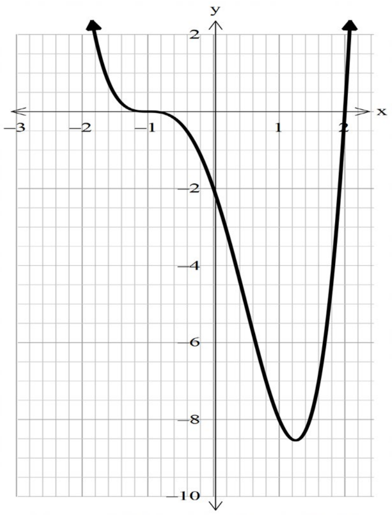
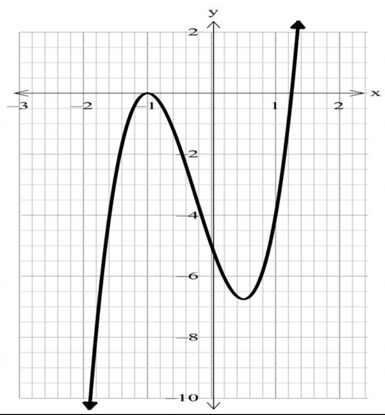
(ii) Use the remainder theorem to confirm your answer for $R(x)$ *1*

5.	(i) Use the factor theorem to find one factor of $P(x) = 6x^3 - 23x^2 + 11x + 12$ (ii) Hence factorise $P(x)$ completely	***1*** ***2***
6.	(i) On the axes provided in the answer booklet, sketch a monic polynomial that has a multiple root and a single root, neither of which is equal to zero, clearly labelling all intercepts. (ii) On the axes provided in the answer booklet, sketch the gradient function of your polynomial (iii) Give a possible equation for your polynomial function (iv) Using calculus, find the equation of your gradient function.	***3*** ***3*** ***1*** ***2***
7.	$P(x)$ is a polynomial of degree 3. When $P(x)$ is divided by $(x + 3)$, the remainder is 2. When $P(x)$ is divided by $(x - 2)$, the remainder is -8 . Find the remainder when $P(x)$ is divided by $x^2 + x - 6$	***3***
8.	α, β and γ are the roots of $x^3 + 2x^2 - 3x + 4 = 0$, find; (i) $\alpha + \beta + \gamma$ (ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ (iii) $\alpha\beta\gamma$ (iv) $\alpha^2 + \beta^2 + \gamma^2$ (v) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ (vi) $\alpha^3 + \beta^3 + \gamma^3$ (vii) $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$ (viii) a polynomial equation with roots α^2, β^2 and γ^2	 ***1*** ***1*** ***1*** ***2*** ***2*** ***2*** ***2*** ***2***
9.	The cubic equation $x^3 - 15x + 4 = 0$ has three roots, two of which are reciprocals of each other. (i) Find the third root (ii) Verify that the other two roots are reciprocals of each other	***1*** ***3***
10.	If the equation $x^3 - 3kx + l = 0$ has a double root, prove that $l^2 = 4k^3$	***3***
11.	It is known that a polynomial has the following properties; • odd function • monic • coefficients are rational • degree is 5 • $P(1 + \sqrt{2}) = 0$ Find the polynomial that satisfies all of these properties. Give your answer in factored form, where no factor has a degree greater than 2.	***3***

END OF EXAM

BAULKHAM HILLS HIGH SCHOOL
YEAR 11 MATHEMATICS EXTENSION ASSESSMENT TASK 2 2019 SOLUTIONS

Solution	Marks	Comments
1 (i) the coefficients of a polynomial are the constant multipliers of the algebraic terms	1	**1 mark** • Correct definition
1 (ii) the degree of a polynomial is the highest index	1	**1 mark** • Correct definition
1 (iii) the leading term of a polynomial is the algebraic term with the highest index	1	**1 mark** • Correct definition
1 (iv) a monic polynomial is a polynomial whose leading coefficient is equal to one	1	**1 mark** • Correct definition
1 (v) the roots are the solutions to the polynomial equation	1	**1 mark** • Correct definition
2. Some reasons would be; • The three roots are all positive, thus $\Sigma\alpha > 0$, this is not possible as the given polynomial has $\Sigma\alpha < 0$ • y intercept of the given polynomial is 0, yet the graph does not go through the origin • the leading coefficient of the given polynomial is positive, however the graph behaves like a polynomial whose leading coefficient is negative	3	**1 mark for each correct reason** • Reasons need to be unique, i.e. not the same reason expressed in a different way • Maximum of 3 marks
3. Some facts would be; • There are only two points of intersection • The graphs cut each other at $x = 1$ • The curves are tangential i.e. touch other, at $x = -3$	3	**1 mark for each correct fact** • Facts need to be unique, i.e. not the same fact expressed in a different way • Maximum of 3 marks
4 (i) $\begin{array}{r} 5x^2 - 12x + 20 \\ x+1 \overline{)\,5x^3 - 7x^2 + 8x - 3} \\ \underline{5x^3 + 5x^2} \qquad\qquad \\ -12x^2 + 8x - 3 \\ \underline{-12x^2 - 12x} \qquad \\ 20x - 3 \\ \underline{20x + 20} \\ -23 \end{array}$ $\therefore 5x^3 - 7x^2 + 8x - 3 = (x+1)(5x^2 - 12x + 20) - 23$	3	**3 marks** • Correct solution **2 marks** • Completes the polynomial division **1 mark** • Divides at least one term of $P(x)$ by $A(x)$
4 (ii) $R(x) = P(-1)$ $= 5(-1)^3 - 7(-1)^2 + 8(-1) - 3$ $= -5 - 7 - 8 - 3$ $= -23$	1	**1 mark** • Correct solution, that includes the substitution step
5 (i) $P(3) = 6(3)^3 - 23(3)^2 + 11(3) + 12$ $= -162 - 207 + 33 + 12$ $= 0$ $\therefore (x - 3)$ is a factor	1	**1 mark** • Substitutes a possible zero into the polynomial and obtains zero • Other possibilities are $P\left(-\frac{1}{2}\right)$ or $P\left(\frac{4}{3}\right)$
5 (ii) $P(x) = 6x^3 - 23x^2 + 11x + 12$ $= (x-3)(6x^2 - 5x - 4)$ $= (x-3)(2x+1)(3x-4)$	2	**2 marks** • Factorises completely **1 mark** • Finds the other quadratic factor • Other possibilities are $3x^2 - 13x + 12$ or $2x^2 - 5x - 3$

Solution	Marks	Comments
6 (i) 	**3**	**1 mark for each of these features** • A curve with a labelled single root, not at $x = 0$ • A curve with a labelled multiple root, not at $x = 0$ • Correctly labels their y-intercept
6 (ii) 	**3**	**1 mark for each of these features** • Multiple root is reduced by an order of one • x-intercepts of gradient function match the stationary points of the polynomial function • Gradient function is positive/negative when the polynomial function is increasing/decreasing
6 (iii) $y = (x+1)^3(x-2)$	**1**	**1 mark** • An equation that correctly matches their sketch
6 (iv) $y = (x+1)^3(x-2)$ $\frac{dy}{dx} = (x+1)^3(1) + (x-2)[3(x+1)^2(1)]$ $= (x+1)^2(x+1+3x-6)$ $= (x+1)^2(4x-5)$	**2**	**2 marks** • Gradient function matches the graph drawn **1 mark** • Finds the gradient function
7. As the divisor is a quadratic, then the remainder is a linear function or a constant, i.e. $R(x) = ax + b$ $\therefore P(x) = (x^2 + x - 6)Q(x) + ax + b$ $= (x+3)(x-2)Q(x) + ax + b$ $P(-3) = 2$ $\quad$ $P(2) = -8$ $-3a + b = 2$ $\quad$ $2a + b = -8$ $-3a + b = 2$ $2a + b = -8$ $-5a = 10$ $a = -2$ $\therefore b = -4$ thus the remainder is $-2x - 4$	**3**	**3 marks** • Correct solution **2 marks** • Uses $P(2)$ and $P(-3)$ to find two simultaneous equations **1 mark** • Establishing the remainder could be a linear function
8 (i) $\alpha + \beta + \gamma = -2$	**1**	**1 mark** • Correct answer
8 (ii) $\alpha\beta + \alpha\gamma + \beta\gamma = -3$	**1**	**1 mark** • Correct answer
8 (iii) $\alpha\beta\gamma = -4$	**1**	**1 mark** • Correct answer

Solution	Marks	Comments
8 (iv) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $=(-2)^2-2(-3)$ $=4+6$ $=10$	**2**	**2 marks** • Correct solution **1 mark** • Rearranges the perfect square to obtain a formula for $\alpha^2+\beta^2+\gamma^2$
8 (v) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$ $=\frac{-3}{-4}$ $=\frac{3}{4}$	**2**	**2 marks** • Correct solution **1 mark** • Forms a single fraction that previous answers can be substituted into
8 (vi) If α, be and γ are the roots of $x^3+2x^2-3x+4=0$, then; $\alpha^3+2\alpha^2-3\alpha+4=0$ $\beta^3+2\beta^2-3\beta+4=0$ $\gamma^3+2\gamma^2-3\gamma+4=0$ $\alpha^3+\beta^3+\gamma^3+2(\alpha^2+\beta^2+\gamma^2)-3(\alpha+\beta+\gamma)+12=0$ $\alpha^3+\beta^3+\gamma^3=-2(10)+3(-2)-12$ $=-38$	**2**	**2 marks** • Correct solution **1 mark** • Substitutes α,β and γ into the polynomial, or equivalent • rearranges the perfect cube to obtain a formula for $\alpha^3+\beta^3+\gamma^3$
8 (vii) $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$ $=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)$ $=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$ $=(-3)^2-2(-4)(-2)$ $=9-16$ $=-7$	**2**	**2 marks** • Correct solution **1 mark** • Uses the $\Sigma\alpha^2$ formula to create an expression to substitute known answers into.
8 (viii) $\alpha^2+\beta^2+\gamma^2=10$ $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2=-7$ $\alpha^2\beta^2\gamma^2=(-4)^2=16$ $\therefore$ equation is $x^3-10x^2-7x-16=0$	**2**	**2 marks** • Correct solution **1 mark** • evaluates $\alpha^2\beta^2\gamma^2$ • uses $\Sigma\alpha^2$ and $\Sigma\alpha^2\beta^2$ to create a polynomial
9 (i) $x^3-15x+4=0$ Roots are $\alpha,\frac{1}{\alpha}$ and β Using product of roots $(\alpha)\left(\frac{1}{\alpha}\right)(\beta)=-4$ $\beta=-4$	**1**	**1 mark** • Correct answer

Solution	Marks	Comments
9 (ii) using sum of roots $\alpha + \frac{1}{\alpha} + \beta = 0$ $\alpha + \frac{1}{\alpha} - 4 = 0$ $\alpha^2 - 4\alpha + 1 = 0$ $\alpha = \frac{4 \pm \sqrt{16-4}}{2}$ $= \frac{4 \pm \sqrt{12}}{2}$ $= 2 \pm \sqrt{3}$ Product of the other two roots $= (2+\sqrt{3})(2-\sqrt{3})$ $= 4 - 3$ $= 1$ Since the product of the two roots is one, they must be reciprocals of each other.	3	**3 marks** • Correct solution **2 marks** • Finds the other two roots **1 mark** • Finds the other quadratic factor
10. $P(x) = x^3 - 3kx + l$ $P'(x) = 3x^2 - 3k$ $P'(x) = 0$ $3x^2 - 3k = 0$ $x^2 = k$ $x = \pm\sqrt{k}$ $P(x) = 0$ $(\pm\sqrt{k})^3 - 3k(\pm\sqrt{k}) + l = 0$ $\pm k\sqrt{k} \mp 3k\sqrt{k} + l = 0$ $l = \pm 2k\sqrt{k}$ $l^2 = 4k^2 \times k$ $l^2 = 4k^3$	3	**3 marks** • Correct solution **2 marks** • Finds the possible double roots **1 mark** • Uses the fact that $P'(x) = 0$
11. All odd function s have rotation al symmetry about the origin, thus $f(-x) = -f(x)$ $P(0) = -P(0)$ $2 \times P(0) = 0$ $P(0) = 0 \Rightarrow x$ is a factor As all of the coefficients of the polynomial are real; If $-1-\sqrt{2}$ is a root, its conjugate must also be a root By rotational symmetry $1+\sqrt{2}$ is also a root, and its conjugate must also be a root. The five roots are $0, 1\pm\sqrt{2}, -1\pm\sqrt{2}$ $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ $\alpha + \beta = 2$ $\alpha\beta = -1$ $\therefore x^2 - 2x - 1$ is a factor $\gamma = -1+\sqrt{2}$ and $\delta = -1-\sqrt{2}$ $\gamma + \delta = -2$ $\gamma\delta = -1$ $\therefore x^2 + 2x - 3$ is a factor Thus $P(x) = x(x^2 - 2x - 1)(x^2 + 2x - 1)$	3	**3 marks** • Correct solution **2 marks** • Finds a quadratic factor • Establishes two of the points below **1 mark** • Establishes that x is a factor • Establishes $1-\sqrt{2}$ i.e. conjugate surd is a root • Establishes $\frac{1}{1+\sqrt{2}}$ i.e. reciprocal is a root • Uses the property of odd functions to establish $P(-1-\sqrt{2}) = 0$ i.e. $1+\sqrt{2}$ is a root