
Student Number

Barker College

2020	YEAR 12 ASSESSMENT BLOCK SEMESTER 1

Mathematics Extension 2

Section I – Multiple Choice

Write your student number at the top of this page.

Remove this sheet.

Select the alternative A, B C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = A 2 B 6 C 8 D 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

correct

A ● B ~~●~~ C ○ D ○

Start Here →

1.	A○	B○	C○	D○	6.	A○	B○	C○	D○
2.	A○	B○	C○	D○	7.	A○	B○	C○	D○
3.	A○	B○	C○	D○	8.	A○	B○	C○	D○
4.	A○	B○	C○	D○	9.	A○	B○	C○	D○
5.	A○	B○	C○	D○	10.	A○	B○	C○	D○

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Barker College

Student Number

2020	YEAR 12 ASSESSMENT TASK 2

Mathematics Extension 2

Staff Involved:

WMD* ARM*

40 copies

12:00 NOON
FRIDAY 19 JUNE

TOPICS COVERED
- **Complex Numbers I & II**
- **Proof**
- **Integration**
- **Vectors**

General Instructions:

- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- In Questions 11-15, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale

Total marks: 80

Section I – 10 marks (pages 4-8)
- Attempt Questions 1-10
- Allow about 20 minutes for this section

Section II – 70 marks (pages 9-14)
- Attempt Questions 11-15
- Allow about 1 hour and 40 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 20 minutes for this section

Use the multiple choice answer sheet for Questions 1–10.

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1. If $z = x + iy$, what is the value of $z\bar{z}$?

A. $x - iy$

B. $x^2 + y^2$

C. $x^2 - y^2$

D. $x^2 + iy^2$

2. At which point does the line $\underset{\sim}{r} = (1 + \lambda)\underset{\sim}{i} + (4 - 2\lambda)\underset{\sim}{j} + (3 + \lambda)\underset{\sim}{k}$ cut the x-z plane?

A. $(-2, 10, 0)$

B. $(3, 0, 5)$

C. $(0, 6, 2)$

D. $(1, 0, 3)$

Section I continues on the next page

Section I continued

3. For the statement $P \Rightarrow Q$, which of the following represents its contrapositive?

A. $\neg P \Rightarrow \neg Q$

B. $Q \Rightarrow P$

C. $\neg Q \Rightarrow \neg P$

D. $\neg P \Rightarrow Q$

4. What is the vector equation of the line segment between (–2, 3) and (7, 5)?

A. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 8 \end{pmatrix}$ for $0 \le \lambda \le 1$

B. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 2 \end{pmatrix}$ for $0 \le \lambda \le 1$

C. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -14 \\ 15 \end{pmatrix}$ for $-1 \le \lambda \le 1$

D. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 2 \end{pmatrix}$ for $0 \le \lambda \le 1$

Section I continues on the next page

5. In which quadrant of the complex plane does the complex number $3e^{17i}$ lie?

A. I

B. II

C. III

D. IV

6. Two vectors are given by $\underset{\sim}{a} = 4\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + n\underset{\sim}{j} - \underset{\sim}{k}$, where m and n are positive real numbers. If $|\underset{\sim}{a}| = 10$ and $\underset{\sim}{a} \perp \underset{\sim}{b}$, then what are the values of m and n?

A. $m = 5\sqrt{3}, \quad n = \dfrac{\sqrt{3}}{3}$

B. $m = 5\sqrt{3}, \quad n = \sqrt{3}$

C. $m = -5\sqrt{3}, \quad n = \sqrt{3}$

D. $m = 5, \quad n = 1$

Section I continues on the next page

7. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero, non-parallel vectors such that $x(\underset{\sim}{a} + \underset{\sim}{b}) = 2y\underset{\sim}{a} + (y + 3)\underset{\sim}{b}$, what are the values of x and y?

A. $x = 6,\ y = 3$

B. $x = 3,\ y = 6$

C. $x = -6,\ y = -3$

D. $x = 2,\ y = 1$

8. Given $z = \dfrac{1 + i\sqrt{3}}{1 + i}$, which of the following is equal to z^5?

A. $2\sqrt{2}e^{\frac{5\pi}{6}i}$

B. $4\sqrt{2}e^{\frac{5\pi}{12}i}$

C. $4\sqrt{2}e^{\frac{7\pi}{12}i}$

D. $2\sqrt{2}e^{\frac{5\pi}{12}i}$

Section I continues on the next page

9. What are the integer values of n for which $\left(3-\sqrt{3}i\right)^n$ is purely imaginary?

A. $n = 12k + 3$, where k is an integer

B. $n = 6k + 3$, where k is an integer

C. $n = 12k - 3$, where k is an integer

D. $n = 6k$, where k is an integer

10. Which expression is equivalent to $\int \frac{6x^2 \tan^{-1}\left(x^3\right)}{1+x^6} dx$?

A. $\left(\tan^{-1} x^3\right)^2 + C$

B. $\frac{1}{2}\left(\tan^{-1} x^4\right)^3 + C$

C. $6\tan^{-1}\left(\tan^{-1} x^3\right) + C$

D. $3\tan^{-1}\left(\tan^{-1} x^3\right)^2 + C$

End of Section I

Section II

70 marks
Attempt Questions 11–15
Allow about 1 hour and 40 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–15, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE writing booklet. **Marks**

(a) For $w = 2 + 3i$ and $z = 4 - 5i$, express the following in the form $x + iy$:

(i) $w - 2iz$ **1**

(ii) $\dfrac{w}{z}$ **2**

(b) Consider the complex number $z = -2\sqrt{3} + 2i$.

(i) Express z in the form $r(\cos\theta + i\sin\theta)$. **1**

(ii) Find z^3. **1**

(c) Solve $x^2 - 4x + (1 - 4i) = 0$. **3**

(d) Sketch the region on the complex plane that satisfies the following conditions: **3**

$$|z - (2 + i)| \le 1 \quad \text{and} \quad -\frac{\pi}{4} \le \operatorname{Arg}(z - (1 + i)) \le \frac{\pi}{4}$$

(e) It is known that $1 - 3i$ is a zero of $P(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$. **3**
Express $P(x)$ as a product of four linear factors.

End of Question 11

Question 12 (14 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \frac{x+2}{x-2}dx$. **2**

(b) Find $\int \frac{dx}{\sqrt{-x^2-2x+3}}$. **2**

(c) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x}dx$. **3**
Leave your answer in simplest exact form.

(d) Find the exact value of $\int_1^e x\ln x\, dx$. **3**

(e) Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$ for integers $n \geq 0$,

(i) Show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ for integers $n \geq 2$. **3**

(ii) Hence, or otherwise, evaluate I_4. **1**

End of Question 12

Question 13 (14 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $z^4 = -2 - 2\sqrt{3}i$. **3**

Give your solutions in exponential form with principal arguments.

(b) Prove that $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$. **1**

(c) The base of a pyramid is the parallelogram $ABCD$ with vertices at points $A(2, -1, 3)$, $B(4, -2, 1)$, $C(a, b, c)$, $D(4, 3, -1)$. The apex (top) of the pyramid is located at $P(4, -4, 9)$.

(i) Find the values of a, b and c. **2**

(ii) Find the cosine of the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$. **2**

(iii) Find the exact area of the base of the pyramid. **2**

(iv) Show that $\underset{\sim}{v} = 6\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AD}$. **2**

Hence find a unit vector that is perpendicular to the base of the pyramid.

(v) Find the volume of the pyramid. You may use the formula $V = \frac{1}{3}Ah$. **2**

End of Question 13

Question 14 (14 marks) Use a SEPARATE writing booklet. **Marks**

(a) The following statement is true, and its converse is also true: **1**

If x is a prime number, then it has only two divisors (itself and 1).

Write a true statement whose converse is **false**. Your statement must begin with "If x is divisible by 6, then…" (Assume x is a positive integer.)

(b) Mr Eel and Mr Nolnah are discussing Mersenne primes which are of the form $2^n - 1$ for some positive integer n. **1**

Mr Eel says, "if n is prime, then $2^n - 1$ must be prime."

Mr Nolnah is sure Mr Eel is wrong but is unable to justify his position.

Help Mr Nolnah by writing a proof showing Mr Eel is wrong.

(c) Consider the following "proof" that 2 = 1. Identify the error in the proof and explain why the proof is invalid. (You may use the letters to identify particular lines.) **1**

A	$-2 = -2$	(initial true statement)
B	$4 - 6 = 1 - 3$	(equivalent arithmetic expressions)
C	$4 - 6 + \frac{9}{4} = 1 - 3 + \frac{9}{4}$	(adding $\frac{9}{4}$ to both sides)
D	$\left(2 - \frac{3}{2}\right)^2 = \left(1 - \frac{3}{2}\right)^2$	(expressing as a perfect square)
E	$2 - \frac{3}{2} = 1 - \frac{3}{2}$	(taking square roots of both sides)
F	$2 = 1$	(adding $\frac{3}{2}$ to both sides)

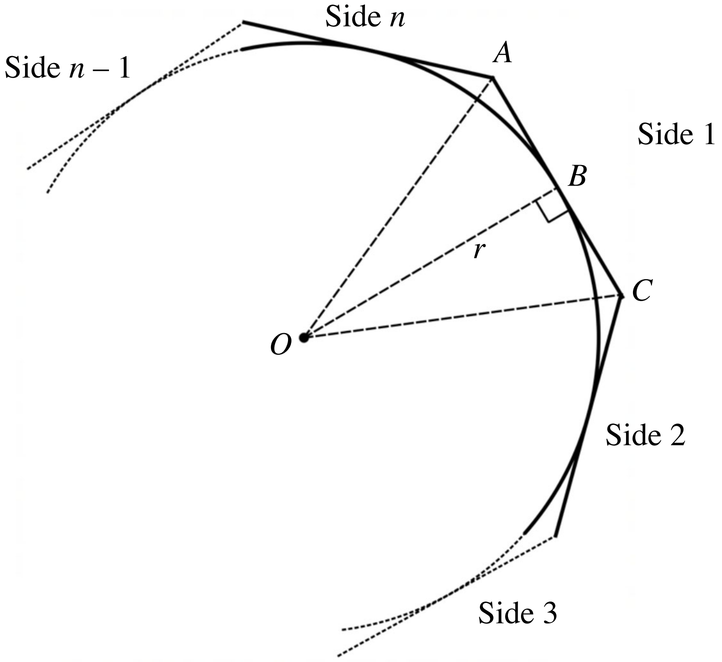
(d) Prove the following statement by contradiction, or otherwise, for $a, b, c \in \mathbb{Z}$: **3**

If $a^2 + b^2 = c^2$, then at least one of a or b is even.

You may assume that the square of an odd number is odd, and the square of an even number is even.

Question 14 continues on the next page

(e) Consider a regular n-sided polygon drawn around the outside of a circle of radius r and centre O such that each side is tangent to the circle at its midpoint, as shown.



(i) Show that the area of the triangle AOC is $r^2 \tan \dfrac{\pi}{n}$. **2**

(ii) Hence, evaluate $\lim\limits_{n \to \infty} n \tan \dfrac{\pi}{n}$ giving justification for your working. **2**

(f) (i) By considering the sign of $(p-q)^2$, prove that $\dfrac{p}{q} + \dfrac{q}{p} \geq 2$ for positive real numbers p and q. **1**

(ii) Hence, prove $\dfrac{p+q}{r} + \dfrac{q+r}{p} + \dfrac{r+p}{q} \geq 6$ for positive real numbers p, q and r. **1**

(iii) Hence, by using appropriate substitutions, or otherwise, prove that **2**

$\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b} \geq \dfrac{3}{2}$ for positive real numbers a, b and c.

End of Question 14

Question 15 (14 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = x + iy$ and $w = 2iz$.

(i) Prove algebraically that $|z|^2 + |w|^2 = |w - z|^2$. **2**

(ii) Sketch z and w on an Argand plane and hence justify the result in (i) geometrically. **2**

(b) Use the substitution $x = 5\sin^2\theta + \cos^2\theta$ to evaluate $\displaystyle\int_2^3 \frac{1}{2\sqrt{(x-1)(5-x)}}\,dx$. **3**

(c) z is a complex number such that $|z - 1| = 1$ and $\arg z = \theta$.

(i) Using an Argand diagram, or otherwise, show why $\arg(z - 1) = 2\theta$. **2**

(ii) Hence, find $\arg(z^2 - 3z + 2)$ in terms of θ. **2**

(d) Evaluate $\displaystyle\int_0^a \frac{3x^2 - ax}{(x-2a)(x^2+a^2)}\,dx$. **3**

End of Question 15

End of Paper

2020 Extension 2 Assessment 2

1. $z\bar{z} = (x+iy)(x-iy)$
$= x^2 - ixy + ixy - i^2y^2$
$= x^2 + y^2$ $\boxed{B}$

2. on the x-z plane, $y=0$
i.e. $4-2\lambda = 0$ (for $0\underset{\sim}{j}$)
$\lambda = 2$
for $\underset{\sim}{i}$: $1+\lambda = 1+2 = 3$
for $\underset{\sim}{k}$: $3+\lambda = 3+2 = 5$
$\therefore (3,0,5)$ $\boxed{B}$

3. $\boxed{C}$

4. Test values of λ at either end of the allowable interval to find $(-2,3)$ and $(7,5)$.
For D: when $\lambda = 0$
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + 0 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$
when $\lambda = 1$
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 9 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ $\boxed{D}$

5. Argument is 17:
$17 - 4\pi \approx 4.4$
$\pi < 4.4 < \frac{3\pi}{2}$

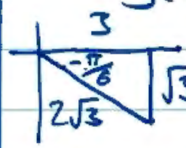
$\therefore$ III $\boxed{C}$

6. $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ $(\underset{\sim}{a} \perp \underset{\sim}{b})$
$4x - 2 + mn - 3x - 1 = 0$
$mn = 5$
$|\underset{\sim}{a}| = 10$
$\sqrt{4^2 + m^2 + (-3)^2} = 10$
$m^2 + 25 = 100$
$m = 5\sqrt{3}$ $(m>0)$
$n = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\boxed{A}$

7. $x\underset{\sim}{a} + x\underset{\sim}{b} = 2y\underset{\sim}{a} + (y+3)\underset{\sim}{b}$
So $x = 2y$ $\quad x = y+3$
$2y = y+3$
$y = 3$, $x = 6$ $\boxed{A}$

8. $z = \frac{1+i\sqrt{3}}{1+i} \times \frac{1-i}{1-i}$
$= \frac{1+\sqrt{3} + (\sqrt{3}-1)i}{2}$
$|z| = \sqrt{\left(\frac{1+\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}-1}{2}\right)^2} = \sqrt{2}$
$\arg(z) = \tan^{-1}\left(\frac{\sqrt{3}-1}{2} \Big/ \frac{\sqrt{3}+1}{2}\right) = 15^\circ = \frac{\pi}{12}$
$\therefore z^5 = \left(\sqrt{2}e^{\frac{\pi}{12}i}\right)^5 = 4\sqrt{2}e^{\frac{5\pi}{12}i}$ $\boxed{B}$

9. Purely imaginary when
$\arg(z) = \frac{-\pi}{2} - k\pi$ $(k \in \mathbb{Z})$



$\arg(3 - \sqrt{3}i) = \frac{-\pi}{6}$
$\arg((3-\sqrt{3}i)^n) = -\frac{n\pi}{6}$
So, $\frac{-n\pi}{6} = \frac{-\pi}{2} - k\pi$
$n = 3 + 6k$
Or consider multiples of $\frac{-\pi}{6}$ that give $\pm\frac{\pi}{2} \pm k\pi$: 3, 9, 15, −3, −9, −15, etc which fit $n = 6k+3$ $\boxed{B}$

10. Try differentiating answers.
A: $\frac{d}{dx}\left((\tan^{-1}x^3)^2 + C\right)$
$= 2(\tan^{-1}x^3)^1 \cdot \frac{3x^2}{1+(x^3)^2}$
$= \frac{6x^2\tan^{-1}(x^3)}{1+x^6}$ $\boxed{A}$

Question 11

(a) (i) $w - 2iz$

$= 2 + 3i - 2i(4 - 5i)$

$= 2 + 3i - 8i - 10$

$= -8 - 5i$

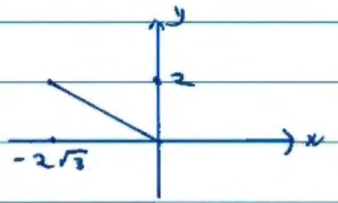
(ii) $\frac{w}{z} = \frac{(2+3i)}{(4-5i)} \times \frac{(4+5i)}{(4+5i)}$

$= \frac{8 + 10i + 12i - 15}{4^2 + 5^2}$

$= \frac{-7 + 22i}{41}$

$= -\frac{7}{41} + \frac{22}{41} i$

(b) (i)



$r = \sqrt{(-2\sqrt{3})^2 + 2^2}$

$= 4$

$\tan\theta = \frac{2}{-2\sqrt{3}}$

$= -\frac{1}{\sqrt{3}}$

$\theta = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

$z = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(ii) $z^3 = 4^3 \operatorname{cis}\left(3 \times \frac{5\pi}{6}\right)$

$= 64\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$= 64i$

(c) $x^2 - 4x + (1 - 4i) = 0$

$x^2 - 4x = -1 + 4i$

$x^2 - 4x + 4 = 3 + 4i$

$(x-2)^2 = 3 + 4i$

let $(a+ib)^2 = 3 + 4i$

$a^2 - b^2 = 3$, $2ab = 4$, $a^2 + b^2 = \sqrt{3^2+4^2} = 5$

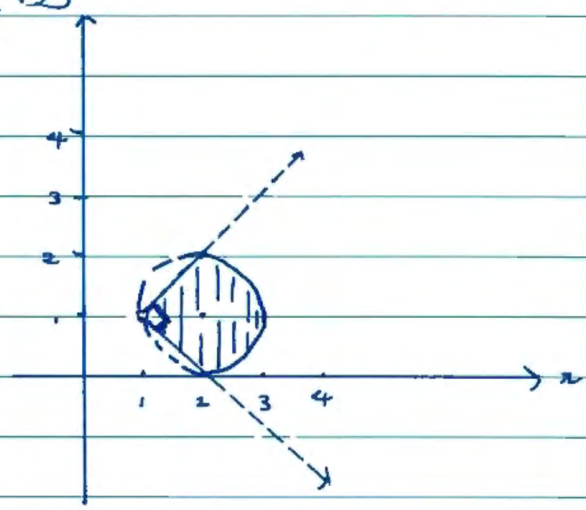
$\therefore 2a^2 = 8$, $a^2 = 4$, $a = 2 \Rightarrow b = 1$
$a = -2 \Rightarrow b = -1$

$\therefore x - 2 = \pm(2+i)$

$x = 2 + (2+i),\ 2 - (2+i)$

$= 4 + i,\ -i$

(d)



(e) $1+3i$ is also a zero of $P(x)$.

$(1-3i) + (1+3i) = 2$

$(1-3i)(1+3i) = 10$

$P(x) = (x^2 - 2x + 10)(x^2 + bx + c)$

$= x^4 - 3x^3 + 6x^2 + 2x - 60$

Comparing of coeff's of x^3:

$b - 2 = -3$

$\therefore b = -1$

Comparing constants:

$10c = -60$

$\therefore c = -6$

$\therefore P(x) = (x - (1+3i))(x - (1-3i))(x^2 - x - 6)$

$= (x-1-3i)(x-1+3i)(x+2)(x-3)$

Question 12

(a) $$\int \frac{x+2}{x-2}\,dx$$

$$= \int \frac{(x-2)+4}{x-2}\,dx$$

$$= \int \left(1 + \frac{4}{x-2}\right) dx$$

$$= x + 4\ln|x-2| + c$$

(b) $$\int \frac{dx}{\sqrt{-x^2-2x+3}}$$

$$= \int \frac{dx}{\sqrt{3-(x^2+2x)}}$$

$$= \int \frac{dx}{\sqrt{4-(x^2+2x+1)}}$$

$$= \int \frac{dx}{\sqrt{2^2-(x+1)^2}}$$

$$= \sin^{-1}\left(\frac{x+1}{2}\right) + c$$

(c) $$\int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x}\,dx,$$

$t = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1} t$

$dx = \frac{2\,dt}{1+t^2}$

$x = 0,\ t = 0$

$x = \frac{\pi}{2},\ t = 1$

$$= \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{5+3\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_0^1 \frac{2\,dt}{5+5t^2+3-3t^2}$$

$$= \int_0^1 \frac{2\,dt}{2t^2+8}$$

$$= \int_0^1 \frac{dt}{t^2+2^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{t}{2}\right]_0^1$$

$$= \frac{1}{2}\left\{\left(\tan^{-1}\frac{1}{2}\right) - \left(\tan^{-1} 0\right)\right\}$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2}$$

(d) $$\int_1^e x\ln x\,dx$$

$u = \ln x \qquad v = \frac{x^2}{2}$

$u' = \frac{1}{x} \qquad v' = x$

$$= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\times\frac{1}{x}\,dx$$

$$= \left(\frac{e^2}{2}\right) - (0) - \int_1^e \frac{x}{2}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$$

$$= \frac{e^2}{2} - \left\{\left(\frac{e^2}{4}\right) - \left(\frac{1}{4}\right)\right\}$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{1}{4}\left(e^2+1\right)$$

(e) (i) $$I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

$u = \cos^{n-1} x$

$u' = -(n-1)\cos^{n-2} x \sin x$

$v = \sin x$

$v' = \cos x$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x\,dx$$

$$= \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int \cos^{n-2} x \sin^2 x\,dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}(1-\cos^2 x)\,dx$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore I_n(1+(n-1)) = (n-1)I_{n-2}$$

$$\therefore I_n = \left(\frac{n-1}{n}\right)I_{n-2}$$

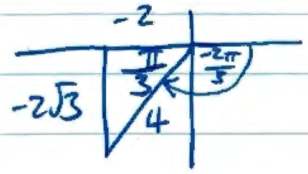
(ii) $$I_0 = \int_0^{\frac{\pi}{2}} 1\,dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$I_2 = \left(\frac{2-1}{2}\right)\times I_0 = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$$

$$I_4 = \left(\frac{4-1}{4}\right)\times I_2 = \frac{3}{4}\times\frac{\pi}{4} = \frac{3\pi}{16}$$

Question 13

a) Let $z = re^{i\theta}$



So $-2-2\sqrt{3}i = 4e^{i\left(\frac{-2\pi}{3}+2k\pi\right)}, \ k \in \mathbb{Z}$

Now $\left(re^{i\theta}\right)^4 = 4e^{i\left(\frac{-2\pi}{3}+2k\pi\right)}$

$r^4 e^{4i\theta} = (\sqrt{2})^4 e^{i\left(\frac{-2\pi+6k\pi}{3}\right)}$

So $r = \sqrt{2}$

$4\theta = \frac{-2\pi+6k\pi}{3}$

$\theta = \frac{(6k-2)\pi}{12} = \frac{(3k-1)\pi}{6}$

But $-\pi < \theta \leq \pi$

$-\pi < \frac{(3k-1)\pi}{6} \leq \pi$

$-6 < 3k-1 \leq 6$

$\frac{-5}{3} < k \leq \frac{7}{3}$

$k = -1, 0, 1, 2$

$\theta = \frac{-4\pi}{6}, \frac{-\pi}{6}, \frac{2\pi}{6}, \frac{5\pi}{6}$

$z = \sqrt{2}e^{\frac{-2\pi}{3}i}, \sqrt{2}e^{\frac{-\pi}{6}i},$

$\sqrt{2}e^{\frac{\pi}{3}i}, \sqrt{2}e^{\frac{5\pi}{6}i}$

b) Since $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

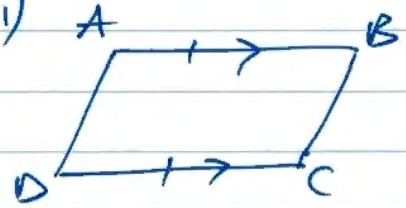
Now RHS $= \frac{\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)}{2i}$

$= \frac{\cos\theta - \cos\theta + i\sin\theta + i\sin\theta}{2i}$

$= \frac{2i\sin\theta}{2i}$

$= \sin\theta = $ LHS.

c) i)



$\overrightarrow{AB} = \overrightarrow{DC}$

$\overrightarrow{AB} = \begin{pmatrix}4\\-2\\1\end{pmatrix} - \begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\-1\\-2\end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix}a\\b\\c\end{pmatrix} - \begin{pmatrix}4\\3\\-1\end{pmatrix} = \begin{pmatrix}a-4\\b-3\\c+1\end{pmatrix}$

So $a-4 = 2 \longrightarrow a = 6$

$b-3 = -1 \longrightarrow b = 2$

$c+1 = -2 \longrightarrow c = -3$

ii) $\overrightarrow{AD} = \begin{pmatrix}4\\3\\-1\end{pmatrix} - \begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\4\\-4\end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{2^2+(-1)^2+(-2)^2} = 3$

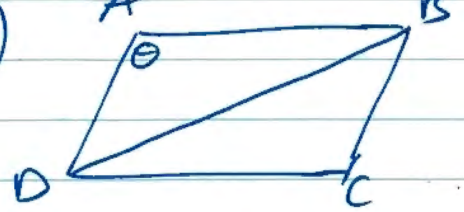
$|\overrightarrow{AD}| = \sqrt{2^2+4^2+(-4)^2} = 6$

$\overrightarrow{AB} \cdot \overrightarrow{AD} = |\overrightarrow{AB}| \times |\overrightarrow{AD}| \times \cos\theta$

$\begin{pmatrix}2\\-1\\-2\end{pmatrix} \cdot \begin{pmatrix}2\\4\\-4\end{pmatrix} = 18\cos\theta$

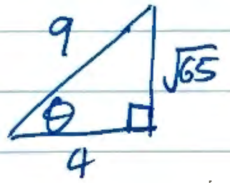
$4 - 4 + 8 = 18\cos\theta$

$\cos\theta = \frac{4}{9}$

iii) 

Area ABCD is twice Area $\triangle ABD$ (since $\triangle ABD \equiv \triangle CDB$)

$$\text{Area } ABCD = 2 \times \frac{1}{2} |\vec{AB}| |\vec{AD}| \sin\theta$$



so $\sin\theta = \frac{\sqrt{65}}{9}$

$$\text{Area } ABCD = 3 \times 6 \times \frac{\sqrt{65}}{9}$$
$$= 2\sqrt{65} \text{ units}^2$$

iv) $$\underset{\sim}{v} \cdot \vec{AB} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$
$$= 12 - 2 - 10$$
$$= 0$$
$$\underset{\sim}{v} \cdot \vec{AD} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}$$
$$= 12 + 8 - 20$$
$$= 0$$

so $\underset{\sim}{v} \perp \vec{AB}$ and $\underset{\sim}{v} \perp \vec{AB}$.

$$|\underset{\sim}{v}| = \sqrt{6^2 + 2^2 + 5^2} = \sqrt{65}$$

$$\therefore \ \hat{\underset{\sim}{v}} = \frac{1}{\sqrt{65}} \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{6}{\sqrt{65}} \\ \frac{2}{\sqrt{65}} \\ \frac{5}{\sqrt{65}} \end{pmatrix}$$

v) Perpendicular height vector of pyramid is the projection onto $\underset{\sim}{v}$ (or $\hat{\underset{\sim}{v}}$) of any of the edges from A, B, C or D to P. Let's choose $\vec{AP}$.

$$\vec{AP} = \begin{pmatrix} 4 \\ -4 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$$

$$\text{proj}_{\underset{\sim}{v}} \vec{AP} = (\vec{AP} \cdot \hat{\underset{\sim}{v}}) \hat{\underset{\sim}{v}}$$

So, the length of the projection (i.e. the height of the pyramid) is:

$$h = \vec{AP} \cdot \hat{\underset{\sim}{v}}$$
$$= \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \cdot \frac{1}{\sqrt{65}} \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$$
$$= \frac{1}{\sqrt{65}} (12 - 6 + 30)$$
$$= \frac{36}{\sqrt{65}}$$

So, $$V = \frac{1}{3} \times 2\sqrt{65} \times \frac{36}{\sqrt{65}}$$
$$= \frac{72}{3}$$
$$= 24 \text{ units}^3$$

Question 14

(a) For example:
'If x is divisible by 6, then x is divisible by 3' is true whereas the converse 'If x is divisible by 3, then x is divisible by 6' is not necessarily true.

(b) 11 is a prime and $2^{11}-1 = 23 \times 89$ which is not a prime. The counter example disproves Mr Eel's conjecture.

(c) In D, $2-\frac{3}{2} = \frac{1}{2}$ and $(1-\frac{3}{2}) = -\frac{1}{2}$ so it is valid to say that $(2-\frac{3}{2})^2 = (1-\frac{3}{2})^2$; however it is not valid in E to then say that $2-\frac{3}{2} = 1-\frac{3}{2}$ as the square of one number equalling the square of another number does not mean the two numbers are equal, just their absolute values, as is the case here.

(d) Assume that both a and b are odd integers.
Let $a = 2m+1$ and $b = 2n+1$ where $m, n \in \mathbb{Z}$

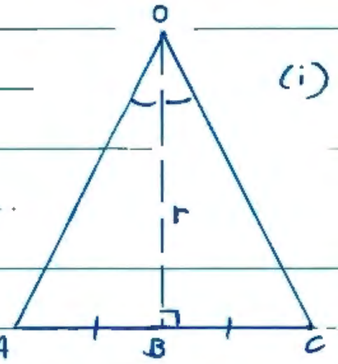
$$\therefore a^2 + b^2 = (2m+1)^2 + (2n+1)^2$$
$$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$$
$$= 2(2m^2 + 2m + 2n^2 + 2n + 1) \quad \{2 \times \text{an odd integer}\}$$

which is not the square of an integer, as it only has one factor of 2.

Hence the initial assumption is incorrect, meaning a and b can't both be odd, i.e. at least one of a or b is even.

N.B. $3^2+4^2=5^2$, $6^2+8^2=10^2 \Rightarrow$ both a & b can be even or one can be even & the other odd.

(e)



(i) $\angle AOC = \frac{2\pi}{n}$

Since ΔAOC is isosceles and $AC \perp OB$,
$\angle AOB = \angle COB = \frac{\pi}{n}$ & $AB = BC$

$\therefore \frac{AB}{r} = \tan\frac{\pi}{n} \Rightarrow AB = r\tan\frac{\pi}{n}$

$$\text{Area } \Delta AOC = \tfrac{1}{2} \times AC \times OB$$
$$= r\tan\tfrac{\pi}{n} \times r$$
$$= r^2 \tan\tfrac{\pi}{n}$$

Question 14 (continued).

(e) (ii) $\left.\begin{array}{l}\text{Area of the} \\ \text{regular } n\text{-sided polygon}\end{array}\right\} = \text{Area } \Delta AOC \times n$

$= r^2 n \tan \frac{\pi}{n}$

$$\lim_{n \to \infty} \begin{pmatrix}\text{area of} \\ \text{polygon}\end{pmatrix} = r^2 \lim_{n \to \infty} n \tan \frac{\pi}{n} = \pi r^2 \begin{pmatrix}\text{area of} \\ \text{circle}\end{pmatrix}$$

$$\therefore \lim_{n \to \infty} n \tan \frac{\pi}{n} = \pi$$

(f) (i) $(p-q)^2 \geqslant 0$

$p^2 + q^2 \geqslant 2pq$

$\frac{p^2}{pq} + \frac{q^2}{pq} \geqslant 2$

$\frac{p}{q} + \frac{q}{p} \geqslant 2.$

(ii) $\frac{p}{q} + \frac{q}{p} \geqslant 2$ from (i)

similarly:

$\frac{p}{r} + \frac{r}{p} \geqslant 2$

& $\frac{q}{r} + \frac{r}{q} \geqslant 2$

$\therefore \left(\frac{p}{q} + \frac{q}{p}\right) + \left(\frac{p}{r} + \frac{r}{p}\right) + \left(\frac{q}{r} + \frac{r}{q}\right) \geqslant 6$

i.e. $\left(\frac{p+q}{r}\right) + \left(\frac{q+r}{p}\right) + \left(\frac{r+p}{q}\right) \geqslant 6$

(iii) let $r = b+c$, $p = a+c$, $q = a+b$ in result proved in (ii):

$$\frac{a+c+a+b}{b+c} + \frac{a+b+b+c}{a+c} + \frac{b+c+a+c}{a+b} \geqslant 6$$

$$\therefore \left(1 + \frac{2a}{b+c}\right) + \left(1 + \frac{2b}{a+c}\right) + \left(1 + \frac{2c}{a+b}\right) \geqslant 6$$

$$\therefore 2\left[\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}\right] \geqslant 3$$

$$\therefore \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geqslant \frac{3}{2}$$

Question 15

(a)(i) $z = x + iy$

$w = 2iz$

$= 2i(x+iy)$

$= -2y + 2ix$

$w - z = (-2y + 2ix) - (x + iy)$

$= -(x+2y) + i(2x - y)$

$\therefore |z|^2 + |w|^2$

$= (x^2 + y^2) + ((-2y)^2 + (2x)^2)$

$= 5x^2 + 5y^2$

$|w - z|^2$

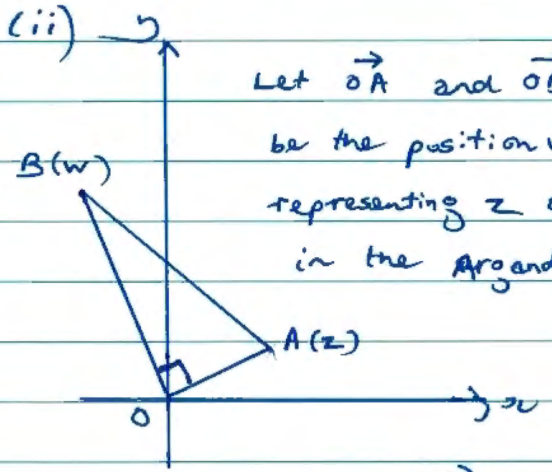
$= [-(x+2y)]^2 + (2x-y)^2$

$= x^2 + 2xy + 4y^2 + 4x^2 - 4xy + y^2$

$= 5x^2 + 5y^2$

$= |z|^2 + |w|^2$

(ii)



Let $\vec{OA}$ and $\vec{OB}$ be the position vectors representing z and w in the Argand diagram.

$w = 2iz \Rightarrow \vec{OB} = i\,2\vec{OA}$

$\Rightarrow OB \perp OA$

By Pythagoras' thm:

$OA^2 + OB^2 = BA^2$

i.e. $|z|^2 + |w|^2 = |w - z|^2$

(b) $x = 5\sin^2\theta + 1 - \sin^2\theta$

$= 4\sin^2\theta + 1$

$x - 1 = 4\sin^2\theta$

$\sin^2\theta = \frac{x-1}{4}$

let $\sin\theta = \frac{\sqrt{x-1}}{2}$

& $\theta = \sin^{-1}\left[\frac{\sqrt{x-1}}{2}\right]$

$x = 2,\ \theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$x = 3,\ \theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$dx = 8\sin\theta\cos\theta\, d\theta$

$\sqrt{(x-1)(5-x)} = \sqrt{4\sin^2\theta(4 - 4\sin^2\theta)}$

$= \sqrt{16\sin^2\theta\cos^2\theta}$

$= 4\sin\theta\cos\theta$

$\left\{\text{noting } \frac{\pi}{6} \le \theta \le \frac{\pi}{4}\right\}$

$\therefore \int_2^3 \frac{1}{2\sqrt{(x-1)(5-x)}}\, dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{8\sin\theta\cos\theta\, d\theta}{8\sin\theta\cos\theta}$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta$

$= [\theta]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

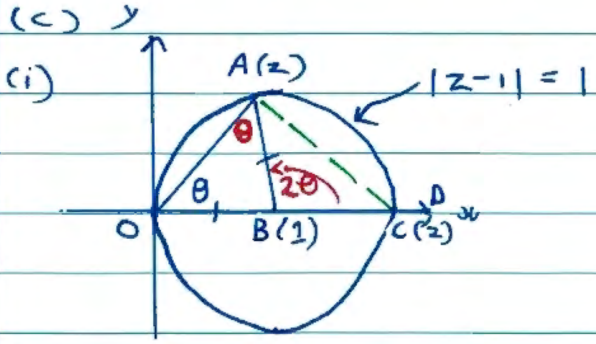
$= \frac{\pi}{4} - \frac{\pi}{6}$

$= \frac{3\pi - 2\pi}{12}$

$= \frac{\pi}{12}$

(c)

(i)



Question 15 (continued)

(c) (i) (continued)

Let $\vec{OA}$ be the position vector representing z on the circle shown.

$\angle AOB = \arg z = \theta$

$OB = BA$ (radii)

$\therefore \angle OAB = \theta$ (base $\angle$'s of isos. ΔOBA)

$\therefore \angle ABC = 2\theta$ (ext. $\angle$ of isos. ΔOBA)

$= \arg(z-1)$

(ii) $\arg(z^2 - 3z + 2)$

$= \arg[(z-1)(z-2)]$

$= \arg(z-1) + \arg(z-2)$

Now $\angle ACB = \dfrac{\pi - 2\theta}{2}$ (isos. ΔABD)

$= \frac{\pi}{2} - \theta$

$\therefore \angle ACD = \frac{\pi}{2} + \theta$ (straight angle BCD)

$= \arg(z-2)$

$\therefore \arg(z^2 - 3z + 2) = 2\theta + \frac{\pi}{2} + \theta$

$= \underline{\underline{3\theta + \frac{\pi}{2}}}$

OR

$\angle OAC = 90^\circ$ ($\angle$ in a semicircle)

$\therefore \angle ACD = \angle AOC + \angle OAC$ (ext. $\angle$ of ΔOAC)

$= \theta + \frac{\pi}{2}$

$\therefore \arg(z^2 - 3z + 2) = 3\theta + \frac{\pi}{2}$

(d)

Let $\dfrac{3x^2 - ax}{(x-2a)(x^2+a^2)} = \dfrac{A}{x-2a} + \dfrac{Bx+C}{x^2+a^2}$

$\therefore 3x^2 - ax = A(x^2+a^2) + (Bx+C)(x-2a)$

$x = 2a$: $12a^2 - 2a^2 = 5a^2 A$

$\therefore A = 2$

equating coeff'ts of x^2:

$3 = A + B$

$\therefore B = 1$

$x = 0$: $0 = 2a^2 + C \times (-2a)$

$\therefore C = a$

$$\int_0^a \frac{3x^2 - ax}{(x-a)(x^2+a^2)}\,dx$$

$$= \int_0^a \left\{\frac{2}{x-2a} + \frac{x+a}{x^2+a^2}\right\} dx$$

$$= \int_0^a \left\{\frac{2}{x-2a} + \frac{1}{2}\left(\frac{2x}{x^2+a^2}\right) + \frac{a}{x^2+a^2}\right\} dx$$

$$= \left[2\ln|x-2a| + \tfrac{1}{2}\ln|x^2+a^2| + \tan^{-1}\frac{x}{a}\right]_0^a$$

$$= \left(2\ln|a| + \tfrac{1}{2}\ln(2a^2) + \tan^{-1}1\right) - \left(2\ln|2a| + \tfrac{1}{2}\ln a^2 + 0\right)$$

$$= 2\ln\left|\frac{a}{2a}\right| + \frac{1}{2}\ln\left(\frac{2a^2}{a^2}\right) + \frac{\pi}{4}$$

$$= 2\ln\tfrac{1}{2} + \tfrac{1}{2}\ln 2 + \frac{\pi}{4}$$

$$= -2\ln 2 + \tfrac{1}{2}\ln 2 + \frac{\pi}{4}$$

$$= \underline{\underline{\frac{\pi}{4} - \frac{3}{2}\ln 2.}}$$