Centre Number

Student Number

NSW Education Standards Authority

2023 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 100

Section I – 10 marks (pages 2–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8–18)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is equal to $(a + ib)^3$?

A. $\left(a^3 - 3ab^2\right) + i\left(3a^2b + b^3\right)$

B. $\left(a^3 + 3ab^2\right) + i\left(3a^2b + b^3\right)$

C. $\left(a^3 - 3ab^2\right) + i\left(3a^2b - b^3\right)$

D. $\left(a^3 + 3ab^2\right) + i\left(3a^2b - b^3\right)$

2 Consider the following statement.

'If an animal is a herbivore, then it does not eat meat.'

Which of the following is the converse of this statement?

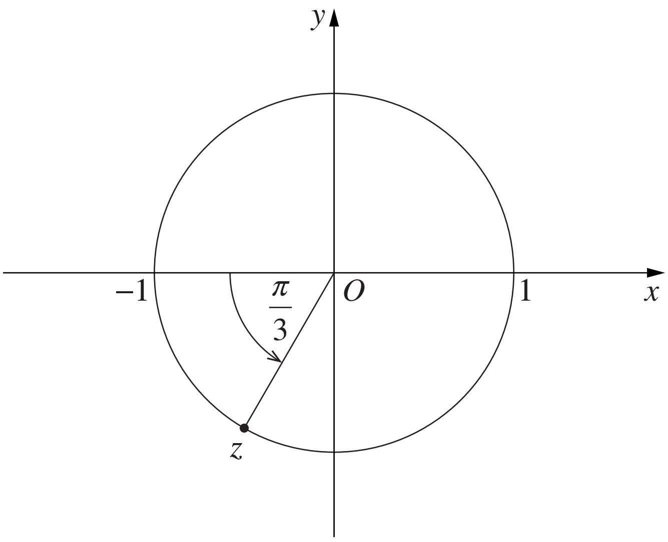
A. If an animal is a herbivore, then it eats meat.

B. If an animal is not a herbivore, then it eats meat.

C. If an animal eats meat, then it is not a herbivore.

D. If an animal does not eat meat, then it is a herbivore.

3 A complex number z lies on the unit circle in the complex plane, as shown in the diagram.



Which of the following complex numbers is equal to $\overline{z}$?

A. $-z$

B. z^2

C. $-z^3$

D. z^4

4 Consider the following statement about real numbers.

> 'Whichever positive number r you pick, it is possible to find a number x greater than 1 such that $\dfrac{\ln x}{x^3} < r$.'

When this statement is written in the formal language of proof, which of the following is obtained?

A. $\forall\, x > 1 \qquad \exists\, r > 0 \qquad \dfrac{\ln x}{x^3} < r$

B. $\exists\, x > 1 \qquad \forall\, r > 0 \qquad \dfrac{\ln x}{x^3} < r$

C. $\forall\, r > 0 \qquad \exists\, x > 1 \qquad \dfrac{\ln x}{x^3} < r$

D. $\exists\, r > 0 \qquad \forall\, x > 1 \qquad \dfrac{\ln x}{x^3} < r$

5 Which of the following is a true statement about the lines $\ell_1 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ and $\ell_2 = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$?

A. ℓ_1 and ℓ_2 are the same line.

B. ℓ_1 and ℓ_2 are not parallel and they intersect.

C. ℓ_1 and ℓ_2 are parallel and they do not intersect.

D. ℓ_1 and ℓ_2 are not parallel and they do not intersect.

6 Which of the following functions does NOT describe simple harmonic motion?

A. $x = \cos^2 t - \sin 2t$

B. $x = \sin 4t + 4\cos 2t$

C. $x = 2\sin 3t - 4\cos 3t + 5$

D. $x = 4\cos\left(2t + \frac{\pi}{2}\right) + 5\sin\left(2t - \frac{\pi}{4}\right)$

7 Which of the following statements about complex numbers is true?

A. For all real numbers x, y, θ with $x \neq 0$,

$$\tan\theta = \frac{y}{x} \quad \Rightarrow \quad x + iy = re^{i\theta},$$

for some real number r.

B. For all non-zero complex numbers z_1 and z_2,

$$\operatorname{Arg}(z_1) = \theta_1 \text{ and } \operatorname{Arg}(z_2) = \theta_2 \quad \Rightarrow \quad \operatorname{Arg}(z_1 z_2) = \theta_1 + \theta_2,$$

where Arg denotes the principal argument.

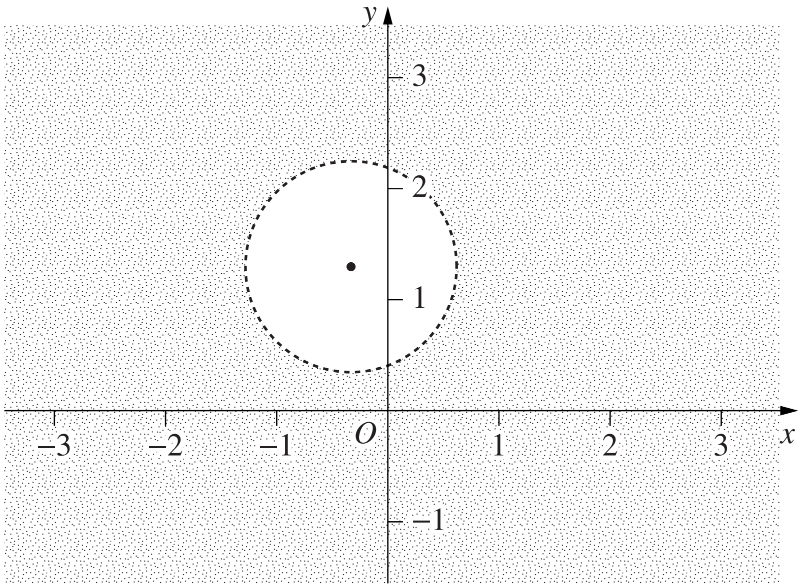
C. For all real numbers $r_1, r_2, \theta_1, \theta_2$ with $r_1, r_2 > 0$,

$$r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \quad \Rightarrow \quad r_1 = r_2 \text{ and } \theta_1 = \theta_2.$$

D. For all real numbers x, y, r, θ with $r > 0$ and $x \neq 0$,

$$x + iy = re^{i\theta} \quad \Rightarrow \quad \theta = \arctan\left(\frac{y}{x}\right).$$

8 A shaded region on a complex plane is shown.



Which relation best describes the region shaded on the complex plane?

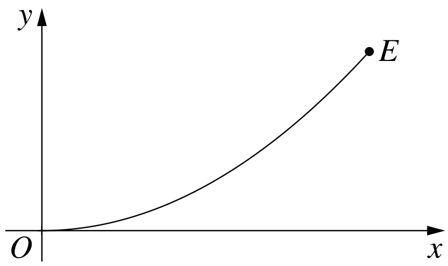
A. $|z-i| > 2|z-1|$

B. $|z-i| < 2|z-1|$

C. $|z-1| > 2|z-i|$

D. $|z-1| < 2|z-i|$

9 A particle travels along a curve from O to E in the xy-plane, as shown in the diagram.



The position vector of the particle is $\boldsymbol{r}$, its velocity is $\boldsymbol{v}$, and its acceleration is $\boldsymbol{a}$.

While travelling from O to E, the particle is always slowing down.

Which of the following is consistent with the motion of the particle?

A. $\boldsymbol{r} \cdot \boldsymbol{v} \leq 0$ and $\boldsymbol{a} \cdot \boldsymbol{v} \geq 0$

B. $\boldsymbol{r} \cdot \boldsymbol{v} \leq 0$ and $\boldsymbol{a} \cdot \boldsymbol{v} \leq 0$

C. $\boldsymbol{r} \cdot \boldsymbol{v} \geq 0$ and $\boldsymbol{a} \cdot \boldsymbol{v} \geq 0$

D. $\boldsymbol{r} \cdot \boldsymbol{v} \geq 0$ and $\boldsymbol{a} \cdot \boldsymbol{v} \leq 0$

10 Consider any three-dimensional vectors $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{OB}$ and $\underset{\sim}{c} = \overrightarrow{OC}$ that satisfy these three conditions

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 1$$
$$\underset{\sim}{b} \cdot \underset{\sim}{c} = 2$$
$$\underset{\sim}{c} \cdot \underset{\sim}{a} = 3.$$

Which of the following statements about the vectors is true?

A. Two of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ could be unit vectors.

B. The points A, B and C could lie on a sphere centred at O.

C. For any three-dimensional vector $\underset{\sim}{a}$, vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ can be found so that $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ satisfy these three conditions.

D. $\forall$ $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ satisfying the conditions, $\exists$ r, s and t such that r, s and t are positive real numbers and $r\underset{\sim}{a} + s\underset{\sim}{b} + t\underset{\sim}{c} = \underset{\sim}{0}$.

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Solve the quadratic equation **2**

$$z^2 - 3z + 4 = 0,$$

where z is a complex number. Give your answers in Cartesian form.

(b) Find the angle between the vectors **3**

$$\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$$
$$\underset{\sim}{b} = -\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k},$$

giving your answer to the nearest degree.

(c) Find a vector equation of the line through the points $A(-3, 1, 5)$ and $B(0, 2, 3)$. **2**

(d) The quadrilaterals $ABCD$ and $ABEF$ are parallelograms. **2**

By considering $\overrightarrow{AB}$, show that $CDFE$ is also a parallelogram.

(e) A particle moves in simple harmonic motion described by the equation **2**

$$\ddot{x} = -9(x - 4).$$

Find the period and the central point of motion.

(f) Find $\displaystyle\int_0^2 \frac{5x - 3}{(x + 1)(x - 3)}\,dx$. **4**

Question 12 (15 marks) Use the Question 12 Writing Booklet

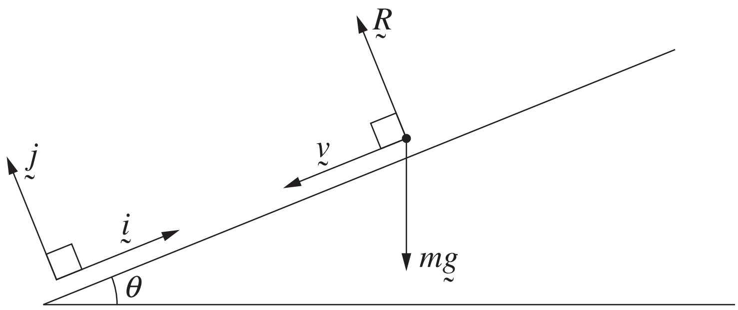
(a) Prove that $\sqrt{23}$ is irrational. **3**

(b) Prove that for all real numbers x and y, where $x^2 + y^2 \neq 0$, **2**

$$\frac{(x+y)^2}{x^2+y^2} \leq 2.$$

(c) An object with mass m kilograms slides down a smooth inclined plane with velocity $\underset{\sim}{v}(t)$, where t is the time in seconds after the object started sliding down the plane. The inclined plane makes an angle θ with the horizontal, as shown in the diagram. The normal reaction force is $\underset{\sim}{R}$. The acceleration due to gravity is $\underset{\sim}{g}$ and has magnitude g. No other forces act on the object.

The vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors parallel and perpendicular, respectively, to the plane, as shown in the diagram.



(i) Show that the resultant force on the object is $\underset{\sim}{F} = -(mg\sin\theta)\underset{\sim}{i}$. **2**

(ii) Given that the object is initially at rest, find its velocity $\underset{\sim}{v}(t)$ in terms of g, θ, t and $\underset{\sim}{i}$. **2**

Question 12 continues on page 10

(d) Find the cube roots of $2 - 2i$. Give your answer in exponential form. **3**

(e) The complex number $2 + i$ is a zero of the polynomial

$$P(z) = z^4 - 3z^3 + cz^2 + dz - 30$$

where c and d are real numbers.

(i) Explain why $2 - i$ is also a zero of the polynomial $P(z)$. **1**

(ii) Find the remaining zeros of the polynomial $P(z)$. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Find $\displaystyle\int \frac{1-x}{\sqrt{5-4x-x^2}}\,dx$. **3**

(b) (i) Show that $k^2 - 2k - 3 \geq 0$ for $k \geq 3$. **1**

(ii) Hence, or otherwise, use mathematical induction to prove that $2^n \geq n^2 - 2$, for all integers $n \geq 3$. **3**

Question 13 continues on page 12

Question 13 (continued)

(c) A particle of mass 1 kg is projected from the origin with speed 40 m s^{-1} at an angle 30° to the horizontal plane.

(i) Use the information above to show that the initial velocity of the particle is $\mathbf{v}(0) = \begin{pmatrix} 20\sqrt{3} \\ 20 \end{pmatrix}$. **1**

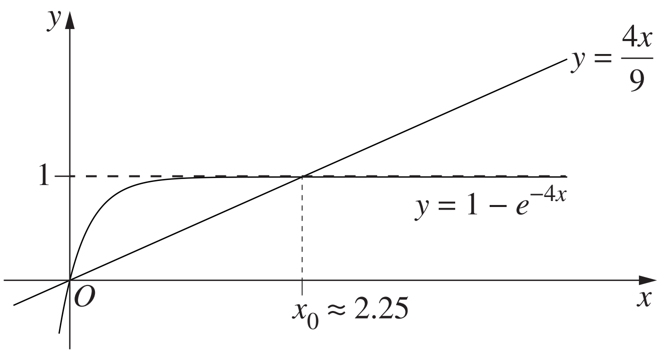
The forces acting on the particle are gravity and air resistance. The air resistance is proportional to the velocity vector with a constant of proportionality 4. Let the acceleration due to gravity be 10 m s^{-2}.

The position vector of the particle, at time t seconds after the particle is projected, is $\mathbf{r}(t)$ and the velocity vector is $\mathbf{v}(t)$.

(ii) Show that $\mathbf{v}(t) = \begin{pmatrix} 20\sqrt{3}e^{-4t} \\ \frac{45}{2}e^{-4t} - \frac{5}{2} \end{pmatrix}$. **3**

(iii) Show that $\mathbf{r}(t) = \begin{pmatrix} 5\sqrt{3}\left(1 - e^{-4t}\right) \\ \frac{45}{8}\left(1 - e^{-4t}\right) - \frac{5}{2}t \end{pmatrix}$. **2**

(iv) The graphs $y = 1 - e^{-4x}$ and $y = \frac{4x}{9}$ are given in the diagram below. **2**



Using the diagram, find the horizontal range of the particle, giving your answer rounded to one decimal place.

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Let z be the complex number $z = e^{\frac{i\pi}{6}}$ and w be the complex number $w = e^{\frac{3i\pi}{4}}$.

(i) By first writing z and w in Cartesian form, or otherwise, show that $|z+w|^2 = \dfrac{4-\sqrt{6}+\sqrt{2}}{2}$. **3**

(ii) The complex numbers z, w and $z+w$ are represented in the complex plane by the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively, where O is the origin. **2**

Show that $\angle AOC = \dfrac{7\pi}{24}$.

(iii) Deduce that $\cos\dfrac{7\pi}{24} = \dfrac{\sqrt{8-2\sqrt{6}+2\sqrt{2}}}{4}$. **1**

(b) The point P is 4 metres to the right of the origin O on a straight line. **3**

A particle is released from rest at P and moves along the straight line in simple harmonic motion about O, with period 8π seconds.

After 2π seconds, another particle is released from rest at P and also moves along this straight line in simple harmonic motion about O, with period 8π seconds.

Find when and where the two particles first collide.

(c) A projectile of mass M kg is launched vertically upwards from the origin with an initial speed v_0 m s^{-1}. The acceleration due to gravity is g m s^{-2}.

The projectile experiences a resistive force of magnitude kMv^2 newtons, where k is a positive constant and v is the speed of the projectile at time t seconds.

(i) The maximum height reached by the particle is H metres. **3**

Show that $H = \dfrac{1}{2k}\ln\left(\dfrac{kv_0^{\,2}+g}{g}\right)$.

(ii) When the projectile lands on the ground, its speed is v_1 m s^{-1}, where v_1 is less than the magnitude of the terminal velocity. **3**

Show that $g\left(v_0^{\,2}-v_1^{\,2}\right) = kv_0^{\,2}v_1^{\,2}$.

Question 15 (16 marks) Use the Question 15 Writing Booklet

(a) (i) Let $J_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$ where $n \geq 0$ is an integer. **3**

Show that $J_n = \dfrac{n-1}{n} J_{n-2}$ for all integers $n \geq 2$.

(ii) Let $I_n = \displaystyle\int_0^1 x^n(1-x)^n \, dx$ where n is a positive integer. **4**

By using the substitution $x = \sin^2\theta$, or otherwise,

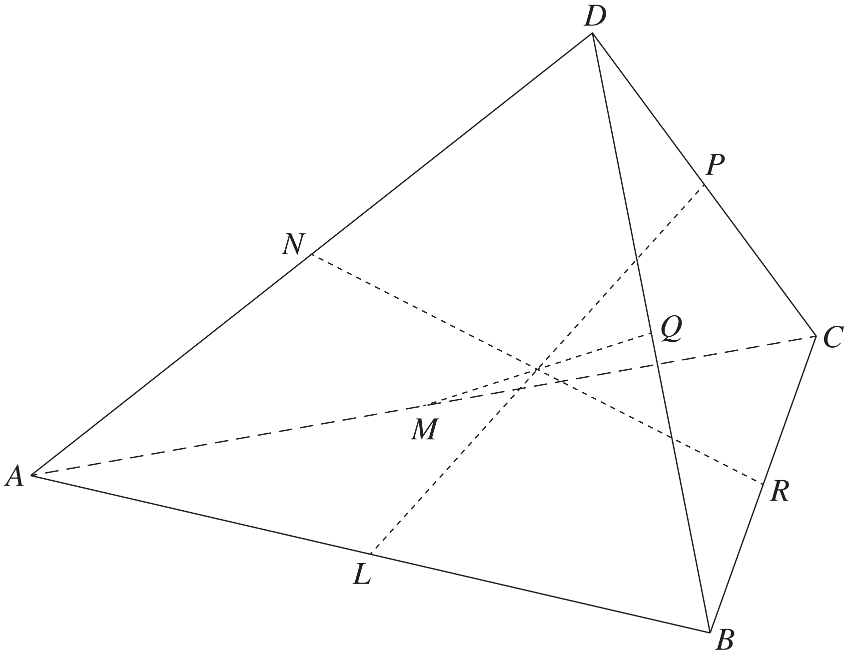
show that $I_n = \dfrac{1}{2^{2n}} \displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta$.

(iii) Hence, or otherwise, show that $I_n = \dfrac{n}{4n+2} I_{n-1}$, for all integers $n \geq 1$. **2**

Question 15 continues on page 15

Question 15 (continued)

(b) On the triangular pyramid $ABCD$, L is the midpoint of AB, M is the midpoint of AC, N is the midpoint of AD, P is the midpoint of CD, Q is the midpoint of BD and R is the midpoint of BC.



Let $\underset{\sim}{b} = \overrightarrow{AB}$, $\underset{\sim}{c} = \overrightarrow{AC}$ and $\underset{\sim}{d} = \overrightarrow{AD}$.

(i) Show that $\overrightarrow{LP} = \frac{1}{2}(-\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d})$. **1**

(ii) It can be shown that **3**

$$\overrightarrow{MQ} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{c} + \underset{\sim}{d})$$

and $$\overrightarrow{NR} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{d}).$$

(Do NOT prove these.)

Prove that

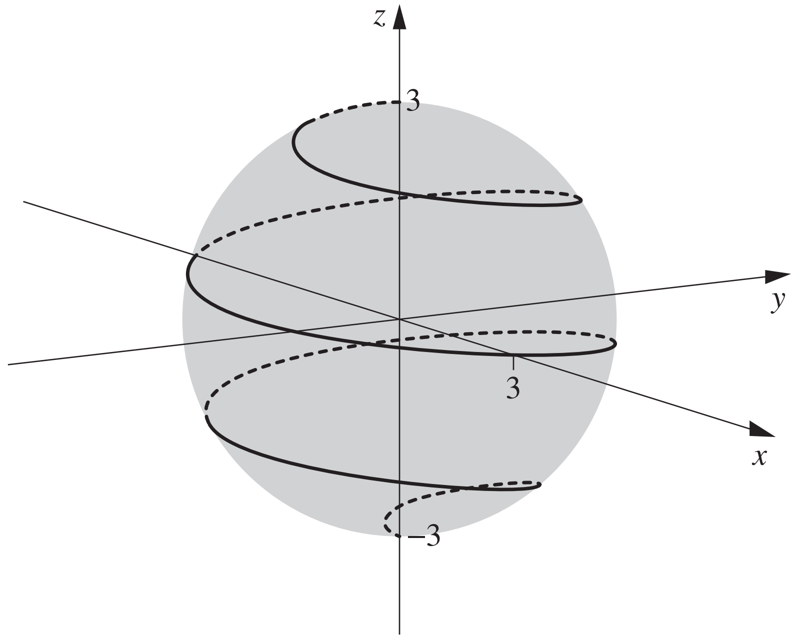
$$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{BD}\right|^2 + \left|\overrightarrow{CD}\right|^2 = 4\left(\left|\overrightarrow{LP}\right|^2 + \left|\overrightarrow{MQ}\right|^2 + \left|\overrightarrow{NR}\right|^2\right).$$

Question 15 continues on page 16

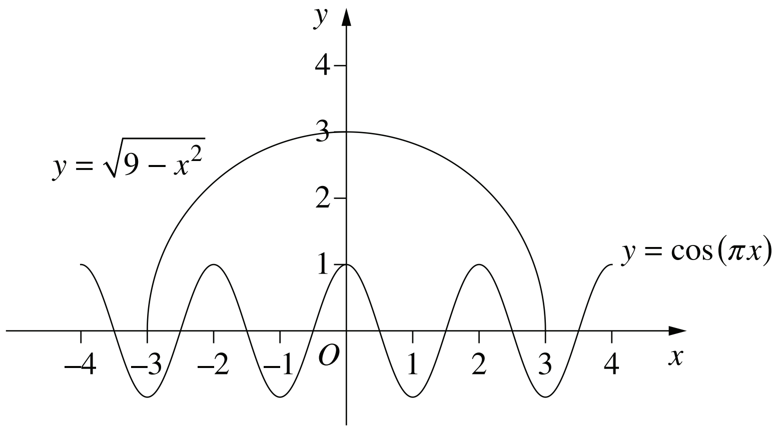
Question 15 (continued)

(c) A curve $\mathscr{C}$ spirals 3 times around the sphere centred at the origin and with radius 3, as shown. **3**

A particle is initially at the point $(0, 0, -3)$ and moves along the curve $\mathscr{C}$ on the surface of the sphere, ending at the point $(0, 0, 3)$.



By using the diagram below, which shows the graphs of the functions $f(x) = \cos(\pi x)$ and $g(x) = \sqrt{9 - x^2}$, and considering the graph $y = f(x)g(x)$, give a possible set of parametric equations that describe the curve $\mathscr{C}$.



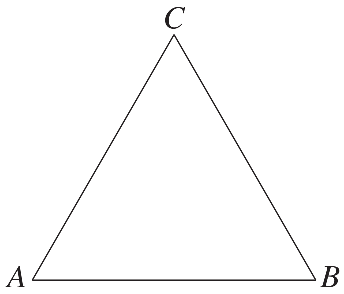
End of Question 15

Question 16 (14 marks) Use the Question 16 Writing Booklet

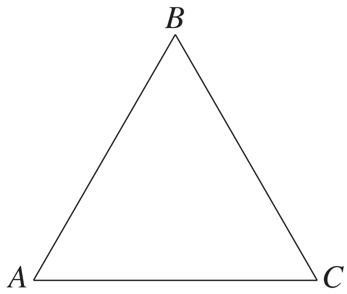
(a) Let w be the complex number $w = e^{\frac{2i\pi}{3}}$.

(i) Show that $1 + w + w^2 = 0$. **2**

The vertices of a triangle can be labelled A, B and C in anticlockwise or clockwise direction, as shown.



ABC is anticlockwise



ABC is clockwise

Three complex numbers a, b and c are represented in the complex plane by points A, B and C respectively.

(ii) Show that if triangle ABC is anticlockwise and equilateral, then $a + bw + cw^2 = 0$. **2**

(iii) It can be shown that if triangle ABC is clockwise and equilateral, then $a + bw^2 + cw = 0$. (Do NOT prove this.) **2**

Show that if ABC is an equilateral triangle, then

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

Question 16 continues on page 18

Question 16 (continued)

(b) (i) Prove that $x > \ln x$, for $x > 0$. **2**

(ii) Using part (i), or otherwise, prove that for all positive integers n, **3**

$$e^{n^2+n} > (n!)^2 .$$

(c) The complex numbers w and z both have modulus 1, and $\dfrac{\pi}{2} < \operatorname{Arg}\left(\dfrac{z}{w}\right) < \pi$, **3**

where Arg denotes the principal argument.

For real numbers x and y, consider the complex number $\dfrac{xz + yw}{z}$.

On an xy-plane, clearly sketch the region that contains all points (x, y) for which $\dfrac{\pi}{2} < \operatorname{Arg}\left(\dfrac{xz + yw}{z}\right) < \pi$.

End of paper

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2023 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

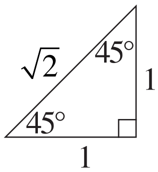
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

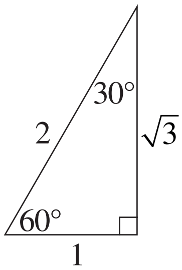


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

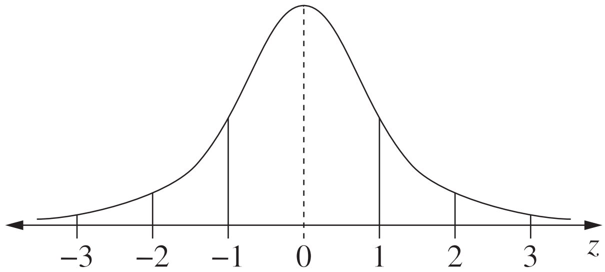
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$|\underset{\sim}{u}| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

2023 HSC Mathematics Extension 2 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	D
3	B
4	C
5	A
6	B
7	A
8	D
9	D
10	B

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Completes the square OR • Uses quadratic formula	1

Sample answer:

$$
\begin{aligned}
z &= \frac{3 \pm \sqrt{9-16}}{2} \\
&= \frac{3 \pm \sqrt{-7}}{2} \\
&= \frac{3}{2} \pm \frac{\sqrt{7}}{2} i
\end{aligned}
$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly evaluates $\underset{\sim}{a} \cdot \underset{\sim}{b}$ and $\lvert \underset{\sim}{a} \rvert$ and $\lvert \underset{\sim}{b} \rvert$, or equivalent merit	2
• Evaluates $\underset{\sim}{a} \cdot \underset{\sim}{b}$ or $\lvert \underset{\sim}{a} \rvert$ or $\lvert \underset{\sim}{b} \rvert$, or equivalent merit	1

Sample answer:

$$\begin{aligned}\underset{\sim}{a} \cdot \underset{\sim}{b} &= \big(1 \times (-1)\big) + (2 \times 4) + (-3 \times 2)\\ &= -1 + 8 - 6\\ &= 1\end{aligned}$$

$$\begin{aligned}\lvert \underset{\sim}{a} \rvert &= \sqrt{1^2 + 2^2 + (-3)^2}\\ &= \sqrt{14}\end{aligned}$$

$$\begin{aligned}\lvert \underset{\sim}{b} \rvert &= \sqrt{(-1)^2 + 4^2 + 2^2}\\ &= \sqrt{21}\end{aligned}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \lvert \underset{\sim}{a} \rvert \lvert \underset{\sim}{b} \rvert \cos\theta$$

$$\therefore \quad \cos\theta = \frac{1}{\sqrt{14} \cdot \sqrt{21}}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{1}{\sqrt{14} \cdot \sqrt{21}}\right)\\ &= 86.656\ldots\\ &\approx 87^\circ \qquad \text{(nearest degree)}\end{aligned}$$

Question 11 (c)

Criteria	Marks
• Provides a correct equation	2
• Finds $\overline{AB}$, or equivalent merit	1

Sample answer:

$$\overline{AB} = \overline{AO} + \overline{OB}$$

$$= \overrightarrow{OB} + \overrightarrow{OA}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$$

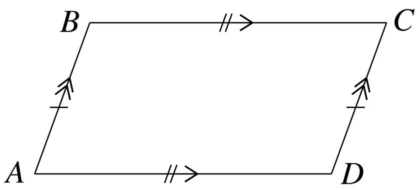
$$= \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

$\therefore$ A vector equation of the line is $\underset{\sim}{r} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \ \lambda \in \mathbb{R}$

Question 11 (d)

Criteria	Marks
• Provides correct proof	2
• Explains why $\overline{AB} = \overline{DC}$, or equivalent merit	1

Sample answer:

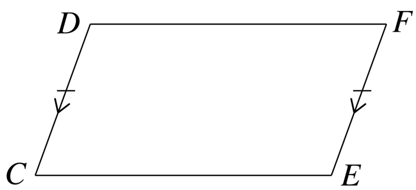


$\overline{AB} = \overline{DC}$ (opposite sides of a parallelogram are equal in length and parallel)

But in parallelogram $ABEF$

$\overline{AB} = \overline{FE}$ (opposite sides of a parallelogram are equal in length and parallel)

$\therefore \quad \overline{DC} = \overline{FE}$



$\therefore \quad CDFE$ is a parallelogram (one pair of opposite sides both equal in length and parallel)

Question 11 (e)

Criteria	Marks
• Provides correct period and central point of motion	2
• Provides correct period or centre, or equivalent merit	1

Sample answer:

$$\ddot{x} = -3^2(x-4)$$

$\therefore \; n = 3$ and $c = 4$

$$\therefore \; T = \frac{2\pi}{n} = \frac{2\pi}{3}$$

$\therefore$ Period $= \dfrac{2\pi}{3}$ and centre is $x = 4$

Question 11 (f)

Criteria	Marks
• Provides correct solution	4
• Obtains the antiderivative, or equivalent merit	3
• Shows $\frac{5x-3}{(x+1)(x-3)} = \frac{2}{x+1} + \frac{3}{x-3}$, or equivalent merit	2
• Attempts to use partial fractions, or equivalent merit	1

Sample answer:

Let $\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$$\begin{aligned}\therefore \quad 5x-3 &= A(x-3)+B(x+1)\\ &= Ax-3A+Bx+B\end{aligned}$$

Equating coefficients

$$5 = A + B \qquad (1)$$
$$-3 = -3A + B \qquad (2)$$

(1) – (2): $8 = 4A$

$\therefore A = 2$

In (1) $B = 3$

$$\begin{aligned}\therefore \quad \int_0^2 \frac{5x-3}{(x+1)(x-3)}\,dx &= \int_0^2 \frac{2}{x+1} + \frac{3}{x-3}\,dx\\ &= \Big[2\ln|x+1| + 3\ln|x-3|\Big]_0^2\\ &= \big(2\ln 3 + 3\ln|-1|\big) - \big(2\ln 1 + 3\ln|-3|\big)\\ &= 2\ln 3 - 3\ln 3\\ &= -\ln 3 \quad \text{or} \quad \ln\left(\frac{1}{3}\right)\end{aligned}$$

Question 12 (a)

Criteria	Marks
• Provides correct proof	3
• Shows that 23 is a factor of a relevant integer, or equivalent merit	2
• Attempts a proof by contradiction or writes $\sqrt{23} = \dfrac{p}{q}$ for integers p, q, or equivalent merit	1

Sample answer:

Assume the contradiction, that $\sqrt{23}$ is rational.

Let $\sqrt{23} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$.

Also assume that p and q are coprime.

$$\left(\sqrt{23}\right)^2 = \left(\frac{p}{q}\right)^2$$

$$23 = \frac{p^2}{q^2}$$

$$23q^2 = p^2$$

23 divides LHS, so 23 divides p^2.

Since 23 is prime, 23 also divides p.

So $p = 23m$, for some $m \in \mathbb{Z}$

$$\therefore \quad 23q^2 = (23m)^2$$

$$q^2 = 23m^2$$

So similarly 23 also divides q.

But now p and q have a common factor of 23, contradicting the assumption. Therefore, the assumption is false and $\sqrt{23}$ is irrational.

Question 12 (b)

Criteria	Marks
• Provides correct proof	2
• Expands the numerator of the LHS and splits into two fractions, or equivalent merit	1

Sample answer:

$$\frac{(x+y)^2}{x^2+y^2} = \frac{x^2+2xy+y^2}{x^2+y^2}$$

$$= 1 + \frac{2xy}{x^2+y^2}$$

But $\dfrac{a+b}{2} \geq \sqrt{ab}$, therefore $\dfrac{x^2+y^2}{2} \geq \sqrt{x^2y^2}$

and $\quad x^2+y^2 \geq 2|xy| \geq 2xy \qquad$ (since $\sqrt{x^2} = |x|$ and here we have $a,\ b \geq 0$

$$1 \geq \frac{2xy}{x^2+y^2}$$

$$\therefore \quad \frac{(x+y)^2}{x^2+y^2} \leq 1+1$$

$$\leq 2$$

<u>Alternative solution</u>

$$(x-y)^2 \geq 0 \qquad \forall x,\ y \in \mathbb{R}$$

$$x^2 - 2xy + y^2 \geq 0$$

$$2x^2 - 2xy + 2y^2 \geq x^2 + y^2$$

$$2\left(x^2+y^2\right) \geq x^2 + y^2 + 2xy$$

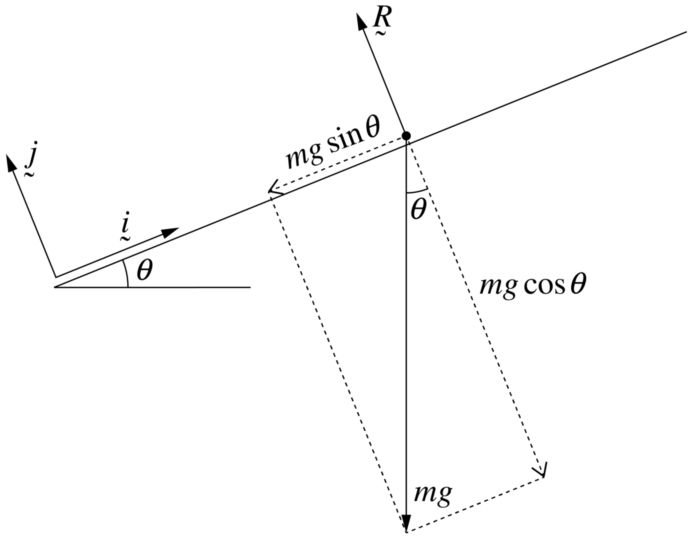
$$\geq (x+y)^2$$

$$\therefore \quad 2 \geq \frac{(x+y)^2}{x^2+y^2}$$

Question 12 (c) (i)

Criteria	Marks
• Provides correct proof	2
• Attempts to resolve a force into two perpendicular components or writes $\underset{\sim}{F} = \underset{\sim}{R} + m\underset{\sim}{g}$, or equivalent merit	1

Sample answer:



Resolving parallel and perpendicular to the slope gives

$$\underset{\sim}{R} - mg\cos\theta\,\underset{\sim}{j} = 0$$

and $\quad \underset{\sim}{F} = -(mg\sin\theta)\underset{\sim}{i} = 0$

$\therefore\quad$ Resultant force is $\underset{\sim}{F} = -(mg\sin\theta)\underset{\sim}{i}$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds a correct equation for acceleration and attempts to integrate, or equivalent merit	1

Sample answer:

$$m\underset{\sim}{a} = -mg\sin\theta\,\underset{\sim}{i}$$
$$\underset{\sim}{a} = -g\sin\theta\,\underset{\sim}{i}$$
$$\frac{d\underset{\sim}{v}}{dt} = -g\sin\theta\,\underset{\sim}{i}$$
$$\underset{\sim}{v} = -gt\sin\theta\,\underset{\sim}{i} + C$$

But when $t = 0$ $\quad \underset{\sim}{v} = 0$

So $\underset{\sim}{v} = -gt\sin\theta\,\underset{\sim}{i}$

Question 12 (d)

Criteria	Marks
• Provides correct solution	3
• Finds one correct cube root, or equivalent merit	2
• Writes $2-2i$ in exponential form, or equivalent merit	1

Sample answer:

Let $z^3 = 2-2i$ where $z = re^{i\theta}$, $r \in \mathbb{R}$

$\therefore \quad r^3 e^{3i\theta} = 2-2i$

But $2-2i = \sqrt{8}e^{-\frac{i\pi}{4}}$

$$\therefore \quad r^3 = \sqrt{8} \quad \text{and} \quad 3\theta = -\frac{\pi}{4} \text{ or } \frac{7\pi}{4} \text{ or } \frac{15\pi}{4}$$

$$r = \sqrt{2} \quad \text{and} \quad \theta = -\frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{15\pi}{12}$$

$$\therefore \quad z = \sqrt{2}e^{-\frac{i\pi}{12}} \text{ or } \sqrt{2}e^{\frac{7i\pi}{12}} \text{ or } \sqrt{2}e^{\frac{5i\pi}{4}}$$

Question 12 (e) (i)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

Since $P(z)$ has real coefficients, zeros occur in complex conjugate pairs

$\therefore \quad 2 + i$ a zero $\Rightarrow \overline{2+i} = 2 - i$ is also a zero

Question 12 (e) (ii)

Criteria	Marks
• Provides correct solution	2
• Writes down a correct equation involving the sum of the roots or the product of the roots, or attempts a long division of polynomials, or equivalent merit	1

Sample answer:

$P(z) = z^4 - 3z^3 + cz^2 + dz - 30$

Sum of roots $= -\dfrac{b}{a} = 3$

Product of roots $= \dfrac{e}{a} = -30$

Let other two zeros be α and β

$$\therefore \quad (2+i) + (2-i) + \alpha + \beta = 3$$
$$4 + \alpha + \beta = 3$$
$$\alpha + \beta = -1 \qquad (1)$$

$$(2+i)(2-i)\alpha\beta = -30$$
$$(4+1)\alpha\beta = -30$$
$$\alpha\beta = -6 \qquad (2)$$

$\therefore \quad$ Other zeros are $z = 2$ and $z = -3$

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Splits the integral into the sum of two integrals and correctly integrates one, or equivalent merit	2
• Completes the square for $5-4x-x^2$ or writes $1-x=\frac{1}{2}(-4-2x)+3$ or equivalent merit	1

Sample answer:

$$\int \frac{1-x}{\sqrt{5-4x-x^2}}\,dx$$

$$= \int \frac{-2-x}{\sqrt{5-4x-x^2}}\,dx + \int \frac{3}{\sqrt{5-4x-x^2}}\,dx$$

$$= \sqrt{5-4x-x^2} + \int \frac{3}{\sqrt{9-\left(4+4x+x^2\right)}}\,dx$$

$$= \sqrt{5-4x-x^2} + 3\sin^{-1}\left(\frac{x+2}{3}\right) + C$$

$$\frac{d}{dx}\sqrt{5-4x-x^2} = \frac{1}{2\sqrt{5-4x-x^2}} \times (-4-2x)$$

$$= \frac{-2-x}{\sqrt{5-4x-x^2}}$$

Question 13 (b) (i)

Criteria	Marks
• Provides correct proof	1

Sample answer:

$$\begin{aligned} k^2 - 2k - 3 &= k^2 - 2k + 1 - 4 \\ &= (k-1)^2 - 4 \end{aligned}$$

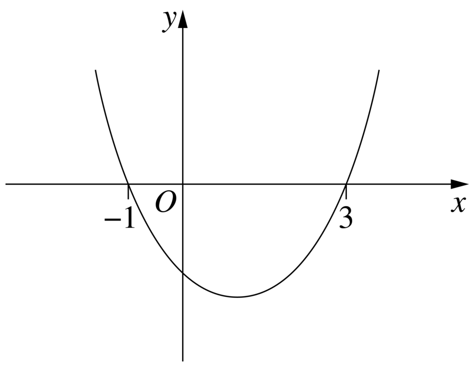
$(k-1)^2 - 4 \geq 0 \quad$ for all $k \geq 3$

$\therefore \; k^2 - 2k - 3 \geq 0 \quad$ for all $k \geq 3$

Alternative solution

$$\begin{aligned} k^2 - 2k - 3 &= (k-3)(k+1) \\ &\geq 0 \qquad \text{for } k \geq 3 \text{ or } k \leq -1 \end{aligned}$$

Since $k \geq 3$, we can ignore $k \leq -1$



Question 13 (b) (ii)

Criteria	Marks
• Provides correct proof	3
• Establishes the inductive step, or equivalent merit	2
• Establishes the base case, or equivalent merit	1

Sample answer:

Let $P(n)$ be the statement $2^n \geq n^2 - 2$

Consider $n = 3$ case

LHS $= 2^3 = 8$

RHS $= 3^2 - 2 = 9 - 2 = 7$

$8 \geq 7$ so $P(3)$ is true

Assume $P(k)$ is true for some $k \geq 3$

Thus, $2^k \geq k^2 - 2,\ k \geq 3$

Prove that $P(k + 1)$ is true.

ie $\quad 2^{k+1} \geq (k+1)^2 - 2$

$\quad\quad 2^{k+1} \geq k^2 + 2k - 1$

$$
\begin{aligned}
\text{LHS} &= 2^{k+1} \\
&= 2\left(2^k\right) \\
&\geq 2\left(k^2 - 2\right) && \text{since we assumed } P(k) \text{ is true} \\
&= 2k^2 - 4 \\
&= k^2 + 2k - 1 + k^2 - 2k - 3 \\
\therefore \text{LHS} &\geq k^2 + 2k - 1 && \text{using part (i)} \\
&= \text{RHS}
\end{aligned}
$$

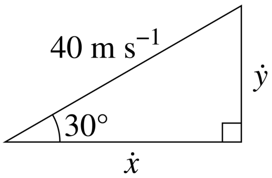
Therefore $P(k + 1)$ is true

Hence, using the principle of mathematical induction, $2^n \geq n^2 - 2$ for all integers $n \geq 3$.

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:



$$t = 0 \Rightarrow \quad \frac{\dot{x}}{40} = \cos 30°$$
$$\dot{x} = 40\cos 30°$$
$$= 20\sqrt{3}$$

$$t = 0 \Rightarrow \quad \frac{\dot{y}}{40} = \sin 30°$$
$$\dot{y} = 40\sin 30°$$
$$= 20$$

$$\therefore \quad \mathbf{v}(0) = \begin{pmatrix} 20\sqrt{3} \\ 20 \end{pmatrix}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds one component of $\underset{\sim}{v}(t)$, or equivalent merit	2
• Provides a relevant force or acceleration equation, or equivalent merit	1

Sample answer:

The resulting force $\underset{\sim}{F} = -4\underset{\sim}{v} + m\underset{\sim}{g}$

$$\begin{aligned} \underset{\sim}{F} &= 1 \times \underset{\sim}{a}(t) \\ &= \underset{\sim}{a}(t) \end{aligned}$$

$$\underset{\sim}{a}(t) = \begin{pmatrix} -4\dot{x} \\ -4\dot{y} - 10 \end{pmatrix} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$$

$\ddot{x} = -4\dot{x}$

$$\begin{aligned} \frac{d\dot{x}}{dt} = -4\dot{x} \quad &\Rightarrow \quad \frac{d\dot{x}}{\dot{x}} = -4dt \\ &\Rightarrow \quad \ln \dot{x} = -4t + C \\ &\Rightarrow \quad \dot{x} = Ae^{-4t} \\ &\qquad\ \ \dot{x} = 20\sqrt{3}e^{-4t} \qquad \text{since } \dot{x} = 20\sqrt{3} \text{ when } t = 0 \end{aligned}$$

$$\begin{aligned} \ddot{y} &= -4\dot{y} - 10 \\ &= -4\left(\dot{y} - \frac{5}{2}\right) \end{aligned}$$

So $\dot{y} = -\frac{5}{2} + Be^{-4t}$

When $t = 0$ $\quad \dot{y} = 20$

$20 = -\frac{5}{2} + B$

$\frac{45}{2} = B$

$\therefore \quad \dot{y} = -\frac{5}{2} + \frac{45}{2}e^{-4t}$

$$\underset{\sim}{v}(t) = \begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \end{pmatrix} = \begin{pmatrix} 20\sqrt{3}e^{-4t} \\ \frac{45}{2}e^{-4t} - \frac{5}{2} \end{pmatrix}$$

Question 13 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Finds one component of $\underset{\sim}{\mathbf{r}}(t)$, or equivalent merit	1

Sample answer:

By integrating $\mathbf{v}(t)$ with respect to t, we get

$$\mathbf{r}(t) = \begin{pmatrix} \dfrac{20\sqrt{3}}{-4}e^{-4t} + C_1 \\ \\ \dfrac{45}{-8}e^{-4t} - \dfrac{5}{2}t + C_2 \end{pmatrix} \qquad \text{Where } C_1 \text{ and } C_2 \text{ are constants}$$

$\mathbf{r}(0) = \mathbf{0}$ therefore

$-5\sqrt{3} + C_1 = 0 \quad \text{so} \quad C_1 = 5\sqrt{3}$

$-\dfrac{45}{8} + C_2 = 0 \quad \text{so} \quad C_2 = \dfrac{45}{8}$

Substituting back in $\mathbf{r}(t)$ yields

$$\mathbf{r}(t) = \begin{pmatrix} -5\sqrt{3}\,e^{-4t} + 5\sqrt{3} \\ \\ -\dfrac{45}{8}e^{-4t} - \dfrac{5}{2}t + \dfrac{45}{8} \end{pmatrix} = \begin{pmatrix} 5\sqrt{3}\left(1 - e^{-4t}\right) \\ \\ \dfrac{45}{8}\left(1 - e^{-4t}\right) - \dfrac{5}{2}t \end{pmatrix}$$

as required.

Question 13 (c) (iv)

Criteria	Marks
• Provides correct answer	2
• Finds time of flight, or equivalent merit	1

Sample answer:

The particle lands on the ground at a time t which satisfies $y = 0$, that is,

$$\frac{45}{8}\left(1-e^{-4t}\right)-\frac{5}{2}t=0$$

$$\frac{45}{8}\left(1-e^{-4t}\right)=\frac{5}{2}t$$

$$\left(1-e^{-4t}\right)=\frac{5}{2}\times\frac{8}{45}t$$

$$1-e^{-4t}=\frac{4}{9}t$$

Solution $t \approx 2.25$ according to the diagram provided

$$\begin{aligned}\therefore \quad \text{range} &= 5\sqrt{3}\left(1-e^{-4\times 2.25}\right)\\ &= 8.65918\ldots\\ &= 8.7\text{ m} \qquad \text{(rounded to 1 decimal place)}\end{aligned}$$

Question 14 (a) (i)

Criteria	Marks
• Provides correct proof	3
• Obtains $\lvert z+w\rvert^2 = \frac{1}{4}\left(\left(\sqrt{3}-\sqrt{2}\right)^2 + \left(1+\sqrt{2}\right)^2\right)$, or equivalent merit	2
• Finds z or w in Cartesian form, or equivalent merit	1

Sample answer:

We have $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $w = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

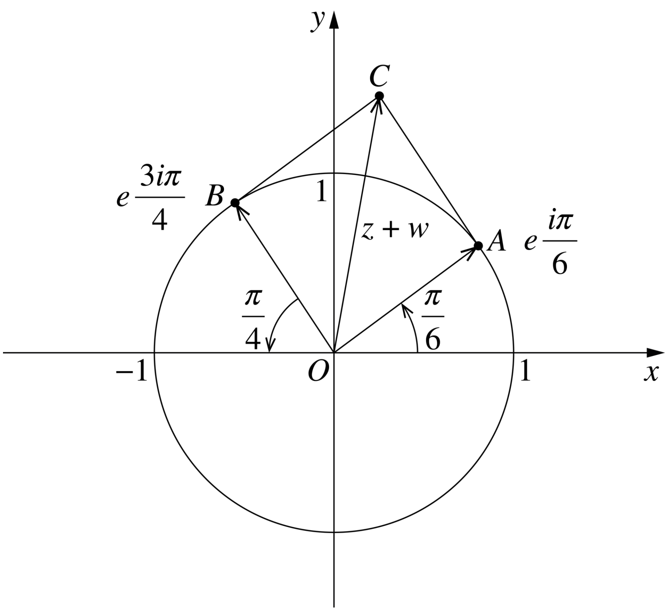
$$\begin{aligned} z+w &= \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)+\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) \\ &= \left(\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)i \\ &= \frac{\sqrt{3}-\sqrt{2}}{2}+\left(\frac{1+\sqrt{2}}{2}\right)i \end{aligned}$$

$$\begin{aligned} \lvert z+w\rvert^2 &= \frac{1}{4}\left[\left(\sqrt{3}-\sqrt{2}\right)^2+\left(1+\sqrt{2}\right)^2\right] \\ &= \frac{1}{4}\left[3-2\sqrt{6}+2+1+2\sqrt{2}+2\right] \\ &= \frac{1}{4}\left[8-2\sqrt{6}+2\sqrt{2}\right] \\ &= \frac{1}{2}\left[4-\sqrt{6}+\sqrt{2}\right] \end{aligned}$$

Question 14 (a) (ii)

Criteria	Marks
• Provides correct proof	2
• Provides why – $\angle AOB = \frac{7\pi}{12}$ OR – $\angle AOC = \text{Arg}(w) - \text{Arg}(z+w)$ OR – $OACB$ is a rhombus OR – OC bisects $\angle AOB$ Or equivalent merit	1

Sample answer:



$$\angle AOB = \pi - \frac{\pi}{4} - \frac{\pi}{6} = \frac{12\pi - 3\pi - 2\pi}{12} = \frac{7\pi}{12}$$

$OACB$ is a parallelogram by definition of vector addition.

Since it has two adjacent sides of equal length, it is a rhombus.

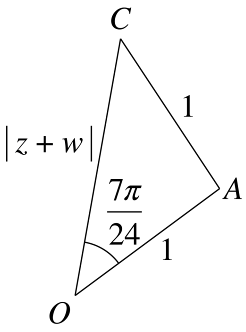
A diagonal of a rhombus bisects the angle at each vertex
so $\angle AOC = \frac{1}{2}\angle AOB = \frac{1}{2} \times \frac{7\pi}{12} = \frac{7\pi}{24}$

$$\angle AOC = \frac{7\pi}{24}$$

Question 14 (a) (iii)

Criteria	Marks
• Provides correct proof	1

Sample answer:



$$\cos\frac{7\pi}{24} = \frac{1^2 + |z+w|^2 - 1^2}{2\times 1\times |z+w|}$$

$$= \frac{1}{2}|z+w|$$

$$= \frac{1}{2}\times\frac{\sqrt{4-\sqrt{6}+\sqrt{2}}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$$

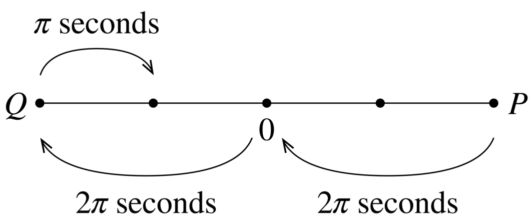
$$= \frac{\sqrt{8-2\sqrt{6}+2\sqrt{2}}}{2\times 2}$$

$$= \frac{\sqrt{8-2\sqrt{6}+2\sqrt{2}}}{4}$$

Question 14 (b)

Criteria	Marks
• Provides correct solution	3
• Finds when the two particles first collide, or equivalent merit	2
• Obtains equation of motion for the first particle or sketches a relevant diagram related to the motion of the first particle or equivalent merit	1

Sample answer:



Since period is 8π, first particle is at 0 when second particle leaves P.

First particle is at Q when second particle is at 0.

$\therefore$ Time of collision is 5π after first particle is released.

Amplitude = 4 m, starts at maximum point

Period = 8π $\quad \therefore \quad \dfrac{2\pi}{4} = 8\pi$

$$n = \frac{1}{4}$$

$\therefore$ Equation of motion is $x = 4\cos\left(\dfrac{1}{4}t\right)$

When $\quad t = 5\pi,\ x = 4\cos\left(\dfrac{5\pi}{4}\right)$

$$= -\frac{4}{\sqrt{2}}$$

$$= -2\sqrt{2}$$

$\therefore$ Collide $2\sqrt{2}$ metres left of 0 when $t = 5\pi$.

Alternative solution

Period $= 8\pi \quad \Rightarrow \quad \text{Period} = \dfrac{2\pi}{n}$

$$n = \frac{2\pi}{8\pi} = \frac{1}{4}$$

Particle 1

$$X_1 = 4\cos\left(\frac{1}{4}t\right)$$

Particle 2

$$X_2 = 4\cos\left(\frac{1}{4}(t - 2\pi)\right) \qquad t \ge 2\pi$$

Particles meet if $X_1 = X_2$

$$\therefore \quad 4\cos\left(\frac{1}{4}t\right) = 4\cos\left(\frac{1}{4}(t-2\pi)\right)$$

$$\therefore \quad \frac{1}{4}t = \frac{1}{4}(t-2\pi) + 2k\pi$$

$$t = t - 2\pi + 8k\pi$$

$$2\pi = 8k\pi$$

Which is impossible since k is an integer.

OR

$$\frac{1}{4}t = -\frac{1}{4}(t-2\pi) + 2k\pi$$

$$t = -t + 2\pi + 8k\pi$$

$$2t = 2\pi + 8k\pi$$

$$t = \pi + 4k\pi$$

- When $k = 0$, $t = \pi$ so the second particle has not been released yet.
- When $k = 1$, $t = 5\pi$ and that is the first time the particles collide.

$$X = 4\cos\left(\frac{5\pi}{4}\right) = 4 \times -\frac{1}{\sqrt{2}} = -2\sqrt{2}$$

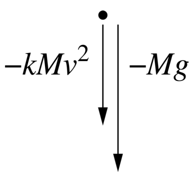
$\therefore$ Meet when $t = 5\pi$ at $X = -2\sqrt{2}$

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Obtains *x* as a function of *v* for upwards motion, or equivalent merit	2
• Obtains force equation for upwards part of flight, or equivalent merit	1

Sample answer:

Upward motion



$$M\ddot{x} = -kMv^2 - Mg \qquad \begin{aligned} t &= 0 \\ x &= 0 \\ \dot{x} &= v_0 \end{aligned}$$

$$\ddot{x} = -kv^2 - g$$

$$v\frac{dv}{dx} = -kv^2 - g$$

$$\int -\frac{v\,dv}{kv^2 + g} = \int dx$$

$$-\frac{1}{2k}\ln\left| kv_0{}^2 + g \right| = x + c$$

At $x = 0$, $v = v_0$ therefore

$$-\frac{1}{2k}\ln\left| kv_0{}^2 + g \right| = x + c$$

$$\therefore \quad x = \frac{1}{2k}\ln\left| kv_0{}^2 + g \right| - \frac{1}{2k}\ln\left| kv^2 + g \right|$$

$$= \frac{1}{2k}\ln\left| \frac{kv_0{}^2 + g}{kv^2 + g} \right|$$

When $x = H$, $v = 0$

$$\therefore \quad H = \frac{1}{2k}\ln\left| \frac{kv_0{}^2 + g}{g} \right|$$

But $kv_0{}^2 + g$ and $g > 0$ so

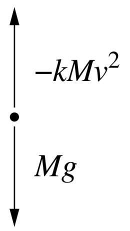
$$H = \frac{1}{2k}\ln\left(\frac{kv_0{}^2 + g}{g} \right) \qquad \text{———— (1)}$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Equates correct expressions for *H* from parts (i) and (ii), with absolute values correctly dealt with, or equivalent merit	2
• Finds the force or acceleration equation of the downwards motion	1

Sample answer:

Downward motion (downward direction positive)



$x = 0,\ v = 0$

$$\therefore \quad M\ddot{x} = Mg - kMv^2$$

$$\ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\int \frac{v\,dv}{g - kv^2} = \int dx$$

$$-\frac{1}{2k}\ln\left| g - kv^2 \right| = x + c$$

$$-\frac{1}{2k}\ln\left| g \right| = 0 + c \qquad x = 0,\ v = 0$$

$$\therefore \quad \frac{1}{2k}\ln\left| g \right| - \frac{1}{2k}\ln\left| g - kv^2 \right| = x$$

$$\frac{1}{2k}\ln\left| \frac{g}{g - kv^2} \right| = x$$

$x = H, \quad v = v_1$

$$\frac{1}{2k}\ln\left| \frac{g}{g - kv_1^{\,2}} \right| = H \qquad \text{———— (1)}$$

The particle never reaches terminal velocity so $g - kv_1^{\,2} > 0$

$$\frac{1}{2k}\ln\left(\frac{g}{g - kv_1^{\,2}} \right) = H \qquad \text{———— (2)}$$

Question 14 (c) (ii) (continued)

Equate equation (1) with (2)

$$\frac{1}{2k}\ln\left(\frac{kv_0{}^2+g}{g}\right) = \frac{1}{2k}\ln\left(\frac{g}{g-kv_1{}^2}\right)$$

$$\therefore \quad \frac{kv_0{}^2+g}{g} = \frac{g}{g-kv_1{}^2}$$

$$\left(kv_0{}^2+g\right)\left(g-kv_1{}^2\right) = g^2$$

$$kv_0{}^2g - k^2v_0{}^2v_1{}^2 + g^2 - gkv_1{}^2 = g^2$$

$$kv_0{}^2g - k^2v_0{}^2v_1{}^2 - gkv_1{}^2 = 0$$

$$kg\left(v_0{}^2 - v_1{}^2\right) = k^2v_0{}^2v_1{}^2$$

$$g\left(v_0{}^2 - v_1{}^2\right) = kv_0{}^2v_1{}^2$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct proof	3
• Applies integration by parts using $\sin\theta \times \sin^{n-1}\theta$ and the limits of integration, or equivalent merit	2
• Writes $\sin^n\theta = \sin\theta \times \sin^{n-1}\theta$ and attempts integration by parts on these factors OR • Applies integration by parts, or equivalent merit	1

Sample answer:

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \sin^{n-1}\theta \, d\theta$$

$$= \left[-\cos\theta \sin^{n-1}\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos\theta)(n-1)\sin^{n-2}\theta \cos\theta \, d\theta$$

$$= 0 - 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^2\theta \sin^{n-2}\theta \, d\theta$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \left(1 - \sin^2\theta\right)\sin^{n-2}\theta \, d\theta$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta - (n-1)\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$$

$$J_n = (n-1)J_{n-2} - (n-1)J_n$$

$$nJ_n = (n-1)J_{n-2}$$

$$J_n = \frac{n-1}{n} J_{n-2}$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct proof	4
• Uses an appropriate substitution to obtain a correct definite integral from 0 to π in terms of $\phi = 2\theta$, or equivalent merit	3
• Obtains a definite integral in terms of $\sin 2\theta$, or equivalent merit	2
• Correctly uses given substitution to obtain a definite integral in terms of θ or equivalent merit	1

Sample answer:

$$I_n = \int_0^1 x^n (1-x)^n \, dx$$

If $x = \sin^2\theta$

then $\dfrac{dx}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta$

at $\quad x = 0, \quad \theta = 0$

at $\quad x = 1, \quad \theta = \dfrac{\pi}{2}$

$$\begin{aligned}
I_n &= \int_0^1 x^n (1-x)^n \, dx \\
&= \int_0^{\frac{\pi}{2}} \sin^{2n}\theta \left(1 - \sin^2\theta\right)^n \sin 2\theta \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^{2n}\theta \cos^{2n}\theta \sin 2\theta \, d\theta \\
&= \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n}(2\theta) \sin(2\theta) \, d\theta \\
&= \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta) \, d\theta
\end{aligned}$$

Question 15 (a) (ii) (continued)

If $\phi = 2\theta$

$$\frac{d\phi}{d\theta} = 2$$

at $\theta = 0$, $\phi = 0$

at $\theta = \frac{\pi}{2}$, $\phi = \pi$

$$I_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\, d\theta$$

$$= \frac{1}{2^{2n}} \int_0^{\pi} \sin^{2n+1}\phi \times \frac{1}{2} d\phi$$

$$= \frac{1}{2^{2n+1}} \int_0^{\pi} \sin^{2n+1}\phi\, d\phi$$

but $\displaystyle\int_0^{\pi} \sin^{2n+1}\phi\, d\phi = 2\int_0^{\frac{\pi}{2}} \sin^{2n+1}\phi\, d\phi$

as $\sin^{2n+1}\phi$ is symmetrical about $\phi = \frac{\pi}{2}$

So $\displaystyle I_n = \frac{1}{2^{2n+1}} \int_0^{\pi} \sin^{2n+1}\phi\, d\phi$

$$= \frac{2}{2^{2n+1}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\phi\, d\phi$$

$$= \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\phi\, d\phi$$

which can be rewritten

$$I_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\, d\theta$$

Question 15 (a) (iii)

Criteria	Marks
• Provides correct proof	2
• Obtains I_n in terms of J_{2n-1}, or equivalent merit	1

Sample answer:

$$I_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta$$

$$= \frac{1}{2^{2n}} J_{2n+1}$$

$$= \frac{1}{2^{2n}} \left(\frac{(2n+1)-1}{2n+1} \right) J_{2n-1}$$

$$= \frac{1}{2^{2n}} \times \frac{2n}{2n+1} J_{2(n-1)+1}$$

But
$$\begin{aligned} I_n &= \frac{1}{2^{2n}} J_{2n+1} \\ 2^{2n} I_n &= J_{2n+1} \\ 2^{2(n-1)} I_{n-1} &= J_{2(n-1)+1} \\ &= J_{2n-1} \end{aligned}$$

$$I_n = \frac{1}{2^{2n}} \times \frac{2n}{2n+1} \times 2^{2(n-1)} I_{n-1}$$

$$= \frac{1}{2^2} \times \frac{2n}{2n+1} \times I_{n-1}$$

$$= \frac{n}{4n+2} I_{n-1}$$

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\overline{LP} = \overline{LB} + \overline{BC} + \overline{CP}$$

$$= \frac{1}{2}\underset{\sim}{b} + \left(\underset{\sim}{c} - \underset{\sim}{b}\right) + \frac{1}{2}\left(\underset{\sim}{d} - \underset{\sim}{c}\right)$$

$$= \frac{1}{2}\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{d} - \frac{1}{2}\underset{\sim}{c}$$

$$= -\frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{d}$$

$$= \frac{1}{2}\left(-\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}\right)$$

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly expands LHS or RHS in terms of $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{d}$ using dot products, or equivalent merit	2
• Obtains $\left\|\overrightarrow{LP}\right\|^2 = \left\|\underset{\sim}{c}\right\|^2 + 2\underset{\sim}{c}\cdot\underset{\sim}{d} - 2\underset{\sim}{c}\cdot\underset{\sim}{b} + \left\|\underset{\sim}{d}-\underset{\sim}{b}\right\|^2$, or equivalent merit	1

Sample answer:

$$\overrightarrow{MQ} = \frac{1}{2}\left(\underset{\sim}{d}+\underset{\sim}{b}-\underset{\sim}{c}\right)$$

$$\overrightarrow{NR} = \frac{1}{2}\left(\underset{\sim}{b}+\underset{\sim}{c}-\underset{\sim}{d}\right)$$

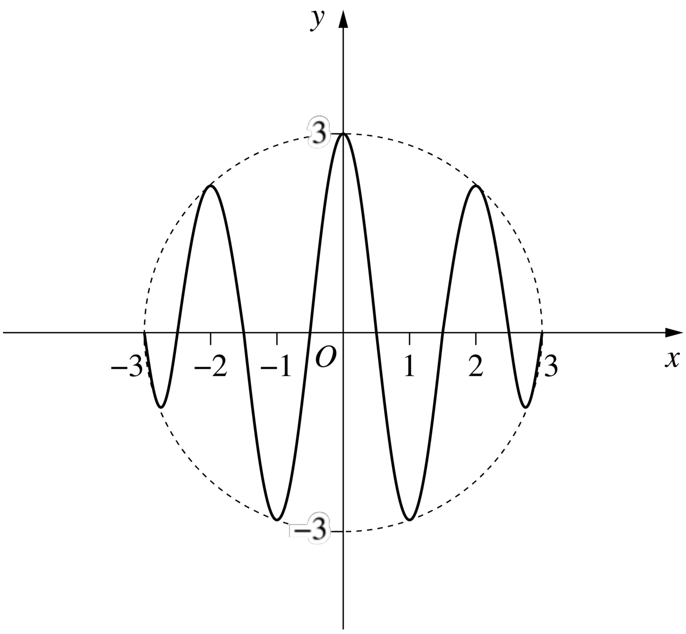
$$\begin{aligned}
\text{RHS} &= 4\left(\left|\overrightarrow{LP}\right|^2 + \left|\overrightarrow{MQ}\right|^2 + \left|\overrightarrow{NR}\right|^2\right)\\
&= 4\left(\overrightarrow{LP}\cdot\overrightarrow{LP} + \overrightarrow{MQ}\cdot\overrightarrow{MQ} + \overrightarrow{NR}\cdot\overrightarrow{NR}\right)\\
&= 4\Big(\frac{1}{4}\left(-\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d}\right)\cdot\left(-\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d}\right) +\\
&\qquad \frac{1}{4}\left(\underset{\sim}{d}+\underset{\sim}{b}-\underset{\sim}{c}\right)\cdot\left(\underset{\sim}{d}+\underset{\sim}{b}-\underset{\sim}{c}\right) +\\
&\qquad \frac{1}{4}\left(\underset{\sim}{b}+\underset{\sim}{c}-\underset{\sim}{d}\right)\cdot\left(\underset{\sim}{b}+\underset{\sim}{c}-\underset{\sim}{d}\right)\Big)\\
&= \left(\underset{\sim}{c}+\left(\underset{\sim}{d}-\underset{\sim}{b}\right)\right)\cdot\left(\underset{\sim}{c}+\left(\underset{\sim}{d}-\underset{\sim}{b}\right)\right) + \left(\underset{\sim}{d}+\left(\underset{\sim}{b}-\underset{\sim}{c}\right)\right)\cdot\left(\underset{\sim}{d}+\left(\underset{\sim}{b}-\underset{\sim}{c}\right)\right) + \left(\underset{\sim}{b}+\left(\underset{\sim}{c}-\underset{\sim}{d}\right)\right)\cdot\left(\underset{\sim}{b}+\left(\underset{\sim}{c}-\underset{\sim}{d}\right)\right)\\
&= \left|\underset{\sim}{c}\right|^2 + 2\underset{\sim}{c}\cdot\left(\underset{\sim}{d}-\underset{\sim}{b}\right) + \left|\underset{\sim}{d}-\underset{\sim}{b}\right|^2 +\\
&\qquad \left|\underset{\sim}{d}\right|^2 + 2\underset{\sim}{d}\cdot\left(\underset{\sim}{b}-\underset{\sim}{c}\right) + \left|\underset{\sim}{b}-\underset{\sim}{c}\right|^2 +\\
&\qquad \left|\underset{\sim}{b}\right|^2 + 2\underset{\sim}{b}\cdot\left(\underset{\sim}{c}-\underset{\sim}{d}\right) + \left|\underset{\sim}{c}-\underset{\sim}{d}\right|^2\\
&= \left|\underset{\sim}{b}\right|^2 + \left|\underset{\sim}{c}\right|^2 + \left|\underset{\sim}{d}\right|^2 + \left|\underset{\sim}{b}-\underset{\sim}{c}\right|^2 + \left|\underset{\sim}{d}-\underset{\sim}{b}\right|^2 + \left|\underset{\sim}{c}-\underset{\sim}{d}\right|^2 +\\
&\qquad 2\left[\underset{\sim}{c}\cdot\underset{\sim}{d} - \underset{\sim}{c}\cdot\underset{\sim}{b} + \underset{\sim}{d}\cdot\underset{\sim}{b} - \underset{\sim}{d}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{b}\cdot\underset{\sim}{d}\right]\\
&= \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{BD}\right|^2 + \left|\overrightarrow{CD}\right|^2 + 2\times 0\\
&= \text{LHS}
\end{aligned}$$

Question 15 (c)

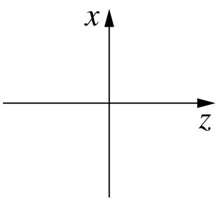
Criteria	Marks
• Provides correct solution	3
• Provides parametric equations of a curve that lies on the sphere, or equivalent merit	2
• Sketches the graph $y = f(x)g(x)$, or equivalent merit	1

Sample answer:

Graph of $y = \cos(\pi x)\sqrt{9 - x^2}$ for $-3 \le x \le 3$



Projecting curve $\mathscr{C}$ on sphere onto xz-plane gives the curve above



So if $z = t$

$$x = \cos(\pi t)\sqrt{9 - t^2}$$

Similarly, projecting curve onto yz-plane gives $y = -\sin(\pi t)\sqrt{9 - t^2}$

$\therefore$ Parametric equations are

$$x = \cos(\pi t)\sqrt{9 - t^2}$$
$$y = -\sin(\pi t)\sqrt{9 - t^2}$$
$$z = t$$

Question 15 (c) (continued)

Alternative solution

Because the curve lies on the sphere, we must have $x^2 + y^2 + z^2 = 3^2$

Given the hint about using $\sqrt{9-t^2}\cos(\pi t)$ and noticing that

$$\left(\sqrt{9-t^2}\cos(\pi t)\right)^2 + \left(\sqrt{9-t^2}\sin(\pi t)\right)^2 + t^2 = 3^2$$

as well as the fact that z increases as the point travels on the curve whereas x and y change signs, one can try the following

$z = t$

One of x and y is $\pm\sqrt{9-t^2}\cos(\pi t)$ and the other one is $\pm\sqrt{9-t^2}\sin(\pi t)$.

- We notice that when $t = z = 0$, x is positive and reaches a maximum value so $x(t) = \sqrt{9-t^2}\cos(\pi t)$.
- This leaves $y = \sqrt{9-t^2}\sin(\pi t)$ or $y = -\sqrt{9-t^2}\sin(\pi t)$. Given that shortly after when $t = z = 0$, $y < 0$ we get $y(t) = -\sqrt{9-t^2}\sin(\pi t)$.

Finally
$$x = \sqrt{9-t^2}\cos(\pi t)$$
$$y = -\sqrt{9-t^2}\sin(\pi t)$$
$$z = t$$

Question 16 (a) (i)

Criteria	Marks
• Provides correct proof	2
• Recognises that $1+w+w^2$ is a factor of $1-w^3$, or equivalent merit	1

Sample answer:

As $w \neq 1$

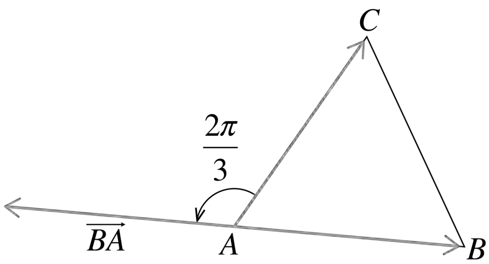
$$1+w+w^2 = \frac{1-w^3}{1-w} = 0 \quad \text{since} \quad w^3 = \left(e^{\frac{2i\pi}{3}}\right)^3 = 1$$

Question 16 (a) (ii)

Criteria	Marks
• Attempts to use the fact that $\overline{AC}$ rotated by $\frac{2\pi}{3}$ is $\overline{BA}$, or equivalent merit	1

Sample answer:

Suppose triangle ABC is equilateral and anticlockwise



Rotating $\overline{AC}$ by $+\frac{2\pi}{3}$ brings it onto $\overline{BA}$ so $w(c-a) = a-b$

$$a(1+w) - b - wc = 0$$
$$\Rightarrow \quad -aw^2 - b - wc = 0 \qquad \text{(by part (i))}$$
$$\Rightarrow \quad a + bw + w^2c = 0 \qquad \text{(multiplying by } -w\text{)}$$

Question 16 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Identifies that in either situation the product is 0, or equivalent merit	1

Sample answer:

Suppose ABC is equilateral.

It is either anticlockwise or clockwise.

Therefore $a + bw + cw^2 = 0$ or $a + bw^2 + cw = 0$

This is equivalent to $\left(a + bw + cw^2\right)\left(a + bw^2 + cw\right) = 0$

Since a product is zero if and only if one of its factors is zero,

$$\left(a + bw + cw^2\right)\left(a + bw^2 + cw\right) = 0$$

$$\Leftrightarrow \quad a^2 + bw^3 + cw^3 + ab\left(w^2 + w\right) + bc\left(w^2 + w^4\right) + ac\left(w + w^2\right) = 0$$

$$\Leftrightarrow \quad a^2 + b^2 + c^2 - (ab + bc + ac) = 0 \qquad \text{(since } w^3 = 1 \text{ and } w + w^2 = -1 \text{ by part (i))}$$

$$\Leftrightarrow \quad a^2 + b^2 + c^2 = ab + bc + ac \qquad \text{as required}$$

Question 16 (b) (i)

Criteria	Marks
• Provides correct proof	2
• Defines a function $f(x) = x - \ln x$, or equivalent merit	1

Sample answer:

Let $f(x) = x - \ln x$ for $x > 0$

$f'(x)$ has the same sign as $x - 1$ when $x > 0$

t	0	1	$-\infty$
$f'(t)$	$-$	0	$+$
f	$\searrow$	1	$\nearrow$

This shows that $\forall x > 0 \quad f(x) \geq 1$
that is, $x - \ln x \geq 1$
This implies $\forall x > 0 \quad x - \ln x > 0 \quad$ so $x > \ln x$

<u>Alternative explanation</u>

$$f'(x) = 1 - \frac{1}{x}$$

$$f''(x) = \frac{1}{x^2} > 0, \qquad \forall x > 0$$

$\therefore$ Curve is concave up for all $x > 0$

$$f'(x) \geq 0 \quad \text{when} \quad 1 \geq \frac{1}{x}$$

$$x \geq 1 \quad (\text{since } x > 0)$$

$$\begin{aligned} f(1) &= 1 - \ln 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \therefore \quad f(x) &\geq 1 \qquad \forall x > 0 \\ &> 0 \end{aligned}$$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct proof	3
• Applies part (i) to each term of $\sum_{i=1}^{n} \ln i$ OR • Establishes inductive step OR • Recognises that the question is equivalent to showing $n^2 + n > 2\left(\sum_{i=1}^{n} \ln i\right)$	2
• Uses part (i) to establish base case of induction, or equivalent merit	1

Sample answer:

Assume n is a positive integer

$$\left.\begin{array}{c} n > \ln n \\ n-1 > \ln(n-1) \\ \vdots \\ 2 > \ln(2) \\ 1 > \ln(1) \end{array}\right\} \text{by part (i)}$$

Adding all those inequalities gives

$$1 + 2 + \cdots + (n-1) + n > \ln(n) + \ln(n-1) + \cdots + \ln(2) + \ln 1$$

$$\frac{n(n+1)}{2} > \ln(n!)$$

$$n^2 + n > 2\ln(n!) = \ln\left((n!)^2\right)$$

By taking the exponential of both sides, we get

$$e^{n^2+n} > (n!)^2$$

as required.

Question 16 (c)

Criteria	Marks
• Provides correct sketch	3
• Identifies the quadrant in which the region lies, or equivalent merit	2
• Identifies a quadrant that is excluded, or equivalent merit	1

Sample answer:

Let $\theta = \operatorname{Arg}\left(\frac{z}{w}\right)$, then $\frac{\pi}{2} < \theta < \pi$

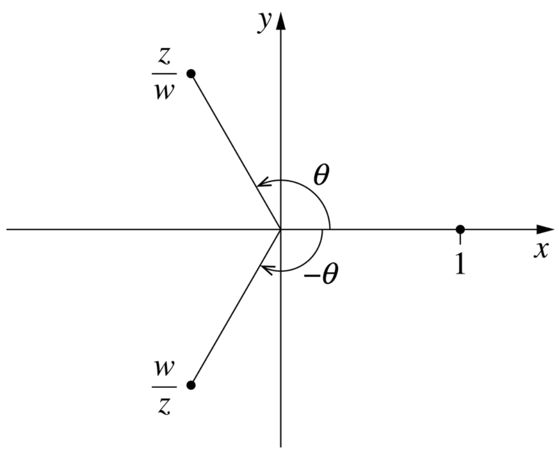
As $\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = 1 \qquad \frac{w}{z} = \overline{\left(\frac{z}{w}\right)}$

So $\operatorname{Arg}\left(\frac{w}{z}\right) = -\theta$

$$\frac{xz + yw}{z} = x + y\,\frac{w}{z}$$
$$= x\,1 + y\,\frac{w}{z}$$

For $\frac{\pi}{2} < \operatorname{Arg}\left(\frac{xz + yw}{z}\right) < \pi$

the number $x\,1 + y\,\frac{w}{z}$ must lie in the second quadrant of the complex plane.



Clearly $y < 0$, otherwise $x + y\,\frac{w}{z}$ lies below the x-axis.

The real part of $x + y\,\frac{w}{z}$ must be negative.

Question 16 (c) (continued)

We have $\operatorname{Re}\left(\dfrac{w}{z}\right) = \cos(-\theta)$

So $\quad x + y\cos(-\theta) < 0$

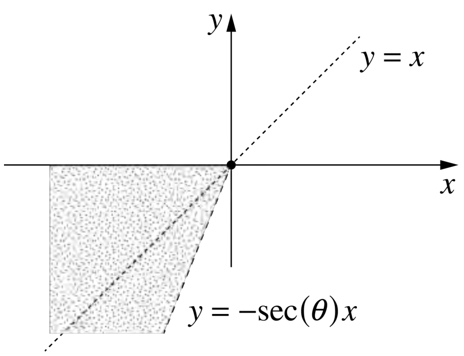
$x < -\cos(\theta)\,y \qquad$ cos is even

$-1 < \cos(\theta) < 0 \qquad \left(\dfrac{\pi}{2} < \theta < \pi\right)$

So $\quad 0 < -\cos(\theta) < 1$

$\Rightarrow \quad -\sec(\theta) > 1$

and $y > -\sec(\theta)\,x$



2023 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX-N1 Introduction to Complex Numbers	MEX12-7
2	1	MEX-P1 The Nature of Proof	MEX12-2
3	1	MEX-N1 Introduction to Complex Numbers	MEX12-4
4	1	MEX-P1 The Nature of Proof	MEX12-8
5	1	MEX-V1 Further Work With Vectors	MEX12-3
6	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
7	1	MEX-N1 Introduction to Complex Numbers	MEX12-4
8	1	MEX-N2 Using Complex Numbers	MEX12-4
9	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
10	1	MEX-V1 Further Work With Vectors	MEX12-8

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	2	MEX-N1 Introduction to Complex Numbers	MEX12-7
11 (b)	3	MEX-V1 Further Work with Vectors	MEX12-3
11 (c)	2	MEX-V1 Further Work with Vectors	MEX12-3
11 (d)	2	MEX-V1 Further Work with Vectors	MEX12-7
11 (e)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
11 (f)	4	MEX-C1 Further Integration	MEX12-5
12 (a)	3	MEX-P1 The Nature of Proof	MEX12-2
12 (b)	2	MEX-P1 The Nature of Proof	MEX12-2
12 (c) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
12 (c) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
12 (d)	3	MEX-N2 Using Complex Numbers	MEX12-1
12 (e) (i)	1	MEX-N2 Using Complex Numbers	MEX12-7
12 (e) (ii)	2	MEX-N2 Using Complex Numbers	MEX12-7
13 (a)	3	MEX-C1 Further Integration	MEX12-5
13 (b) (i)	1	MEX-P1 The Nature of Proof	MEX12-2
13 (b) (ii)	3	MEX-P2 Further Proof by Mathematical Induction	MEX12-2
13 (c) (i)	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
13 (c) (ii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
13 (c) (iii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
13 (c) (iv)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-7

Question	Marks	Content	Syllabus outcomes
14 (a) (i)	3	MEX-N1 Introduction to Complex Numbers MEX-N2 Using Complex Numbers	MEX12-4
14 (a) (ii)	2	MEX-N1 Introduction to Complex Numbers MEX-N2 Using Complex Numbers	MEX12-4
14 (a) (iii)	1	MEX-N1 Introduction to Complex Numbers MEX-N2 Using Complex Numbers	MEX12-4
14 (b)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
14 (c) (i)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
14 (c) (ii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (a) (i)	3	MEX-C1 Further Integration	MEX12-5
15 (a) (ii)	4	MEX-C1 Further Integration	MEX12-5
15 (a) (iii)	2	MEX-C1 Further Integration	MEX12-5
15 (b) (i)	1	MEX-V1 Further Work with Vectors	MEX12-3
15 (b) (ii)	3	MEX-V1 Further Work with Vectors	MEX12-3
15 (c)	3	MEX-V1 Further Work with Vectors	MEX12-3
16 (a) (i)	2	MEX-N2 Using Complex Numbers	MEX12-4
16 (a) (ii)	2	MEX-V1 Further Work with Vectors MEX-N2 Using Complex Numbers	MEX12-3, MEX12-4
16 (a) (iii)	2	MEX-N2 Using Complex Numbers	MEX12-4
16 (b) (i)	2	MEX-P1 The Nature of Proof	MEX12-1
16 (b) (ii)	3	MEX-P1 The Nature of Proof	MEX12-2
16 (c)	3	MEX-N2 Using Complex Numbers	MEX12-4