
Student Number

Barker College

2021 YEAR 12 ASSESSMENT TASK 3

Mathematics Extension 2

Staff Involved:

8:30AM
WEDNESDAY 2 JUNE

BHC* ARM

TOPICS COVERED
- Integration
- Vectors

40 copies

General Instructions:

- Working time – 55 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- Write all answers in the spaces provided
- Show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale
- Extra working space is provided at the end of the paper

Total marks: 48

Question 1 (11 marks) **Marks**

(a) Find the vector $\underset{\sim}{v}$ parallel to $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$ that has a magnitude of 5. **2**

(b) Find the angle between the vectors and , rounded to the nearest degree. **3**

(c) Consider the points $A(1, 3, 2)$ and $B(2, 4, -1)$.

(i) Find a vector equation of the line passing through A and B. **1**

(ii) Prove that the point $C(5, 7, -10)$ lies on the line passing through A and B. **1**

(iii) Using (i), find a vector equation for the interval from A to B. **1**

(d) The points $A\left(5\frac{2}{3}, 3, 4\right)$, $B(5, 6, z)$ and $C(8, 9, 10)$ form a right-angled triangle, right-angled at B. Find the possible values of z. **3**

Question 2 (12 marks) **Marks**

(a) Find $\int \frac{2x+1}{x^2+4x+5}\,dx$. **3**

(b) Use the substitution $x = 2\sin\theta$ to evaluate . **3**

Question 3 (9 marks) **Marks**

(a) Evaluate $\int_0^1 \frac{x^2+3}{(x^2+1)(x^2+2)}\,dx$ using partial fractions, correct to 2 decimal places. **3**

Question 4 (16 marks) **Marks**

(a) Find the point of intersection of the lines **3**

$$\underset{\sim}{r} = \begin{pmatrix}1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\3\end{pmatrix} \quad \text{and} \quad \underset{\sim}{q} = \begin{pmatrix}3\\-1\end{pmatrix} + \mu\begin{pmatrix}8\\7\end{pmatrix}.$$

(b) The point $B(x, y, z)$ is on the interval AC and is twice as far from $A(-1, -3, -5)$ as it is from $C(5, 6, 7)$. Use vectors to find the coordinates of B. **3**

(d) (i) If $I_n = \int_0^1 \left(x^2 - 1\right)^n dx$ for $n \geq 0$, show that for $n \geq 1$. **3**

(ii) Hence, evaluate I_3. **1**

Question 1 (11 marks) **Marks**

(a) Find the vector $\underset{\sim}{v}$ parallel to $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$ that has a magnitude of 5. **2**

Now $|\underset{\sim}{u}| = \sqrt{4+9+16} = \sqrt{29}$

so $\hat{\underset{\sim}{u}} = \begin{pmatrix} 2/\sqrt{29} \\ 3/\sqrt{29} \\ 4/\sqrt{29} \end{pmatrix}$

$\therefore \underset{\sim}{v} = \frac{1}{\sqrt{29}} \begin{pmatrix} 10 \\ 15 \\ 20 \end{pmatrix}$

(b) Find the angle between the vectors $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$, rounded to the nearest degree. **3**

$\cos\theta = \dfrac{\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \circ \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}}{\sqrt{29} \times \sqrt{8}} = \dfrac{2}{\sqrt{29} \times \sqrt{8}}$

$\therefore \cos\theta = 0 \cdot 131306$

$\therefore \theta = 82^\circ$

(c) Consider the points $A(1, 3, 2)$ and $B(2, 4, -1)$.

(i) Find a vector equation of the line passing through A and B. **1**

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$

$\therefore \underset{\sim}{R} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$

OR $\underset{\sim}{R} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$

OR $\underset{\sim}{R} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ OR $\underset{\sim}{R} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$

(ii) Prove that the point $C(5, 7, -10)$ lies on the line passing through A and B. **1**

Does $\begin{pmatrix} 5 \\ 7 \\ -10 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$?

Yes, when $\lambda = 4$

(or equivalent)

(iii) Using (i), find a vector equation for the interval from A to B. **1**

The interval from A to B is

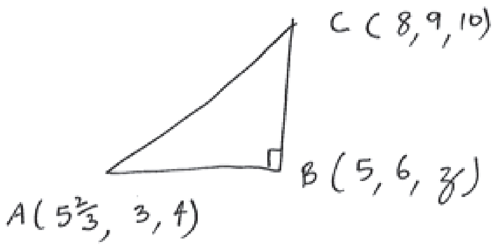
$\underset{\sim}{R} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ for $0 \le \lambda \le 1$

OR $\underset{\sim}{R} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ for $-1 \le \lambda \le 0$

OR $\underset{\sim}{R} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ for $-1 \le \lambda \le 0$

OR $\underset{\sim}{R} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ for $0 \le \lambda \le 1$

(d) The points $A(5\frac{2}{3}, 3, 4)$, $B(5, 6, z)$ and $C(8, 9, 10)$ form a right-angled triangle, right-angled at B. Find the possible values of z. **3**



Now $\overrightarrow{BC} \cdot \overrightarrow{AB} = 0$, so

$$\begin{pmatrix} 3 \\ 3 \\ 10-z \end{pmatrix} \cdot \begin{pmatrix} -\frac{2}{3} \\ 3 \\ z-4 \end{pmatrix} = 0$$

$$\therefore \quad -2 + 9 + (10-z)(z-4) = 0$$

$$7 + 10z - 40 - z^2 + 4z = 0$$

$$-z^2 + 14z - 33 = 0$$

$$z^2 - 14z + 33 = 0$$

$$(z-3)(z-11) = 0$$

$$\therefore \quad z = 3 \text{ or } 11$$

Question 2 (12 marks) **Marks**

(a) Find $\int \frac{2x+1}{x^2+4x+5}\,dx$. **3**

$$= \int \frac{2x+4-3}{x^2+4x+5}$$

$$= \int \frac{2x+4}{x^2+4x+5}\,dx - \int \frac{3}{x^2+4x+5}\,dx$$

$$= \int \frac{2x+4}{x^2+4x+5}\,dx - \int \frac{3}{1+(x+2)^2}\,dx$$

$$= \ln\left|x^2+4x+5\right| - 3\tan^{-1}(x+2) + C$$

(b) Use the substitution $x = 2\sin\theta$ to evaluate $\int_0^{\sqrt{2}} \frac{dx}{\sqrt{(4-x^2)^3}}$. **3**

Let $x = 2\sin\theta$ when $x = \sqrt{2}$, $\theta = \frac{\pi}{4}$

$dx = 2\cos\theta\,d\theta$ when $x = 0$, $\theta = 0$

$$\therefore I = \int_0^{\frac{\pi}{4}} \frac{2\cos\theta\,d\theta}{\sqrt{(4-4\sin^2\theta)^3}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\cos\theta}{8\sqrt{\cos^6\theta}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4}\sec^2\theta\,d\theta$$

$$= \frac{1}{4}\left[\tan\theta\right]_0^{\pi/4} = \frac{1}{4}$$

(c) Find $\int x^2 \sin x \, dx$. **3**

$= -\cos x \cdot x^2 - \int (-\cos x) \cdot 2x \, dx$

$= -x^2 \cos x + 2\int x \cos x \, dx$

$= -x^2 \cos x + 2\left[\sin x \cdot x - \int \sin x \, dx \right]$

$= -x^2 \cos x + 2x \sin x - 2\int \sin x \, dx$

$= -x^2 \cos x + 2x \sin x + 2\cos x + c$

(d) Find $\int (\ln x)^2 \, dx$. **3**

$= x(\ln x)^2 - \int x \cdot 2(\ln x)^{1} \cdot \frac{1}{x} \, dx$

$= x(\ln x)^2 - 2\int \ln x \, dx$

$= x(\ln x)^2 - 2\left[x \ln x - \int x \left(\frac{1}{x}\right) dx \right]$

$= x(\ln x)^2 - 2x \ln x + 2\int dx$

$= x(\ln x)^2 - 2x \ln x + 2x + c$

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Question 3 (9 marks) **Marks**

(a) Evaluate $\int_0^1 \frac{x^2+3}{(x^2+1)(x^2+2)} dx$ using partial fractions, correct to 2 decimal places. **3**

Let $\frac{x^2+3}{(x^2+1)(x^2+2)} = \frac{A}{x^2+1} + \frac{B}{x^2+2}$

$x^2 + 3 = A(x^2+2) + B(x^2+1)$

Let $x = 0$, $3 = 2A + B$

Let $x = 1$, $4 = 3A + 2B$

$6 = 4A + 2B$

$2 = A$

$B = -1$

$\therefore I = \int_0^1 \frac{2}{1+x^2} + \frac{-1}{2+x^2} \, dx$

$= \left[2\tan^{-1} x\right]_0^1 - \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}}\right]_0^1$

$= 2\left(\frac{\pi}{4} - 0\right) - \frac{1}{\sqrt{2}}\left(\tan^{-1} \frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{2} - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= 1.1355\ldots$

$= 1.14$ (2dp)

(b) Let $u=\sqrt{x}$ to evaluate $\int_1^9 e^{\sqrt{x}}\,dx$. **3**

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$
$$du = \frac{1}{2\sqrt{x}}\,dx$$
$$2u\,du = dx$$
when $x=9$, $u=3$

when $x=1$, $u=1$

$$I = \int_1^3 e^u . 2u\,du$$
$$= 2\int_1^3 u e^u\,du$$
$$= 2\left[\left[e^u . u\right]_1^3 - \int_1^3 e^u(1)\,du\right]$$
$$= 2\left[3e^3 - e - \left[e^u\right]_1^3\right]$$
$$= 2\left[3e^3 - e - (e^3 - e)\right]$$
$$= 2(2e^3) = 4e^3$$

(c) Use an appropriate substitution to find $\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$. **3**

Marks will only be awarded for solutions using a substitution.

Let $u^2 = x$
$$\frac{dx}{du} = 2.u$$
$$dx = 2u\,du$$
$$I = \int \frac{2u\,du}{u.\sqrt{1-u^2}}$$
$$= 2\int \frac{du}{\sqrt{1-u^2}}$$
$$= 2\sin^{-1}(u)$$
$$= 2\sin^{-1}(\sqrt{x}) + c$$

OR Let $x = \sin^2\theta$
$$dx = 2\sin\theta\cos\theta\,d\theta$$
$$I = \int \frac{2\sin\theta\cos\theta\,d\theta}{\sin\theta.\cos\theta}$$
$$= \int 2\,d\theta$$
$$= 2\theta$$
but $\sin\theta = \sqrt{x}$ so $\theta = \sin^{-1}(\sqrt{x})$

Hence $I = 2\sin^{-1}(\sqrt{x}) + c$

Question 4 (16 marks) **Marks**

(a) Find the point of intersection of the lines **3**

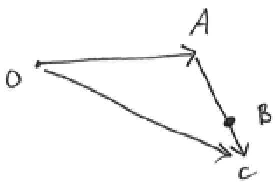
$$\underset{\sim}{r} = \begin{pmatrix}1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\3\end{pmatrix} \quad \text{and} \quad \underset{\sim}{q} = \begin{pmatrix}3\\-1\end{pmatrix} + \mu\begin{pmatrix}8\\7\end{pmatrix}.$$

$$\begin{pmatrix}1+2\lambda\\1+3\lambda\end{pmatrix} = \begin{pmatrix}3+8\mu\\-1+7\mu\end{pmatrix}$$

$$2\lambda - 8\mu = 2$$
$$3\lambda - 7\mu = -2$$
$$\Rightarrow \quad \lambda - 4\mu = 1$$
$$3\lambda - 7\mu = -2$$
$$3\lambda - 12\mu = 3$$
$$\therefore 5\mu = -5$$
$$\therefore \mu = -1 \quad (\text{OR } \lambda = -3)$$

$\therefore$ Point of intersection is $\begin{pmatrix}-5\\-8\end{pmatrix}$

(b) The point $B(x, y, z)$ is on the interval AC and is twice as far from $A(-1, -3, -5)$ as it is from $C(5, 6, 7)$. Use vectors to find the coordinates of B. **3**

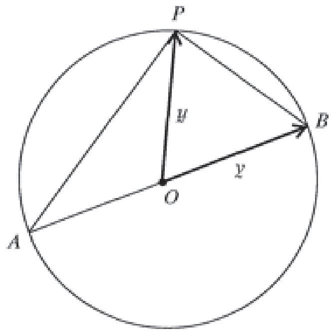


$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}6\\9\\12\end{pmatrix}$$

$$\overrightarrow{OB} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AC}$$
$$= \begin{pmatrix}-1\\-3\\-5\end{pmatrix} + \frac{2}{3}\begin{pmatrix}6\\9\\12\end{pmatrix}$$
$$= \begin{pmatrix}-1\\-3\\-5\end{pmatrix} + \begin{pmatrix}4\\6\\8\end{pmatrix}$$
$$= \begin{pmatrix}3\\3\\3\end{pmatrix}$$

(c) AB is the diameter of a circle, centre O. Let $\overrightarrow{OB} = \underset{\sim}{v}$ and $\overrightarrow{OP} = \underset{\sim}{u}$. **3**

Use vectors to prove that $\angle APB = 90°$.



$$\text{Now } \overrightarrow{OA} = -\underset{\sim}{v}$$

$$\overrightarrow{AP} = \underset{\sim}{u} - (-\underset{\sim}{v}) = \underset{\sim}{u} + \underset{\sim}{v}$$

$$\overrightarrow{PB} = \underset{\sim}{v} - \underset{\sim}{u}$$

$$\overrightarrow{AP} \circ \overrightarrow{PB} = (\underset{\sim}{u} + \underset{\sim}{v}) \circ (\underset{\sim}{v} - \underset{\sim}{u})$$

$$= \underset{\sim}{u} \circ \underset{\sim}{v} - \underset{\sim}{u} \circ \underset{\sim}{u} + \underset{\sim}{v} \circ \underset{\sim}{v} - \underset{\sim}{u} \circ \underset{\sim}{v}$$

$$= \underset{\sim}{v} \circ \underset{\sim}{v} - \underset{\sim}{u} \circ \underset{\sim}{u}$$

$$= |\underset{\sim}{v}|^2 - |\underset{\sim}{u}|^2$$

$$= 0 \quad (\text{since } |\underset{\sim}{v}| = |\underset{\sim}{u}| = \text{radius})$$

$$\therefore \angle APB = 90°$$

(d) (i) If $I_n = \int_0^1 (x^2 - 1)^n \, dx$ for $n \geq 0$, show that $I_n = \frac{-2n}{2n+1} I_{n-1}$ for $n \geq 1$. **3**

$$I_n = \int_0^1 (x^2-1)^n \, dx$$

$$= \left[x(x^2-1)^n\right]_0^1 - \int_0^1 x \cdot n(x^2-1)^{n-1}(2x) \, dx$$

$$= 0 - 2n \int_0^1 x^2 (x^2-1)^{n-1} \, dx$$

$$= -2n \int_0^1 (x^2-1)(x^2-1)^{n-1} + (x^2-1)^{n-1} \, dx$$

$$= -2n \int_0^1 (x^2-1)^n + (x^2-1)^{n-1} \, dx$$

$$= -2n I_n - 2n I_{n-1}$$

$$2n I_n + I_n = -2n I_{n-1}$$

$$I_n(2n+1) = -2n I_{n-1}$$

$$\therefore I_n = \frac{-2n}{2n+1} I_{n-1}$$

(ii) Hence, evaluate I_3. **1**

$$I_3 = -\frac{6}{7} I_2$$

$$I_2 = -\frac{4}{5} I_1$$

$$I_1 = -\frac{2}{3} I_0$$

$$I_0 = \int_0^1 (x^2-1)^0 \, dx$$

$$= [x]_0^1$$

$$= 1$$

$$\therefore I_1 = -\frac{2}{3}$$

$$\therefore I_2 = -\frac{4}{5} \times -\frac{2}{3} = \frac{8}{15}$$

$$\therefore I_3 = -\frac{6}{7} \times \frac{8}{15} = -\frac{48}{105} = \frac{-16}{35}$$

(e) The spheres $(x+2)^2 + (y+3)^2 + (z-4)^2 = 16$ and $(x+2)^2 + (y+3)^2 + (z+2)^2 = 25$ intersect in a circle. What is the centre and radius of this circle? **3**

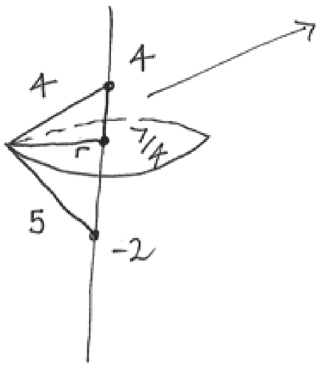
$$(z+2)^2 - (z-4)^2 = 25 - 16$$

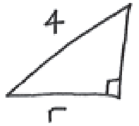
$$z^2 + 4z + 4 - (z^2 - 8z + 16) = 9$$

$$12z - 12 = 9$$

$$z = \frac{21}{12} = \frac{7}{4}$$

$\therefore$ The centre is at $\left(-2, -3, \frac{7}{4}\right)$





$$\Big\}\ 4 - \frac{7}{4} = \frac{9}{4}$$

$$16 = r^2 + \left(\frac{9}{4}\right)^2$$

$$16 - \frac{81}{16} = r^2$$

$$r^2 = \frac{175}{16}$$

$$r = \frac{\sqrt{175}}{4} = \frac{5\sqrt{7}}{4}$$

(OR substitute $z = \frac{7}{4}$ into either of the original equations)

End of Paper