Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS EXTENSION 1

ASSESSMENT TASK NO: 5

Date: 21st August 2012

TIME ALLOWED: *Reading time:* 5 *minutes*

Working time: 60 *minutes*

INSTRUCTIONS: This paper consists of 4 multiple choice and 3 free response questions
ANSWER all 7 questions
Answer questions 1 – 4 on the multiple choice answer sheet provided
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

HE1 - appreciates interrelationships between ideas drawn from different areas of mathematics

HE3 - uses a variety of strategies to investigate mathematical models of situations involving binomial theorem.

PE3 - solves problems involving permutations and combinations

	MARKS
MULTIPLE CHOICE	/4
QUESTION 5	/12
QUESTION 6	/13
QUESTION 7	/ 9
	/38

This assessment task constitutes 10*% of the Higher School Certificate Course Assessment.*

Answer questions 1 – 4 on the multiple choice answer sheet provided

1. The fourth term in the expansion of $(2a-3b)^7$ is

(A) $\binom{7}{4}(2a)^4(3b)^3$

(B) $\binom{7}{3}(2a)^3(3b)^4$

(C) $-\binom{7}{3}(2a)^4(3b)^3$

(D) $-\binom{7}{4}(2a)^4(3b)^3$

2. The letters of the word "*SACHIN*" can be arranged to make 6! words. These words are to be listed in the alphabetical order and numbered from 1 to 6!.

What number would be allocated to the word *SACHIN* ?

(A) 600

(B) 601

(C) 602

(D) 603

3. There are 10 points in a plane, out of which 6 are collinear. If N is the number of triangles formed by joining these points, then

(A) $N \leq 100$

(B) $100 < N \leq 140$

(C) $140 < N \leq 190$

(D) $N > 190$

4. Which of the following statements are correct?

I. $^{n}C_{r-1} = {}^{n}C_{n-r-1}$

II. $^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

III. $\dfrac{^{n}P_{r}}{^{n}C_{r}} = r!$

(A) I and II are correct

(B) II and III are correct

(C) I and III are correct

(D) All of the three are correct

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

a) In a co-educational class there are 4 girls and 7 boys. Their classroom has 5 rows of 5 desks neatly arranged. Each student occupies a desk with a chair.

Find the number of seating arrangements possible, if:

(i) students can sit anywhere. **1**

(ii) all the girls want to occupy the first row. **2**

(iii) Two particular girls and three particular boys fill the back row seated alternately. **2**

b) In an election, there are 10 candidates and 4 are to be elected. A voter may vote for any number of candidates, not greater than the number to be elected. If a voter votes for at least one candidate, then find the number of ways in which he/she can vote. **2**

c) In how many ways can 3 ladies and 3 gentlemen be seated at a round table, so that any two and only two of the ladies can sit together? **3**

d) How many different words can be formed by jumbling the letters of the word *MISSISSIPPI* in which no two *S* are adjacent? **2**

Question 6 (13 marks) Use a SEPARATE writing booklet **Marks**

a) Find the term independent of x in the expansion of $\left(x^3 + \frac{2}{x}\right)^{12}$ **3**

b) (i) Show that $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ **2**

(ii) Find the numerically greatest term in the expansion of $(2-3x)^{10}$ **3**

(iii) Show that **2**

$$\frac{{}^nC_1}{{}^nC_0} + \frac{2.{}^nC_2}{{}^nC_1} + \frac{3.{}^nC_3}{{}^nC_2} + + \ldots.. + \frac{n.{}^nC_n}{{}^nC_{n-1}} = \frac{n}{2}(n+1)$$

c) (i) In how many ways can 52 playing cards be placed in 4 heaps of 13 cards each? **2**

(ii) In how many ways can they be dealt out to four players giving 13 cards each? **1**

Question 7 (9 marks) Use a SEPARATE writing booklet **Marks**

a)

(i) By considering $(1+x)^{n+3} = (1+x)^n \ (1+x)^3$, show that **2**

$$\binom{n+3}{k} = \binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}$$

(ii) Explain in your own words, what each term in this result means to you. **1**

(iii) Between what values must k lie? **1**

b)

(i) Consider the expansion of $(1+x)^n$ and prove that **1**

$$\sum_{r=0}^{n}\binom{n}{r} = 2^n$$

(ii) Consider $(1+x)^{2n} = (1+x)^n \ (1+x)^n$, and prove that **4**

$$\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$$

End of paper

STUDENT NUMBER

Multiple Choice Answer Sheet

Completely fill the response oval representing the most correct answer.

1. A ⬭ B ⬭ C ⬭ D ⬭

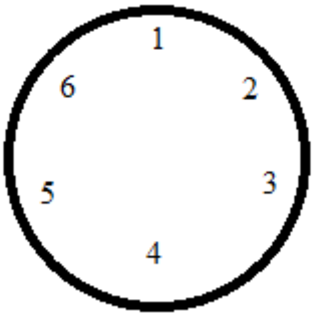
2. A ⬭ B ⬭ C ⬭ D ⬭

3. A ⬭ B ⬭ C ⬭ D ⬭

4. A ⬭ B ⬭ C ⬭ D ⬭

Marking Scheme

1.	$-\binom{7}{3}(2a)^4(3b)^3$ **C**		
2.	The letters are ACHINS No. of words starting with A = 5! No. of words starting with C = 5! No. of words starting with H = 5! No. of words starting with I = 5! No. of words starting with N = 5! No. of words starting with S = 1 Total = 5 × 5!+ 1 = 601 **B**		
3.	$\binom{10}{3}-\binom{6}{3}$ = 120 – 20 = 100 **A**		
4.	**B**		
5 (a)(i)	${}^{25}P_{11}$ =	1 mark: correct answer	
(ii)	The 4 girls can be arranged in the front row of 5 chairs in ${}^{5}P_{4}$ ways. The seven boys can be arranged in the 21 chairs in ${}^{21}P_{7}$ ways. Total number of ways = ${}^{5}P_{4} \times {}^{21}P_{7}$	1 mark: Gives the correct number of ways in which girls can be arranged. 1 mark: gives the correct no. of arrangements of the boys and multiplies with the answer above.	
(iii)	In the last row, the arrangement is B G B G B This can be done in 3! × 2! = 12 The rest of the 6 students can be arranged in ${}^{20}P_{6}$ ways. Total number of arrangements = 3! × 2! × ${}^{20}P_{6}$	1 mark: gives the correct no. of arrangements of the last row. 1 mark: Multiples the answer from above with ${}^{20}P_{6}$	
(b)	$\binom{10}{1}+\binom{10}{2}+\binom{10}{3}+\binom{10}{4}$ = 385	Award 2 marks for correct answer Award only 1 mark if they have given at least two correct terms. *(looking for acknowledgement of the fact that <u>at least one</u> candidate needs to be picked)*	

(c)	 Pick the two ladies in $\binom{3}{2}$ ways Fix them at seats 1 and 2 and they can swap places in 2 ways. Now the third lady can sit at seat 4 or 5. $\therefore$ 2 choices for her. The three men have 3 seats can be arranged in 3! Ways. Total number of ways = $\binom{3}{2} \times 2 \times 2 \times 3! = 72$	1 mark: Finds the no. of ways in which the two particular ladies can be placed. 1 mark: finds the no. of choices for the third lady 1 mark: finds the no. of ways in which the men can be placed and hence gives the total number.	
d)	Other than *S*, seven letters M, I, I, I, P, P, I can be arranged in $\frac{7!}{2!4!}$ Now the four S can be placed in 8 spaces in ${}^{8}C_{4}$ ways. Desired number of ways = $\frac{7!}{2!4!} \times {}^{8}C_{4}$	1 mark: gives the correct no. of ways in which letters other than S can be arranged 1 mark: Gives the correct no. of ways in which S can be placed and multiplies the previous answer.	

Question 6

a)	$\left(x^3+\frac{2}{x}\right)^{12}$ $General\ term\ T_{n+1}=\binom{n}{r}(a)^{n-r}(b)^r$ $=\binom{12}{r}(x^3)^{12-r}\left(\frac{2}{x}\right)^r$ $=\binom{12}{r}(x)^{36-3r-r}(2)^r$ Term is independent of x. Hence power of $x = 0$ ie. $36 - 4r = 0$ r = 9 $\therefore$ Term independent of x is $=\binom{12}{9}(2)^9$	1 mark: gives the correct general term 1 mark: Sets coefficient of $x = 0$ and evaluates the value of r 1 mark: gives the independent term using their value of r	
b)(i)	$\frac{{}^nC_r}{{}^nC_{r-1}}=\frac{n!}{r!(n-r)!}\div\frac{n!}{(r-1)!(n-r+1)!}$ $=\frac{n!}{r!(n-r)!}\times\frac{(r-1)!(n-r+1)!}{n!}$ $=\frac{n-r+1}{r}$	1 mark: Expands ${}^nC_r$ and ${}^nC_{r-1}$ correctly. 1 mark: correct simplification to prove the result. (*must show the cancelling out*)	
(ii)	(i) $\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\cdot\frac{b}{a}$ Substituting, $\frac{T_{r+1}}{T_r}=\frac{11-r}{r}\cdot\frac{-3x}{2}$ **1 mark** We are only interested in the numerically greatest coefficient. $\frac{T_{r+1}}{T_r}=\frac{11-r}{r}\cdot\frac{3}{2}\ \geq\ 1$ $33 - 3r \geq 2r$ $33 \geq 5r$ $\therefore\ r \leq 6.5$ ie. $T_{r+1} \geq T_r$ for $r = 0$, 1,2, …, 6 And $T_r \geq T_{r+1}$ for $r = 7, 8, 9, 10$	1 mark: Gives the correct expression for $\frac{T_{r+1}}{T_r}$ 1 mark: Solves the inequality to give the value of r	

	$\therefore$ T_7 has the greatest coefficient. $T_7 = {}^{10}C_6 . 2^4 . (-3x)^6$ $= 2449\,440\,x^6$	1 mark: Gives the correct coefficient using their value of r	
(iii)	$\frac{^nC_1}{^nC_0} + \frac{2 \cdot {}^nC_2}{^nC_1} + \frac{3 \cdot {}^nC_3}{^nC_2} + + \ldots . + \frac{n \cdot {}^nC_n}{^nC_{n-1}}$ $= \frac{n}{2}(n+1)$ LHS = $\frac{^nC_1}{^nC_0} + \frac{2 \cdot {}^nC_2}{^nC_1} + \frac{3 \cdot {}^nC_3}{^nC_2} + + \ldots . + \frac{n \cdot {}^nC_n}{^nC_{n-1}}$ Using the result from (i), $\frac{n-0}{1} + 2\times\frac{n-1}{2} + 3\times\frac{n-2}{3} + \ldots . + (n-1)\times\frac{n-(n-2)}{n-1} n\times\frac{n-(n-1)}{n}$ **(1 mark)** $= n + n-1 + n-2 + n-3 + \ldots\ldots.. + 3 + 2 + 1$ $= \frac{n}{2}(n+1)$ **1 mark** (1 mark: applying the result from (i)) 1 mark: Writes the sum of the first n counting numbers.		
(c)(i)	No. of ways in which the four heaps can be selected = ${}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}$ Removing the repetitions, no: of heaps = $\frac{{}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}}{4!}$ For interest this is $\frac{52!}{4!(13)^4}$	Award 2 marks for correct answer. Award only 1 mark if not divided by 4!	
(ii)	4 heaps can be given to 4 people in 4! Ways. The total no. of ways = $\frac{{}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}}{4!} \times 4!$ *And this is* $\frac{52!}{(13)^4}$	1 mark: Correct answer.	

Question 7

Marking Scheme

a)(i)	Expands both sides of the equation correctly using Binomial theorem Compares the coefficient of x^k and simplifies to prove the result	**1 mark** **1 mark**	
	$\binom{n+3}{0} + \binom{n+3}{1}x + \binom{n+3}{2}x^2 + \ldots. + \binom{n+3}{k}x^k + \ldots.+\binom{n+3}{n+3}x^{n+3} =$ $\left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{k-3}x^{k-3}+\binom{n}{k-2}x^{k-2}+\binom{n}{k-1}x^{k-1}+\binom{n}{k}x^k\right] \times$ $\left[\binom{3}{0}+\binom{3}{1}x+\binom{3}{2}x^2+\binom{3}{3}x^3\right]$ Comparing the coefficient of x^k, $\binom{n+3}{k} = \binom{n}{k}\binom{3}{0}+\binom{n}{k-1}\binom{3}{1}+\binom{n}{k-2}\binom{3}{2}+\binom{n}{k-3}\binom{3}{3}$ $= \binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}$		
a(ii)	$\binom{n+3}{k}$ represents the number of ways in which k items can be chosen from two lots, n of one kind and 3 of the other kind. This can be done by choosing all k items from n, or $k-1$ from n and 1 from the other lot of 3, or $k-2$ from n and 2 from the other lot of 3, $k-3$ from n and 3 from the other lot of 3.	**1 mark**	
a)(iii)	$k-3 \geq 0$ $\therefore\ k \geq 3$	**1 mark:** correct answer	

<table>
<tr><td>b)(i)</td><td>Correctly expands $(1+x)^n$ using Binomial theorem and sets $x = 1$ to get the result</td><td>1 mark: correct proof</td><td></td></tr>
<tr><td>(ii)</td><td>Correctly expands $(1+x)^{2n} = (1+x)^n \times (1+x)^n$ using Binomial theorem and sets $x = 1$ to get the result

Demonstrates the understanding that $(1+x)^{2n}$ has $2n + 1$ terms and that ${}^nC_r = {}^nC_{n-r}$ to write the equation marked as B in the solution

Uses the result fron question (i) and manipulates the equation B to get 2 $\sum_{k=0}^{n}\binom{2n}{k} - \binom{2n}{n} = 2^{2n}$

Uses the definition of $\binom{2n}{n}$ and simplifies to get the result</td><td>1 mark

1 mark

1 mark

1 mark</td><td></td></tr>
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<table>
<tr><td>b)(i)</td><td>$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$
Substitute $x = 1$, then
$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$</td><td></td><td></td></tr>
<tr><td>(ii)</td><td colspan="3">$(1+x)^{2n} = (1+x)^n \ (1+x)^n$
$\binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \ldots + \binom{2n}{k}x^k + \ldots + \binom{2n}{2n} =$
$\left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right] \times \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right]$
Let $x = 1$,</td></tr>
</table>

$$\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \ldots. + \binom{2n}{n-1} + \binom{2n}{n} + \binom{2n}{n+1} + \ldots. + \binom{2n}{2n-1} + \binom{2n}{2n}$$

$$= \left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}\right]^2 \qquad \textbf{(A)} \qquad \textbf{1 mark}$$

LHS has $2n + 1$ terms (odd numbers) and all except $\binom{2n}{n}$ forms pairs.

$$\binom{2n}{0} = \binom{2n}{2n}, \quad \binom{2n}{1} = \binom{2n}{2n-1}, \quad \binom{2n}{n-1} = \binom{2n}{n+1}$$

$$\binom{2n}{0} = \binom{2n}{2n}, \quad \binom{2n}{1} = \binom{2n}{2n-1}, \quad \binom{2n}{n-1} = \binom{2n}{n+1}$$

(A) Becomes

$$2\left\{\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \ldots. + \binom{2n}{n-1}\right\} + \binom{2n}{n} = \quad \textbf{(B)}$$

1 mark: ***combines the terms to write the equation***

$$\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}\right]^2$$

$$2\left\{\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \ldots. + \binom{2n}{n-1} + \binom{2n}{n}\right\} - \binom{2n}{n} =$$

$$\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}\right]^2$$

$$2\sum_{k=0}^{n}\binom{2n}{k} - \binom{2n}{n} = 2^{2n} \qquad \text{(using(i))} \qquad \textbf{1 mark}$$

$$2\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n} + \binom{2n}{n}$$

$$2\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n} + \frac{(2n)!}{(n)!(2n-n)!}$$

$$2\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n} + \frac{(2n)!}{(n!)(n!)}$$

$$2\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n} + \frac{(2n)!}{(n!)^2}$$

$$\therefore \quad \sum_{k=0}^{n}\binom{2n}{k} = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$$

1 mark

In how many ways letters of the word *EXPERIENCE* can be arranged in a row?

(A) $\frac{6!}{4!}$

(B) $\frac{7!}{4!}$

(C) $\frac{10!}{4}$

(D) $\frac{10!}{4!}$