NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6–14)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the length of the vector $-\underset{\sim}{i} + 18\underset{\sim}{j} - 6\underset{\sim}{k}$?

A. 5

B. 19

C. 25

D. 361

2 Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where p and q are real, what are the values of p and q?

A. $p = -6,\quad q = \sqrt{10}$

B. $p = -6,\quad q = 10$

C. $p = 6,\quad q = \sqrt{10}$

D. $p = 6,\quad q = 10$

3 What is the Cartesian equation of the line $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \end{pmatrix}$?

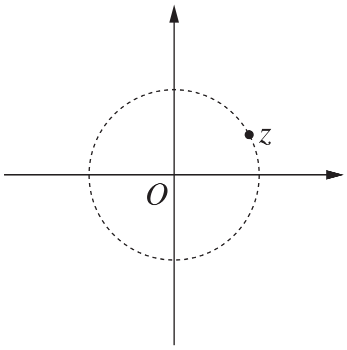
A. $2y + x = 7$

B. $y - 2x = -5$

C. $y + 2x = 5$

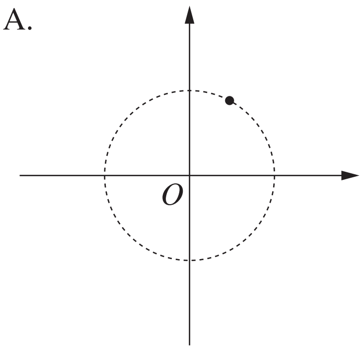
D. $2y - x = -1$

4 The diagram shows the complex number z on the Argand diagram.

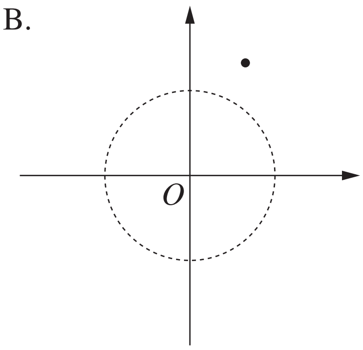


Which of the following diagrams best shows the position of $\dfrac{z^2}{|z|}$?

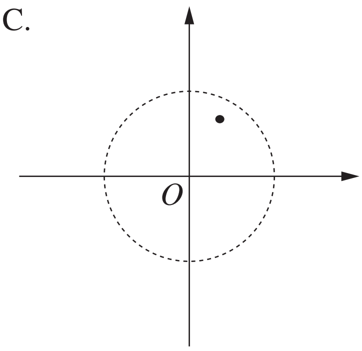
A.



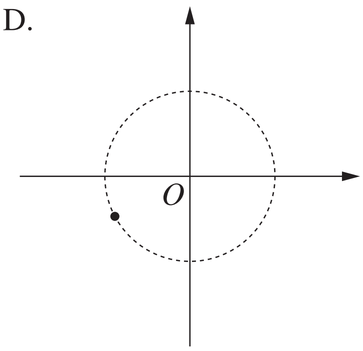
B.



C.



D.



5 A particle undergoing simple harmonic motion has a maximum acceleration of 6 m/s^2 and a maximum velocity of 4 m/s.

What is the period of the motion?

A. π

B. $\dfrac{2\pi}{3}$

C. 3π

D. $\dfrac{4\pi}{3}$

6 Which expression is equal to $\displaystyle\int \frac{1}{x^2+4x+10}\,dx$?

A. $\dfrac{1}{\sqrt{6}}\tan^{-1}\left(\dfrac{x+2}{\sqrt{6}}\right)+c$

B. $\tan^{-1}\left(\dfrac{x+2}{\sqrt{6}}\right)+c$

C. $\dfrac{1}{2\sqrt{6}}\ln\left|\dfrac{x+2-\sqrt{6}}{x+2+\sqrt{6}}\right|+c$

D. $\ln\left|\dfrac{x+2-\sqrt{6}}{x+2+\sqrt{6}}\right|+c$

7 Consider the proposition:

'If 2^n-1 is not prime, then n is not prime'.

Given that each of the following statements is true, which statement disproves the proposition?

A. 2^5-1 is prime

B. 2^6-1 is divisible by 9

C. 2^7-1 is prime

D. $2^{11}-1$ is divisible by 23

8 Consider the statement:

‘If n is even, then if n is a multiple of 3, then n is a multiple of 6’.

Which of the following is the negation of this statement?

A. n is odd and n is not a multiple of 3 or 6.

B. n is even and n is a multiple of 3 but not a multiple of 6.

C. If n is even, then n is not a multiple of 3 and n is not a multiple of 6.

D. If n is odd, then if n is not a multiple of 3 then n is not a multiple of 6.

9 What is the maximum value of $\left|e^{i\theta}-2\right|+\left|e^{i\theta}+2\right|$ for $0\le\theta\le 2\pi$?

A. $\sqrt{5}$

B. 4

C. $2\sqrt{5}$

D. 10

10 Which of the following is equal to $\int_0^{2a} f(x)\,dx$?

A. $\int_0^a f(x)-f(2a-x)\,dx$

B. $\int_0^a f(x)+f(2a-x)\,dx$

C. $2\int_0^a f(x-a)\,dx$

D. $\int_0^a \frac{1}{2}f(2x)\,dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) Consider the complex numbers $w = -1 + 4i$ and $z = 2 - i$.

(i) Evaluate $|w|$. **1**

(ii) Evaluate $w\overline{z}$. **2**

(b) Use integration by parts to evaluate $\displaystyle\int_1^e x \ln x \, dx$. **3**

(c) A particle starts at the origin with velocity 1 and acceleration given by **3**

$$a = v^2 + v,$$

where v is the velocity of the particle.

Find an expression for x, the displacement of the particle, in terms of v.

(d) Consider the two vectors $\underset{\sim}{u} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{v} = p\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$. **3**

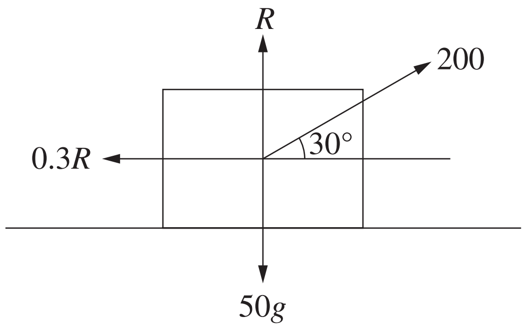
For what values of p are $\underset{\sim}{u} - \underset{\sim}{v}$ and $\underset{\sim}{u} + \underset{\sim}{v}$ perpendicular?

(e) Solve $z^2 + 3z + (3 - i) = 0$, giving your answer(s) in the form $a + bi$, where a and b are real. **4**

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) A 50-kilogram box is initially at rest. The box is pulled along the ground with a force of 200 newtons at an angle of 30° to the horizontal. The box experiences a resistive force of $0.3R$ newtons, where R is the normal force, as shown in the diagram.

Take the acceleration g due to gravity to be 10 m/s^2.



(i) By resolving the forces vertically, show that $R = 400$. **2**

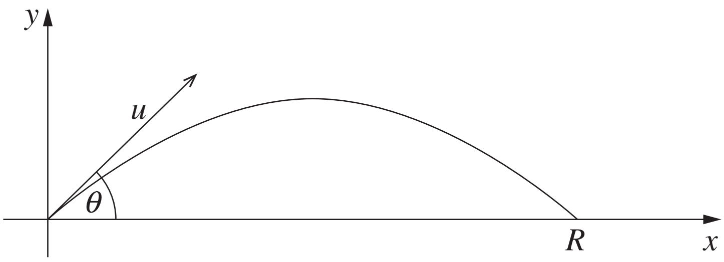
(ii) Show that the net force horizontally is approximately 53.2 newtons. **2**

(iii) Find the velocity of the box after the first three seconds. **2**

Question 12 continues on page 8

Question 12 (continued)

(b) A particle is projected from the origin with initial velocity u m/s at an angle θ to the horizontal. The particle lands at $x = R$ on the x-axis. The acceleration vector is given by $\underset{\sim}{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$, where g is the acceleration due to gravity. (Do NOT prove this.)



(i) Show that the position vector $\underset{\sim}{r}(t)$ of the particle is given by **3**

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}.$$

(ii) Show that the Cartesian equation of the path of flight is given by **3**

$$y = \frac{-gx^2}{2u^2}\left(\tan^2\theta - \frac{2u^2}{gx}\tan\theta + 1\right).$$

(iii) Given $u^2 > gR$, prove that there are 2 distinct values of θ for which the particle will land at $x = R$. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) A particle is undergoing simple harmonic motion with period $\frac{\pi}{3}$. The central point of motion of the particle is at $x = \sqrt{3}$. When $t = 0$ the particle has its maximum displacement of $2\sqrt{3}$ from the central point of motion. **3**

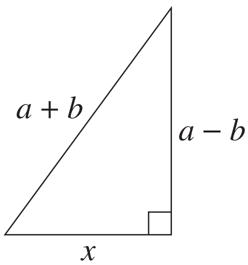
Find an equation for the displacement, x, of the particle in terms of t.

(b) Consider the two lines in three dimensions given by **3**

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}.$$

By equating components, find the point of intersection of the two lines.

(c) (i) By considering the right-angled triangle below, or otherwise, prove that $\frac{a+b}{2} \geq \sqrt{ab}$, where $a > b \geq 0$. **2**



(ii) Prove that $p^2 + 4q^2 \geq 4pq$. **1**

(d) (i) Show that for any integer n, $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$. **1**

(ii) By expanding $\left(e^{i\theta} + e^{-i\theta}\right)^4$, show that **3**

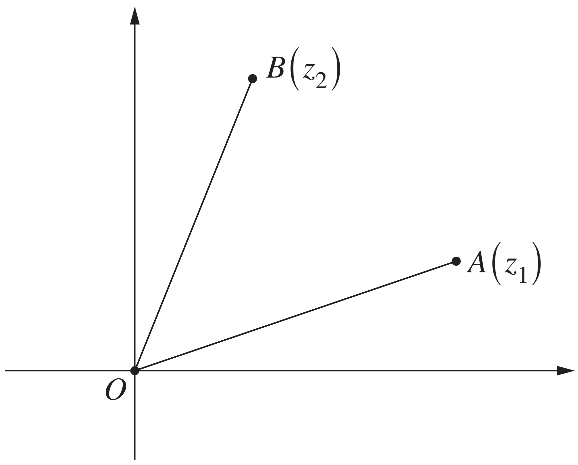
$$\cos^4\theta = \frac{1}{8}\left(\cos(4\theta) + 4\cos(2\theta) + 3\right).$$

(iii) Hence, or otherwise, find $\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$. **2**

Question 14 (16 marks) Use the Question 14 Writing Booklet

(a) Let z_1 be a complex number and let $z_2 = e^{\frac{i\pi}{3}} z_1$.

The diagram shows points A and B which represent z_1 and z_2, respectively, in the Argand plane.



(i) Explain why triangle OAB is an equilateral triangle. **2**

(ii) Prove that $z_1^2 + z_2^2 = z_1 z_2$. **3**

(b) A particle starts from rest and falls through a resisting medium so that its acceleration, in m/s^2, is modelled by **4**

$$a = 10\left(1 - (kv)^2\right),$$

where v is the velocity of the particle in m/s and $k = 0.01$.

Find the velocity of the particle after 5 seconds.

(c) Prove by mathematical induction that, for $n \geq 2$, **4**

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$

(d) Prove that for any integer $n > 1$, $\log_n(n+1)$ is irrational. **3**

Question 15 (13 marks) Use the Question 15 Writing Booklet

(a) In the set of integers, let P be the proposition:

'If $k + 1$ is divisible by 3, then $k^3 + 1$ is divisible by 3'.

(i) Prove that the proposition P is true. **2**

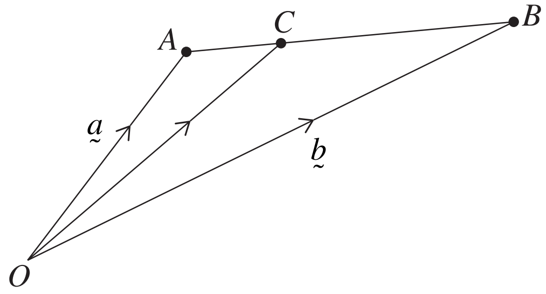
(ii) Write down the contrapositive of the proposition P. **1**

(iii) Write down the converse of the proposition P and state, with reasons, whether this converse is true or false. **3**

Question 15 continues on page 12

Question 15 (continued)

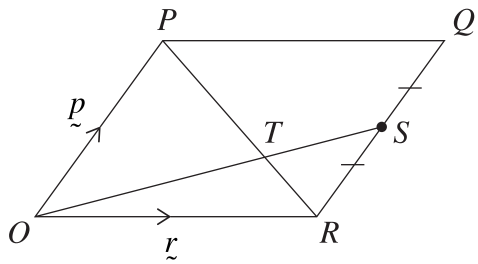
(b) The point C divides the interval AB so that $\dfrac{CB}{AC} = \dfrac{m}{n}$. The position vectors of A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, as shown in the diagram.



(i) Show that $\overrightarrow{AC} = \dfrac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$. **2**

(ii) Prove that $\overrightarrow{OC} = \dfrac{m}{m+n}\underset{\sim}{a} + \dfrac{n}{m+n}\underset{\sim}{b}$. **1**

Let $OPQR$ be a parallelogram with $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OR} = \underset{\sim}{r}$. The point S is the midpoint of QR and T is the intersection of PR and OS, as shown in the diagram.



(iii) Show that $\overrightarrow{OT} = \dfrac{2}{3}\underset{\sim}{r} + \dfrac{1}{3}\underset{\sim}{p}$. **3**

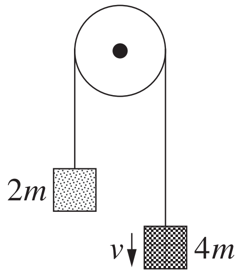
(iv) Using parts (ii) and (iii), or otherwise, prove that T is the point that divides the interval PR in the ratio $2:1$. **1**

End of Question 15

Question 16 (16 marks) Use the Question 16 Writing Booklet

(a) Two masses, $2m$ kg and $4m$ kg, are attached by a light string. The string is placed over a smooth pulley as shown.

The two masses are at rest before being released and v is the velocity of the larger mass at time t seconds after they are released.



The force due to air resistance on each mass has magnitude kv, where k is a positive constant.

(i) Show that $\dfrac{dv}{dt} = \dfrac{gm - kv}{3m}$. **2**

(ii) Given that $v < \dfrac{gm}{k}$, show that when $t = \dfrac{3m}{k}\ln 2$, the velocity of the larger mass is $\dfrac{gm}{2k}$. **3**

Question 16 continues on page 14

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\, d\theta,\ n = 0, 1, \ldots .$

(i) Prove that $I_n = \dfrac{2n}{2n+1} I_{n-1},\ n \geq 1.$ **3**

(ii) Deduce that $I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}.$ **3**

Let $J_n = \int_0^1 x^n (1-x)^n\, dx,\ n = 0, 1, 2, \ldots .$

(iii) Using the result of part (ii), or otherwise, show that $J_n = \dfrac{(n!)^2}{(2n+1)!}.$ **3**

(iv) Prove that $\left(2^n n!\right)^2 \leq \left(2n+1\right)!.$ **2**

End of paper

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2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

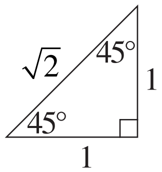
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$



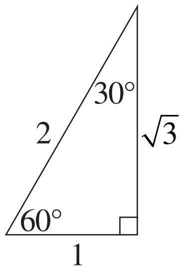
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

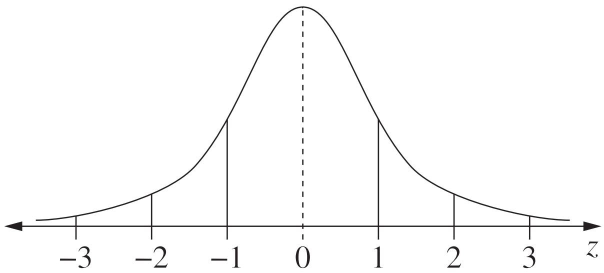
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

2020 HSC Mathematics Extension 2 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	B
3	C
4	A
5	D
6	A
7	D
8	B
9	C
10	B

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$w = -1 + 4i$$
$$|w| = \sqrt{17}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains $\overline{z}$ OR • Correctly evaluates wz	1

Sample answer:

$$\begin{aligned} w\overline{z} &= (-1+4i)(2+i) \\ &= -2-4+8i-i \\ &= -6+7i \end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly applies integration by parts, or equivalent merit	2
• Correctly identifies the two functions to be used in integration by parts, or equivalent merit	1

Sample answer:

$$\int_1^e x \ln x \, dx$$

$$u = \ln x, \qquad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x}, \qquad v = \frac{x^2}{2}$$

$$= \frac{x^2}{2} \ln x \Big|_1^e - \int_1^e \frac{x}{2} \, dx$$

$$= \frac{e^2}{2} - \frac{x^2}{4} \Big|_1^e$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Correctly separates variables, or equivalent merit	2
• Uses $a = v\dfrac{dv}{dx}$, or equivalent merit	1

Sample answer:

$$a = v^2 + v$$

$$\therefore \quad v \times \frac{dv}{dx} = v^2 + v$$

$$\therefore \quad \frac{dx}{dv} = \frac{1}{v+1}$$

$$x = \ln(v+1) + C$$

When $x = 0, \quad v = 1 \quad \Rightarrow \quad C = -\ln 2$

$$\therefore \quad x = \ln(v+1) - \ln 2$$

Question 11 (d)

Criteria	Marks
• Provides correction solution	3
• Correctly obtains $(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v})$, or equivalent merit OR • Attempts to apply $(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = 0$ using correct $(\underset{\sim}{u} - \underset{\sim}{v})$ and $(\underset{\sim}{u} + \underset{\sim}{v})$	2
• Obtains $(\underset{\sim}{u} - \underset{\sim}{v})$ and $(\underset{\sim}{u} + \underset{\sim}{v})$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{u} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\underset{\sim}{v} = p\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$$

$$\underset{\sim}{u} - \underset{\sim}{v} = (-2-p)\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$$

$$\underset{\sim}{u} + \underset{\sim}{v} = (p-2)\underset{\sim}{i} + 5\underset{\sim}{k}$$

$$(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = 0$$

$$-(p+2)(p-2) + 5 = 0$$

$$p^2 - 4 = 5$$

$$p = \pm 3$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	4
• Obtains the square root of the discriminant, or equivalent merit	3
• Makes progress towards finding the square root of the discriminant	2
• Correctly obtains the discriminant, or equivalent merit OR • Attempts to use quadratic formula	1

Sample answer:

$$z^2 + 3z + (3 - i) = 0$$

$$\begin{aligned}\Delta &= 9 - 4(3 - i)\\ &= -3 + 4i\end{aligned}$$

Write $-3 + 4i = (x + iy)^2$

$$\therefore \quad \begin{aligned}x^2 - y^2 &= -3 \quad \text{(real parts)}\\ x^2 + y^2 &= 5 \quad \text{(moduli)}\end{aligned}$$

$$\therefore \quad x^2 = 1 \;\Rightarrow\; x = \pm 1$$
$$\Rightarrow\; \mathrm{y} = \pm 2$$

$$\therefore \quad -3 + 4i = (1 + 2i)^2$$

Hence, $z = \dfrac{-3 \pm (1 + 2i)}{2}$

$= -1 + i, -2 - i$

Question 12 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains the vertical component of the 200 newton force, or equivalent merit	1

Sample answer:

Resolving vertically,

$$200\sin 30° + R = mg$$

$$\therefore R = 50 \times 10 - 200 \times \frac{1}{2}$$

$$= 400 \text{ newtons}$$

Question 12 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains the horizontal component of the 200 newton force, or equivalent merit	1

Sample answer:

Horizontally,

$$\begin{aligned}\text{Net horizontal force } &= 200\cos 30° - 0.3R \\ &= 200 \times \frac{\sqrt{3}}{2} - 0.3 \times 400 \\ &\simeq 53.2 \text{ newtons}\end{aligned}$$

Question 12 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains the acceleration, or equivalent merit	1

Sample answer:

From part (b), using $F = ma$,

$50a = 53.2$

$\therefore \dfrac{dV}{dt} = 1.064$

Hence the velocity after 3 seconds is

$$V = \int_0^3 1.064\,dt$$

$$= 1.064 \times 3$$

$$= 3.192 \text{ m/s}$$

Question 12 (b) (i)

Criteria	Marks
• Provides correct solution	3
• Correctly obtains $\underset{\sim}{v}(t) = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$, or equivalent merit OR • Correctly obtains one component of $\underset{\sim}{r}(t)$	2
• Correctly obtains one component of $\underset{\sim}{v}(t)$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\underset{\sim}{v} = \begin{pmatrix} c_1 \\ -gt + c_2 \end{pmatrix}$$

Now at $t = 0$, $\underset{\sim}{v} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix}$

$$\therefore \quad \underset{\sim}{v} = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$$

$$\therefore \quad \underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta + c_3 \\ ut\sin\theta - \frac{1}{2}gt^2 + c_4 \end{pmatrix}$$

$$\underset{\sim}{r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad c_3 = c_4 = 0$$

$$\therefore \quad \underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$

Question 12 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains a correct Cartesian equation of flight, or equivalent merit	2
• Attempts to eliminate t, or equivalent merit	1

Sample answer:

$$x = ut\cos\theta \quad \Rightarrow \quad t = \frac{x}{u\cos\theta}$$

$$y = ut\sin\theta - \frac{1}{2}gt^2$$

$$= \frac{ux\sin\theta}{u\cos\theta} - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}$$

$$= x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2}\left(1+\tan^2\theta\right)$$

$$= -\frac{gx^2}{2u^2}\left(\tan^2\theta - \frac{2u^2}{gx}\tan\theta + 1\right)$$

Question 12 (b) (iii)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the discriminant $\Delta = \dfrac{4u^4}{g^2x^2} - 4$, or equivalent merit	1

Sample answer:

From part (ii), we require the quadratic to have two distinct real roots, when $x = R$.

$$\Delta = \frac{4u^4}{g^2R^2} - 4$$

$$= \frac{4\left(u^4 - g^2R^2\right)}{g^2R^2}$$

Given $u^2 > gR \quad \Rightarrow \quad u^4 > g^2R^2$

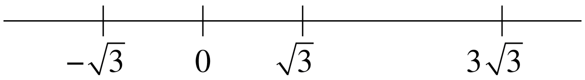
$\therefore \ \Delta > 0$

Hence there are two distinct values for $\tan\theta$, hence two distinct values for θ, as $\tan\theta$ is increasing.

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Correctly finds any two of amplitude, n, function with consistent phase	2
• Correctly finds one of amplitude, n, function with consistent phase	1

Sample answer:



The equation has the form

$x(t) = A\cos(nt) + \sqrt{3}$

Now $A = 2\sqrt{3}$ and $\dfrac{2\pi}{n} = \dfrac{\pi}{3}$, so $n = 6$

$\therefore\ x(t) = 2\sqrt{3}\cos(6t) + \sqrt{3}$

Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly finds either λ_1 or λ_2, or equivalent merit	2
• Finds one equation linking λ_1 and λ_2, or equivalent merit	1

Sample answer:

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

Equating components,

$$3 + \lambda_1 = 3 - 2\lambda_2 \qquad (1)$$
$$-1 + 2\lambda_1 = -6 + \lambda_2 \qquad (2)$$
$$7 + \lambda_1 = 2 + 3\lambda_2 \qquad (3)$$

$(1) \Rightarrow \lambda_1 = -2\lambda_2$

$(2) \Rightarrow -1 - 4\lambda_2 = -6 + \lambda_2 \Rightarrow \lambda_2 = 1 \Rightarrow \lambda_1 = -2.$

As a check, using equation (3),

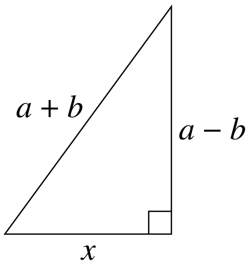
Left hand side = 5, Right hand side = 5

$\therefore$ Point of intersection is $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Applies Pythagoras to find x, or equivalent merit OR • Observes that $x \le a + b$	1

Sample answer:



By Pythagoras,

$$(a+b)^2 = x^2 + (a-b)^2$$
$$\therefore\ x = 2\sqrt{ab}$$

Now the hypotenuse is the longest side so

$$a + b \ge 2\sqrt{ab}$$
$$\therefore\ \frac{a+b}{2} \ge \sqrt{ab}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Put $a = p^2,\ b = (2q)^2$

Then $\dfrac{p^2 + 4q^2}{2} \ge \sqrt{4p^2q^2}$

So $\quad p^2 + 4q^2 \ge 4pq$

Question 13 (d) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$e^{in\theta} + e^{-in\theta}$

$= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$

$= 2\cos(n\theta)$

Question 13 (d) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly uses binomial theorem and groups conjugate pairs, or equivalent merit	2
• Uses part (a) to obtain $\left(e^{i\theta} + e^{-i\theta}\right)^4 = 16\cos^4\theta$ OR • Attempts to use binomial theorem, or equivalent merit	1

Sample answer:

$$\left(e^{i\theta} + e^{-i\theta}\right)^4 = e^{4i\theta} + 4e^{3i\theta} \cdot e^{-i\theta} + 6e^{2i\theta} \cdot e^{-2i\theta} + 4e^{i\theta} \cdot e^{-3i\theta} + e^{-4i\theta}$$

$$= \left(e^{4i\theta} + e^{-4i\theta}\right) + 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6$$

$$= 2\cos 4\theta + 8\cos 2\theta + 6$$

Also $\left(e^{i\theta} + e^{-i\theta}\right)^4 = 2^4 \cos^4\theta$

$$\therefore \quad \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

Question 13 (d) (iii)

Criteria	Marks
• Provides correct solution	2
• Uses part (a) and attempts to integrate, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right) d\theta$$

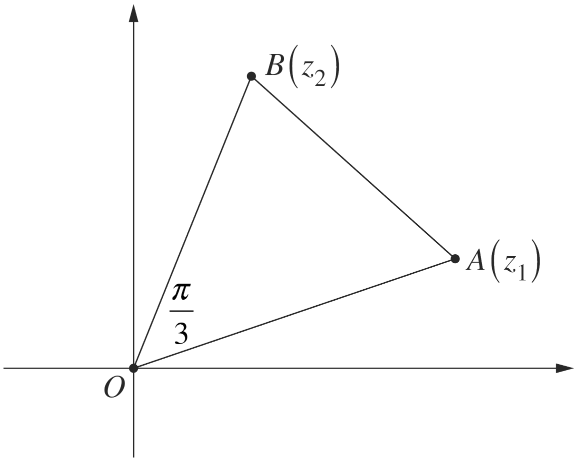
$$= \frac{1}{8}\left(\frac{\sin 4\theta}{4} + 2\sin 2\theta + 3\theta\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left(\frac{3\pi}{2}\right) = \frac{3\pi}{16}$$

Question 14 (a) (i)

Criteria	Marks
• Provides correction solution	2
• Observes either that $\lvert z_2 \rvert = \lvert z_1 \rvert$, or that multiplying by $e^{\frac{i\pi}{3}}$ rotates through an angle of $\frac{\pi}{3}$	1

Sample answer:



Multiplying z_1 by $e^{\frac{i\pi}{3}}$ rotates it anticlockwise through an angle $\frac{\pi}{3}$, giving z_2.

Also $\lvert z_1 \rvert = \lvert z_2 \rvert$.

So $\triangle OAB$ is isosceles giving all angles equal to $\frac{\pi}{3}$ and hence $\triangle OAB$ is equilateral.

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Factors $z_2^3 + z_1^3$, or equivalent merit OR • Simplifies either side of $z_1^2 + z_2^2 = z_1 z_2$ using $z_2 = e^{\frac{i\pi}{3}} z_1$	2
• Observes that $z_2^3 = e^{i\pi} z_1^3$, or equivalent merit OR • Uses $z_2 = e^{\frac{i\pi}{3}} z_1$ in both sides of $z_1^2 + z_2^2 = z_1 z_2$	1

Sample answer:

$$z_2 = e^{\frac{i\pi}{3}} z_1$$

$$\therefore \quad z_2^3 = e^{i\pi} z_1^3 = -z_1^3$$

$$\therefore \quad z_1^3 + z_2^3 = 0$$

$$\left(z_1 + z_2\right)\left(z_1^2 + z_2^2 - z_1 z_2\right) = 0$$

Now $z_1 \neq -z_2$ so $z_1^2 + z_2^2 = z_1 z_2$.

OR

$$\begin{aligned}\text{LHS} &= z_1^2 + z_2^2 = \left(e^{\frac{2\pi i}{3}} + 1\right) z_1^2 \\ &= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + 1\right) z_1^2 \\ &= \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_1^2\end{aligned}$$

$$\begin{aligned}\text{RHS} &= z_1 z_2 = e^{\frac{i\pi}{3}} z_1^2 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) z_1^2 \\ &= \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_1^2\end{aligned}$$

Therefore $z_1^2 + z_2^2 = z_1 z_2$.

Question 14 (b)

Criteria	Marks
• Provides correct solution	4
• Integrates to find v as a function of t, evaluating the constant, or equivalent merit	3
• Correctly applies partial fractions and attempts to integrate, or equivalent merit	2
• Correctly separates variables and attempts to apply partial fractions, or equivalent merit	1

Sample answer:

$$\frac{dv}{dt} = 10\left(1-(kv)^2\right)$$

$$t = \frac{1}{10}\int \frac{1}{(1-kv)(1+kv)}\,dv$$

$$= \frac{1}{20}\int \frac{1}{1-kv} + \frac{1}{1+kv}\,dv \qquad \text{(partial fractions)}$$

$$= \frac{1}{20} \times \frac{1}{k}\left[\ln(1+kv) - \ln(1-kv)\right] + C$$

$$= \frac{1}{20k}\ln\left(\frac{1+kv}{1-kv}\right) + C$$

When $t = 0$, $v = 0$, so $C = 0$.

When $t = 5$, we have

$$\ln\left(\frac{1+kv}{1-kv}\right) = 5 \times 20 \times 0.01 = 1$$

$$\therefore \quad 1 + 0.01v = e(1-0.01v)$$

$$v = 100\left(\frac{e-1}{e+1}\right) \simeq 46.2 \text{ m/s}$$

Question 14 (c)

Criteria	Marks
• Provides correct solution	4
• Correctly proves the inductive step OR • Verifies initial case, $n = 2$, AND assumes true for k and attempts to verify true for $k + 1$	3
• Assumes true for k and attempts to verify true for $k + 1$	2
• Verifies initial case, $n = 2$	1

Sample answer:

Let $P(n)$ be the given proposition.

$P(2)$ is true since $\frac{1}{2^2} < \frac{1}{2}$

Let k be an integer for which $P(k)$ is true.

Thus we assume that

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < \frac{k-1}{k}$$

Consider $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2}$

$$< \frac{1}{(k+1)^2} + \frac{k-1}{k}$$

$$= \frac{k + (k+1)^2(k-1)}{(k+1)^2 k}$$

$$< \frac{(k+1) + (k+1)^2(k-1)}{(k+1)^2 k}$$

$$= \frac{1 + k^2 - 1}{(k+1)k} = \frac{k}{k+1}$$

$\therefore\ P(k+1)$ is true.

$\therefore\ P(n)$ true for all integers $n \geq 2$ by induction.

Question 14 (d)

Criteria	Marks
• Provides correct solution	3
• Assumes $\log_n(n+1)$ is rational and attempts to eliminate the logarithm, or equivalent merit	2
• Attempts to use proof by contradiction, or equivalent merit	1

Sample answer:

Suppose $\log_n(n+1)$ is rational.

Then $\log_n(n+1) = \frac{p}{q}$, where p, q are integers and $q \neq 0$.

$\therefore\ n+1 = n^{\frac{p}{q}}$

so $(n+1)^q = n^p$

If n is even then RHS is even, LHS is odd – a contradiction.

If n is odd then RHS is odd, LHS is even – a contradiction.

Hence in either case we have a contradiction, so $\log_n(n+1)$ is irrational.

Question 15 (a) (i)

Criteria	Marks
• Provides correct proof	2
• Factors $k^3 + 1$ OR • Writes $k + 1 = 3j$, or equivalent merit	1

Sample answer:

Suppose $k + 1$ is divisible by 3, then $k + 1 = 3j$, for some integer j.

$$\text{Now } k^3 + 1 = (k+1)(k^2 - k + 1)$$

$$= 3j(k^2 - k + 1)$$

Hence $k^3 + 1$ is divisible by 3.

Question 15 (a) (ii)

Criteria	Marks
• Provides correct statement	1

Sample answer:

If $k^3 + 1$ is not divisible by 3, then $(k + 1)$ is not divisible by 3.

Question 15 (a) (iii)

Criteria	Marks
• Provides correct solution	3
• Considers cases to justify their statement, or equivalent merit	2
• Correctly states the converse or equivalent merit	1

Sample answer:

The converse states that:

'If $k^3 + 1$ is divisible by 3, then $(k + 1)$ is divisible by 3'.

This statement is true.

The integer k must be of the form $3j$, $3j + 1$ or $3j - 1$, for some integer j.

Then $k^3 + 1 = 27j^3 + 1$ or $(3j + 1)^3 + 1$ or $(3j - 1)^3 + 1$.

Only the third case gives $k^3 + 1$ divisible by 3.

OR

Using proof by contradiction, suppose $k^3 + 1$ is divisible by 3.

But $(k + 1)$ is not divisible by 3.

$\therefore$ $k^2 - k + 1$ is divisible by 3

$\therefore$ $(k + 1)^2 - 3k$ is divisible by 3

$\therefore$ $(k + 1)^2$ and hence $(k + 1)$ is divisible by 3.

This is a contradiction.

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Recognises that $\overrightarrow{CB} = \dfrac{m}{n}\overrightarrow{AC}$, or equivalent merit	1

Sample answer:

$$\overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{OB}$$

$$\therefore\ \underset{\sim}{a} + \overrightarrow{AC} + \frac{m}{n}\overrightarrow{AC} = \overrightarrow{OB}$$

$$\therefore\ \overrightarrow{AC}\left(1 + \frac{m}{n}\right) = \underset{\sim}{b} - \underset{\sim}{a}$$

$$\therefore\ \overrightarrow{AC} = \frac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$$

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \underset{\sim}{a} + \frac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)\\ &= \underset{\sim}{a}\left(1 - \frac{n}{m+n}\right) + \frac{n}{m+n}\left(\underset{\sim}{b}\right)\\ &= \frac{m}{m+n}\underset{\sim}{a} + \frac{n}{m+n}\underset{\sim}{b}\end{aligned}$$

Question 15 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Finds equations of lines OS and PR and attempts to solve for T OR • Writes $\overrightarrow{OT}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{r}$ in two ways and attempts to solve	2
• Finds equation of line OS OR • Writes $\overrightarrow{OT}$ as a multiple of $\overrightarrow{OS} = \underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}$, or equivalent merit	1

Sample answer:

$$\overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{SR}$$

$$= \underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}$$

Equation of line OS is $\underset{\sim}{x} = \left(\underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}\right)\lambda$.

Also equation of line PR is $\underset{\sim}{x} = \underset{\sim}{p} + \left(\underset{\sim}{r} - \underset{\sim}{p}\right)\mu$.

Equating, we have

$$\underset{\sim}{p}\left(\frac{1}{2}\lambda\right) + \underset{\sim}{r}\lambda = \underset{\sim}{p}(1-\mu) + \underset{\sim}{r}\mu$$

$$\therefore\ \frac{1}{2}\lambda = 1 - \mu \text{ and } \lambda = \mu \Rightarrow \lambda = \frac{2}{3}$$

Hence point of intersection is

$$\overrightarrow{OT} = \frac{2}{3}\left(\underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}\right) = \frac{2}{3}\underset{\sim}{r} + \frac{1}{3}\underset{\sim}{p}$$

Question 15 (b) (iv)

Criteria	Marks
• Provides correct solution	1

Sample answer:

By part (ii) the point which divides PR in the ratio $2:1$ is given by $\frac{2}{3}\underset{\sim}{r} + \frac{1}{3}\underset{\sim}{p}$, but this is $\overrightarrow{OT}$.

$\therefore$ T divides the interval PR in the ratio $2:1$.

Question 16 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to balance forces, or equivalent merit	1

Sample answer:

Let a be the acceleration of the masses. Comparing forces,

$$4gm - kv - 2gm - kv = 2ma + 4ma$$

$$\therefore\ 2gm - 2kv = 6am$$

$$\therefore\ a = \frac{gm - kv}{3m}$$

Question 16 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains $t = \frac{3m}{k}\ln\left(\frac{gm}{gm - kv}\right)$, or equivalent merit	2
• Integrates $\frac{dt}{dv}$, or equivalent merit	1

Sample answer:

From part (i), $\frac{dv}{dt} = \frac{gm - kv}{3m}$

$$\therefore t = \int_0^v \frac{3m}{gm - kv}\, dv$$

$$= -\frac{3m}{k}\ln\left(gm - kv\right)\Big|_0^v$$

$$= -\frac{3m}{k}\left[\ln\left(gm - kv\right) - \ln\left(gm\right)\right]$$

$$= \frac{3m}{k}\ln\left(\frac{gm}{gm - kv}\right)$$

When $t = \frac{3m}{k}\ln 2$

We have $\frac{gm}{gm - kv} = 2$

$\Rightarrow 1 - \frac{kv}{gm} = \frac{1}{2}$

$\Rightarrow v = \frac{gm}{2k}$

Question 16 (b) (i)

Criteria	Marks
• Provides correct proof	3
• Correctly applies integration by parts and attempts to reduce to integrals involving only $\sin(2\theta)$, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample answer:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin(2\theta)\,\sin^{2n}(2\theta)\,d\theta$$

$$u = \sin^{2n}(2\theta) \qquad \frac{dv}{d\theta} = \sin 2\theta$$

$$\frac{du}{d\theta} = 4n\sin^{2n-1}(2\theta)\cos 2\theta$$

$$v = -\frac{\cos(2\theta)}{2}$$

$$= \frac{1}{2}\sin^{2n}(2\theta) \times -\cos(2\theta)\Big|_0^{\frac{\pi}{2}} + 2n\int_0^{\frac{\pi}{2}} \sin^{2n-1}(2\theta)\,\cos^2(2\theta)\,d\theta$$

$$\therefore\ I_n = 2n\left[\int_0^{\frac{\pi}{2}} \sin^{2n-1}(2\theta)\,d\theta - \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\,d\theta\right]$$

$$I_n(2n+1) = 2n\,I_{n-1}$$

$$\therefore I_n = \frac{2n}{(2n+1)}I_{n-1}.$$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Writes I_n in terms of I_0 and evaluates I_0, or equivalent merit	2
• Calculates I_0 OR • Writes I_{n} in terms of I_{n-2} OR • Equivalent merit	1

Sample answer:

Applying the reduction formula,

$$I_n = \frac{2n}{2n+1} \times I_{n-1}$$

$$= \frac{2n(2n-2)}{(2n+1)(2n-1)} \times I_{n-2}$$

$$= \frac{2n(2n-2)(2n-4)\ldots\ldots}{(2n+1)(2n-1)(2n-3)\ldots\ldots} \times \frac{2}{3} I_0$$

$$= \frac{[2n(2n-2)\ldots 2][2n(2n-2)\ldots 2] \times I_0}{(2n+1)(2n)(2n-1)(2n-2)\ldots\ldots 3 \times 2}$$

$$= \frac{2^{2n}(n!)^2 \times I_0}{(2n+1)!}$$

Now $I_0 = \int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta$

$$= -\frac{1}{2}\cos(2\theta)\Big|_0^{\frac{\pi}{2}}$$

$$= 1$$

$$\therefore\ I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

Question 16 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Obtains an integral of $\sin^{2n+1} 2\theta$, or equivalent merit	2
• Attempts the substitution $x = \sin^2\theta$, or equivalent merit	1

Sample answer:

$$J_n = \int_0^1 x^n (1-x)^n \, dx$$

Put $x = \sin^2\theta$

$$\frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\therefore\ J_n = \int_0^{\frac{\pi}{2}} 2\sin^{2n+1}(\theta)\,\cos^{2n+1}(\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n}}\left(2\sin\theta\,\cos\theta\right)^{2n+1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n}}\sin^{2n+1}(2\theta)\,d\theta$$

$$= \frac{1}{2^{2n}} I_n$$

$$= \frac{(n!)^2}{(2n+1)!}$$

Question 16 (b) (iv)

Criteria	Marks
• Provides correct solution	2
• Finds the maximum value of $x(1-x)$, for $0 \le x \le 1$, or equivalent merit	1

Sample answer:

The quadratic expression $x(1-x)$ has maximum value of $\dfrac{1}{4}$ when $x = \dfrac{1}{2}$.

$$\therefore \ J_n = \int_0^1 x^n (1-x)^n \, dx \ \le \int_0^1 \left(\frac{1}{4}\right)^n dx$$

$$= \frac{1}{2^{2n}}$$

$$\therefore \ \frac{(n!)^2}{(2n+1)!} \le \frac{1}{2^{2n}}$$

$$\Rightarrow \left(2^n n!\right)^2 \le \left(2n+1\right)!$$

2020 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX-V1 Further Work with Vectors	MEX12–3
2	1	MEX-N2 Using Complex Numbers	MEX12–4
3	1	MEX-V1 Further Work with Vectors	MEX12–3
4	1	MEX-N1 Introduction to Complex Numbers	MEX12–4
5	1	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
6	1	MEX-C1 Further Integration	MEX12–5
7	1	MEX-P1 The Nature of Proof	MEX12–2
8	1	MEX-P1 The Nature of Proof	MEX12–2
9	1	MEX-N1 Introduction to Complex Numbers	MEX12–4
10	1	MEX-C1 Further Integration	MEX12–8

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	MEX-N1 Introduction to Complex Numbers	MEX12–4
11 (a) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12–4
11 (b)	3	MEX-C1 Further Integration	MEX12–5
11 (c)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
11 (d)	3	MEX-V1 Further Work with Vectors	MEX12–3
11 (e)	4	MEX-N2 Using Complex Numbers	MEX12–4
12 (a) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (a) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (a) (iii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (b) (i)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (b) (ii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (b) (iii)	2	MEX-M1 Applications of Calculus to Mechanics 4	MEX12–7
13 (a)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
13 (b)	3	MEX-V1 Further Work with Vectors	MEX12–3
13 (c) (i)	2	MEX-P1 The Nature of Proof	MEX12–2

Question	Marks	Content	Syllabus outcomes
13 (c) (ii)	1	MEX-P1 The Nature of Proof	MEX12–2
13 (d) (i)	1	MEX-N1 Introduction To Complex Numbers	MEX12–4
13 (d) (ii)	3	MEX-N1 Introduction To Complex Numbers	MEX12–4
13 (d) (iii)	2	MEX-N1 Introduction To Complex Numbers	MEX12–5
14 (a) (i)	2	MEX-N2 Using Complex Numbers	MEX12–4
14 (a) (ii)	3	MEX-N2 Using Complex Numbers	MEX12–4
14 (b)	4	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
14 (c)	4	MEX-P2 Further Proof by Mathematical Induction	MEX12–2, MEX12–8
14 (d)	3	MEX-P1 The Nature of Proof	MEX12–2
15 (a) (i)	2	MEX-P1 The Nature of Proof	MEX12–2
15 (a) (ii)	1	MEX-P1 The Nature of Proof	MEX12–2
15 (a) (iii)	3	MEX-P1 The Nature of Proof	MEX12–2
15 (b) (i)	2	MEX-V1 Further Work with Vectors	MEX12–3
15 (b) (ii)	1	MEX-V1 Further Work with Vectors	MEX12–3
15 (b) (iii)	3	MEX-V1 Further Work with Vectors	MEX12–3
15 (b) (iv)	1	MEX-V1 Further Work with Vectors	MEX12–3
16 (a) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
16 (a) (ii)	3	MEX-C1 Further Integration	MEX12–5
16 (b) (i)	3	MEX-C1 Further Integration	MEX12–5
16 (b) (ii)	3	MEX-C1 Further Integration	MEX12–5
16 (b) (iii)	2	MEX-C1 Further Integration	MEX12–5
16 (b) (iv)	2	MEX-P1 The Nature of Proof	MEX12–2