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Student Number

Barker College

**2021** YEAR 12 ASSESSMENT BLOCK
SEMESTER 1

# Mathematics Extension 2

## Section I – Multiple Choice

Write your student number at the top of this page.

Remove this sheet.

Select the alternative A, B C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = A 2 B 6 C 8 D 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

*correct*

A ● B ⊗ C ○ D ○

Start Here →

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|---|---|---|---|---|---|---|---|---|---|
| 1. | A ○ | B ○ | C ○ | D ○ | 5. | A ○ | B ○ | C ○ | D ○ |
| 2. | A ○ | B ○ | C ○ | D ○ | 6. | A ○ | B ○ | C ○ | D ○ |
| 3. | A ○ | B ○ | C ○ | D ○ | 7. | A ○ | B ○ | C ○ | D ○ |
| 4. | A ○ | B ○ | C ○ | D ○ | 8. | A ○ | B ○ | C ○ | D ○ |

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Barker College

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Student Number

2021 YEAR 12 ASSESSMENT BLOCK

# Mathematics Extension 2

**Staff Involved:**

**BHC* ARM**

**12:00 NOON**
**FRIDAY 26 MARCH**

**40 copies**

| TOPICS COVERED: |
|---|
| Complex Numbers I & II |
| Proof |

**General Instructions**

- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale

**Total marks: 80**

**Section I – 8 marks** (pages 4–7)

- Attempt Questions 1–8
- Allow about 15 minutes for this section

**Section II – 72 marks** (pages 8–13)

- Attempt Questions 9–14
- Allow about 1 hour and 45 minutes for this section

## Section I

**8 marks**
**Attempt Questions 1–8**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–8.

---

**1** What is the negation of the following statement?

"If all rich people are happy, then all poor people are sad."

A. Not all rich people are happy and not all poor people are sad.

B. All poor people are sad, but there is a rich person who is not happy.

C. All rich people are happy, but there is a poor person who is not sad.

D. Some rich people are not happy or some poor people are not sad.

**2** If $|z-3-3i|=3$, what is the least value of $|z|$?

A. $3\sqrt{2}-3$

B. $3\sqrt{2}-2$

C. $2\sqrt{3}-2$

D. $2\sqrt{3}-3$

**3** Which one of the following statements is true?

A. $(\forall x, y \in \mathbb{R})\left(\frac{x}{y} \in \mathbb{R}\right)$

B. $(\exists x \in \mathbb{R})(x^2 + 4x + 5 = 0)$

C. $(\forall x \in \mathbb{R})(x^2 \geq x)$

D. $(\forall a \in \mathbb{Z}^+)(\exists x \in \text{primes})(a \leq x \leq 2a)$

**4** Given that $z = \frac{\sqrt{3}+i}{1-i}$, what is the smallest positive value of $n$ for which $z^n$ is purely imaginary?

A. $\frac{6}{7}$

B. $\frac{6}{5}$

C. 6

D. 12



**5** When the cube roots of $2 - 2i$ are expressed in the form $re^{i\theta}$, $r = \sqrt{2}$.
Which expression will give the correct values of $\theta$?

A. $\frac{(8k-1)\pi}{4}$ for $k = -1, 0, 1$

B. $\frac{(8k+1)\pi}{4}$ for $k = -1, 0, 1$

C. $\frac{(8k+1)\pi}{12}$ for $k = -1, 0, 1$

D. $\frac{(8k-1)\pi}{12}$ for $k = -1, 0, 1$

**6** If $z = e^{3i\theta}$, which of the following expressions is equal to $1 + z^2$ ?

A. $2 \cos^2 3\theta \operatorname{cis} 3\theta$

B. $2 \cos 3\theta \operatorname{cis} 3\theta$

C. $\cos^2 3\theta \operatorname{cis} 3\theta$

D. $\cos 3\theta \operatorname{cis} 3\theta$

**7** What is the contrapositive of the following statement?

"If Harry studies Biology or Chemistry in 2022,
then Jesse will not take a holiday in Fiji in 2022."

A. If Jesse does not take a holiday in Fiji in 2022,
then Harry will not study Biology or Chemistry in 2022.

B. If Jesse takes a holiday in Fiji in 2022,
then Harry will study neither Biology nor Chemistry in 2022.

C. If Jesse does not take a holiday in Fiji in 2022,
then Harry will study neither Biology nor Chemistry in 2022.

D. If Jesse takes a holiday in Fiji in 2022,
then Harry will not study Biology or not study Chemistry in 2022.

**8** Shanelle correctly wrote:

$$(\cos\theta)^3 = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^3$$

$$\therefore\ \cos^3\theta = \frac{1}{8}\left(e^{3i\theta}+e^{-3i\theta}+3\left(e^{i\theta}+e^{-i\theta}\right)\right)$$

From her last line of working, Shanelle should be able to deduce which one of the following?

A. $\frac{3}{4}\cos\theta = \cos^3\theta - \frac{1}{4}\cos 3\theta$

B. $8\cos^3\theta = \cos 3\theta + 3\cos\theta$

C. $8\cos^3\theta = 2\cos 3\theta + 3\cos\theta$

D. $\cos 3\theta = 4\cos^3\theta + 3\cos\theta$

**End of Section I**



## Section II

**72 marks**
**Attempt Questions 9–14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 9** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) $x-2-2i$ is a factor of $P(x)=x^3-ax^2+20x-24$, where $a$ is a real number.

Find the value of $a$ and express $P(x)$ as a product of complex linear factors. **3**

(b) (i) Sketch the region in the complex plane where the following inequalities hold simultaneously:

$$|z-i|\le 2 \qquad \text{and} \qquad 0\le \operatorname{Arg}(z-1)\le \frac{3\pi}{4}$$ **3**

(ii) What is the maximum value of $\operatorname{Arg}(z)$?

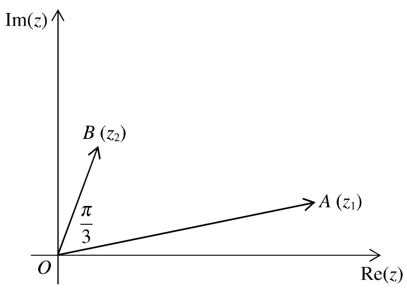
Give your answer in radians, correct to 2 decimal places. **4**

(iii) By considering $z$ as defined in the region above, what is the value of $\operatorname{Arg}(|z|)$?

Give reasons for your answer. **2**

**Question 10** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the Argand diagram below, the point $A$ represents the complex number $z_1$ and the point $B$ represents the complex number $z_2$. Furthermore, $\angle AOB = \dfrac{\pi}{3}$ and $|z_1| = 2 \times |z_2|$.




(i) Express $z_2$ in terms of $z_1$. Give your answer in exponential form. **1**

(ii) If $AOBC$ is a parallelogram, express the complex number represented by point $C$ in terms of $z_1$. Give your answer in exponential form. **1**

(iii) Express the complex number represented by the midpoint of $BC$ in terms of $z_1$. Give your answer in exponential form. **1**

(b) Prove by contradiction that there are no integers $m$ and $n$ which satisfy

$$3m + 21n = 136.$$ **2**

(c) Suppose that $x$ and $y$ are integers. Prove by contraposition that

if $x^2(y + 3)$ is even, then $x$ is even or $y$ is odd. **3**

(d) (i) Starting with $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$, show that $\dfrac{a+b}{2} \geq \sqrt{ab}$. **1**

(ii) Using the result in (i), prove that $xy + yz + zx \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$, where $x, y, z \geq 0$. **3**



**Question 11** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) It is given that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Using the identities above, prove that $e^{i\theta} = \cos\theta + i\sin\theta$. **3**

(b) Sketch the following locus on a complex plane: $\operatorname{Arg}\left(\dfrac{z-1-i}{z+1-i}\right) = 0$. **3**

(c) Solve $z^4 - 8\sqrt{2} + 8\sqrt{2}i = 0$, leaving your answers in exponential form. **3**

(d) It is given that $|z| < \dfrac{1}{2}$.

Use a triangle inequality, or otherwise, to show that $\left|(1+i)z^3 + iz\right| < \dfrac{3}{4}$. **3**

**Question 12** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that $z = \operatorname{cis}\left(\frac{2\pi}{3}\right)$ and $w = \sqrt{3}\operatorname{cis}\left(\frac{\pi}{3}\right)$,

(i) Express $\frac{w}{z}$ in modulus-argument form. **1**

(ii) Use mathematical induction to prove that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all integers $n \geq 0$. **3**

(iii) Hence, by using the result in (ii), find the value of $\left(\frac{w}{z}\right)^{12}$ in simplest form. **2**

(b) Without using induction, prove that $a^3 + 2a$ is divisible by 3 when $a$ is a positive integer. **3**

(c) If $x, y > 0$ and $x > y$, prove that $x^4 + y^4 > x^3y + xy^3$. **3**



**Question 13** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove the following statement, in which $n$ is a positive integer, by using the contrapositive.

If $n$ leaves a remainder of either 2 or 3 after being divided by 4,
then $n$ is not a perfect square. **3**

(b) (i) Show that $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$. **1**

(ii) Show by substitution that $\theta = -i\ln\left(2 + \sqrt{3}\right)$ is a solution to the equation $\cos\theta = 2$. **2**

(iii) By using the substitution $u = e^{i\theta}$, or otherwise, solve the equation

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = 2$$

and hence find another value of $\theta$ that solves the equation $\cos\theta = 2$. **2**

(c) (i) Given that $\frac{d}{dx}(\ln x) = \frac{1}{x}$, show that the Dunstan function, $f(x) = \frac{\ln x}{x}$, is a decreasing function for $x > e$. **2**

(ii) Using the result in (i), explain why $\frac{\ln a}{a} > \frac{\ln b}{b}$ for $e < a < b$. **1**

(iii) Rewrite the result in (ii) without logarithms.
Hence determine if $200^{201} > 201^{200}$ is true or false. **1**

**Question 14** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $(x + 1)(x^2 - x + 1) = x^3 + 1$. **1**

(ii) If $x^2 - x + 1 = 0$, find the value of $x^{2022} + \dfrac{1}{x^{2022}}$. **2**

(b) (i) By expanding $\left(e^{i\theta} - e^{-i\theta}\right)^3$, show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **3**

(ii) Hence, solve $8x^3 - 6x + 1 = 0$. **3**

(c) Suppose that $x$ is a real number. Prove that

if $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$ then $x > 0$. **3**

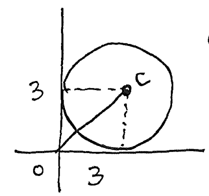
**End of paper**



Extension 2 Mathematics 2021 Friday 26/3/21

Q1// The negation of "if p then q" is "p and not q". (C)

Q2// $|z-(3+3i)| = 3$




$OC = \sqrt{18}$
$= 3\sqrt{2}$
Radius $= 3$

Hence least value of $|z| = 3\sqrt{2} - 3$ (A)

Q3// A) False when $y = 0$

B) False. $\Delta < 0$

C) False. Consider $x = \frac{1}{2}$

D) True by elimination (D)

Q4// $z^n = \left[\dfrac{2 \text{ cis } \frac{\pi}{6}}{\sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)}\right]^n$

$z^n = (\sqrt{2})^n \text{ cis}\left(\dfrac{5n\pi}{12}\right)$

Need $\dfrac{5n\pi}{12} = \dfrac{\pi}{2}$

$n = \dfrac{6}{5}$ (B)

Q5// $z = 2\sqrt{2}\, e^{i\left(-\frac{\pi}{4}\right)}$

$z^{\frac{1}{3}} = \sqrt{2}\, e^{i\left(\frac{-\frac{\pi}{4} + 2k\pi}{3}\right)}$

$\therefore \theta = \dfrac{8k\pi - \pi}{12}$

$\theta = \dfrac{(8k-1)\pi}{12}$ (D)

Q6// $1 + z^2$

$= 1 + (e^{3i\theta})^2$

$= 1 + e^{i6\theta}$

$= 1 + \cos 6\theta + i\sin 6\theta$

$= 1 + 2\cos^2 3\theta - 1 + 2i\sin 3\theta \cos 3\theta$

$= 2\cos 3\theta\,(\cos 3\theta + i\sin 3\theta)$

$= 2\cos 3\theta \text{ cis } 3\theta$ (B)

Q7// The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$. (B)

Q8// From $\cos^3\theta = \frac{1}{8}\left(e^{i3\theta} + e^{-i3\theta} + 3\left(e^{i\theta} + e^{-i\theta}\right)\right)$

$\therefore \cos^3\theta = \frac{1}{8}\left(2\cos 3\theta + 3(2\cos\theta)\right)$

$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{6}{8}\cos\theta$

ie, $\frac{3}{4}\cos\theta = \cos^3\theta - \frac{1}{4}\cos 3\theta$

(A)

Question 9:

(a) $(x-(2+2i))$ is a factor of $P(x)$

Therefore $(x-(2-2i))$ is also a factor since all the coefficients of $P(x)$ are real. Hence

$(x-(2+2i))(x-(2-2i)) = (x^2-4x+8)$ is a factor.

By inspection, the other factor must be $(x-3)$.

To find the value of $a$, compare the coefficient of $x^2$ on both sides of the identity

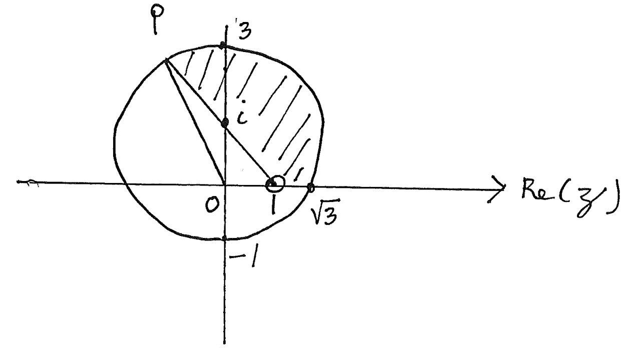
$(x^2-4x+8)(x-3) \equiv x^3 - ax^2 + 20x - 24.$

The coefficient of $x^2$ on the LHS is $(-4-3) = -7$.

Hence $a = 7$.

$P(x) = (x-3)(x-(2+2i))(x-(2-2i))$

(b) (i)




(ii) The maximum value of $\operatorname{Arg}(z)$ is $\angle PORe(z)$.

To find P, solve $y = -x+1$ and $x^2 + (y-1)^2 = 4$.

$$\left.\begin{aligned} x^2 + (-x+1-1)^2 &= 4 \\ 2x^2 &= 4 \\ x &= \pm\sqrt{2} \end{aligned}\right\} \quad P = (-\sqrt{2}, \sqrt{2}+1)$$

The gradient of OP is $\dfrac{\sqrt{2}+1}{-\sqrt{2}}$

$\tan^{-1}\left(\dfrac{\sqrt{2}+1}{-\sqrt{2}}\right) = -1.04089$

$\angle POR = -1.04089 + \pi$

$= 2.10$ radians

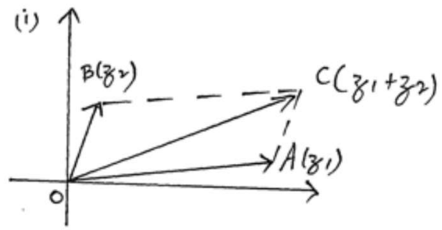
Therefore the maximum value of $\operatorname{Arg}(z)$ is $2.10$ rad.

(iii) $\text{Arg}(|z|) = 0.$

Now the $|z|$ measures the distance of $z$ from the origin.
That will be a positive real number, lying on the real axis to the right of the origin.
And the argument of any such number is zero.

Question 10:

(a) (i)




$$z_2 = \tfrac{1}{2} \times \text{cis}\tfrac{\pi}{3} \times z_1$$

$$z_2 = \tfrac{1}{2} e^{i\frac{\pi}{3}} z_1$$

(ii)
$$\vec{OC} = \vec{OA} + \vec{OB}$$
$$= z_1 + \tfrac{1}{2} e^{i\pi/3} z_1$$
$$= z_1\left(1 + \tfrac{1}{2} e^{i\frac{\pi}{3}}\right)$$

(iii) The midpoint of $\vec{BC}$
$$= \vec{OB} + \tfrac{1}{2} \times \vec{OA}$$
$$= \tfrac{1}{2} e^{i\frac{\pi}{3}} z_1 + \tfrac{1}{2} z_1$$
$$= \tfrac{1}{2} z_1 \left(e^{i\frac{\pi}{3}} + 1\right)$$

(b) Proof by contradiction:

Suppose that there are integers $m$ and $n$ such that
$$3m + 21n = 136$$
Then $$3(m+7n) = 3\times 45 + 1$$
contradiction!

Notice that the LHS is divisible by 3, but the RHS is not divisible by 3.
Therefore the assumption was false.
Therefore there are no integers $m$ and $n$ such that $3m + 21n = 136$.

(c) The contrapositive is:
"if $x$ is odd and $y$ is even then $x^2(y+3)$ is odd".
Let $x = 2k+1$ and $y = 2m$ then
$$x^2(y+3) = (2k+1)^2(2m+3)$$
$$= (4k^2+4k+1)(2m+3)$$
$$= 8k^2m + 8km + 2m + 12k^2 + 12k + 3$$
$$= 2[4k^2m + 4km + m + 6k^2 + 6k + 1] + 1$$
which is odd.

So if $x$ is odd and $y$ is even, then $x^2(y+3)$ is odd.
Therefore if $x^2(y+3)$ is even, then $x$ is even or $y$ is odd.

(d) (i)

$$(\sqrt{a}-\sqrt{b})^2 \geq 0$$

$$a - 2\sqrt{ab} + b \geq 0$$

$$a + b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

(ii) From (i) it follows that

$$\frac{xy+yz}{2} \geq \sqrt{xy^2z} \quad — ①$$

and $\frac{yz+zx}{2} \geq \sqrt{yz^2x}$ - ②

and $\frac{xy+zx}{2} \geq \sqrt{x^2yz}$ - ③

Adding ① + ② + ③ gives

$$\frac{xy+yz}{2} + \frac{yz+zx}{2} + \frac{xy+zx}{2} \geq y\sqrt{xz} + z\sqrt{xy} + x\sqrt{yz}$$

$$\frac{2(xy+yz+xz)}{2} \geq y\sqrt{xz} + z\sqrt{xy} + x\sqrt{yz}$$

$$\therefore \quad xy + yz + xz \geq y\sqrt{xz} + z\sqrt{xy} + x\sqrt{yz}$$

<u>Question 11:</u>

(a) From $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

substitute $x = i\theta$ to get

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)$$

But from

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

we get

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

and from

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

we get

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

Hence from

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

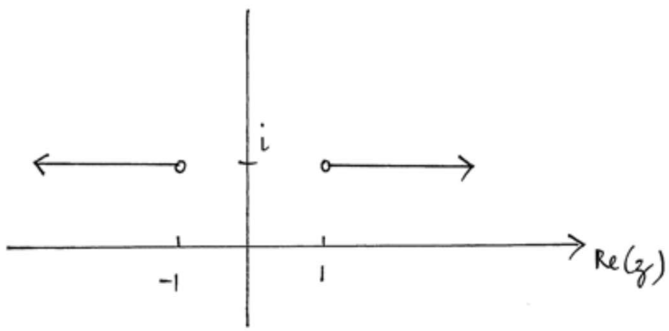
$$\therefore \quad e^{i\theta} = \cos\theta + i\sin\theta$$

as required.

(b)

$$\operatorname{Arg}\left(\frac{z-1-i}{z+1-i}\right)=0$$

ie, $\operatorname{Arg}(z-(1+i))-\operatorname{Arg}(z-(-1+i))=0$




(c) $z^4 = 8\sqrt{2}-8\sqrt{2}i$

$= 16 \text{ cis} \left(\frac{-\pi}{4}\right)$

$= 16 \ e^{i\left(2k\pi - \frac{\pi}{4}\right)}$

$\therefore z = 2e^{i\left(\frac{2k\pi-\frac{\pi}{4}}{4}\right)}$

$z = 2e^{i\left(\frac{8k-1}{16}\right)\pi}$

when $k=-1$, $z_1 = 2e^{-i\frac{9\pi}{16}}$

when $k=0$, $z_2 = 2e^{-i\frac{\pi}{16}}$

when $k=1$, $z_3 = 2e^{i\frac{7\pi}{16}}$

when $k=2$, $z_4 = 2e^{-i\left(\frac{17\pi}{16}\right)} = 2e^{i\left(\frac{15\pi}{16}\right)}$

(d)

RTP that $\left|(1+i)z^3+iz\right| < \frac{3}{4}$, given $|z|<\frac{1}{2}$.

A triangle inequality states that

$$|a+b| \le |a|+|b|.$$

Hence $\left|(1+i)z^3+iz\right| \le \left|(1+i)z^3\right| + |iz|$

$= |(1+i)| \times |z|^3 + |iz|$

$= \sqrt{2}\times|z|^3 + |iz|$

$< \sqrt{2}\left(\frac{1}{2}\right)^3 + \frac{1}{2}$

$= \frac{\sqrt{2}+4}{8}$

$< \frac{2+4}{8}$

$= \frac{3}{4}$

$\therefore \left|(1+i)z^3+iz\right| < \frac{3}{4}$

Question 12:

(a) $z = \text{cis}\left(\frac{2\pi}{3}\right)$

$w = \sqrt{3}\ \text{cis}\left(\frac{\pi}{3}\right)$

(i) $\frac{w}{z} = \frac{\sqrt{3}\ \text{cis}\left(\frac{\pi}{3}\right)}{\text{cis}\left(\frac{2\pi}{3}\right)}$

$\frac{w}{z} = \sqrt{3}\ \text{cis}\left(\frac{-\pi}{3}\right)$

(ii) When $n = 0$,

$LHS = (\cos\theta + i\sin\theta)^0 = 1$

$RHS = \cos(0\theta) + i\sin(0\theta) = 1$

Assume that

$(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

and try to prove that

$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta.$

Now $LHS = (\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k \times (\cos\theta + i\sin\theta)^1$

$= (\cos(k\theta) + i\sin(k\theta)) \times (\cos\theta + i\sin\theta)$ by the assumption

$= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta +$
$i[\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta]$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

$= RHS$

The statement is true by mathematical induction.

(iii) $\left(\frac{w}{z}\right)^{12}$

$= \left(\sqrt{3}\ \text{cis}\left(\frac{-\pi}{3}\right)\right)^{12}$

$= 3^6\ \text{cis}(-4\pi)$

$= 3^6$

$= 729$

(b) Now $a^3 + 2a$

$= a^3 - a + 3a$

$= a(a^2 - 1) + 3a$

$= \underbrace{a(a-1)(a+1)}_{} + 3a$

Note that this expression is the product of three consecutive integers.
It therefore has a factor of 3.

$= 3[j] + 3a$

$= 3[a + j]$, which is divisible by 3.

Hence $a^3 + 2a$ is divisible by 3 for a an positive integer.

(c) RTP that

$$x^4+y^4 > x^3y + xy^3$$

Consider LHS − RHS

$$= x^4+y^4-x^3y-xy^3$$
$$= x^4-x^3y+y^4-xy^3$$
$$= x^3(x-y)+y^3(y-x)$$
$$= x^3(x-y)-y^3(x-y)$$
$$= (x^3-y^3)(x-y)$$

but $x > y > 0$ so we have the product of two non-negative terms.

Hence LHS − RHS > 0

$$\therefore x^4+y^4 > x^3y + xy^3$$

#

Question 13.

(a) Proof by contrapositive.

The contrapositive is "if $n$ is a perfect square, then $n$ leaves a remainder of either 0 or 1 upon division by 4".

Suppose $n = (2j)^2$

then $n = 4j^2$ and so will leave a remainder of 0 upon division by 4.

Suppose $n = (2j+1)^2$

then $n = 4j^2+4j+1$

$n = 4(j^2+j)+1$

$n = 4m+1$ and so will leave a remainder of 1 upon division by 4.

So, if $n$ is a perfect square, then $n$ leaves a remainder of either 0 or 1 upon division by 4.

Hence if $n$ leaves a remainder of either 2 or 3 upon division by 4, then $n$ is not a perfect square.

(b) (i) Now $e^{i\theta} = \cos\theta + i\sin\theta$ — ①

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$ — ②

① + ② gives $e^{i\theta}+e^{-i\theta} = 2\cos\theta$

So $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

as required.

(b) (ii)

Now $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ — ①

substitute $\theta = -i\ln(2+\sqrt{3})$ into ① to get

$$\cos\theta = \frac{e^{-i(i\ln(2+\sqrt{3}))} + e^{i(i\ln(2+\sqrt{3}))}}{2}$$

$$= \frac{e^{\ln(2+\sqrt{3})} + e^{\ln(2+\sqrt{3})^{-1}}}{2}$$

$$= \{(2+\sqrt{3}) + (2+\sqrt{3})^{-1}\}/2$$

$$= \left\{2+\sqrt{3} + \frac{2-\sqrt{3}}{4-3}\right\}/2$$

$$= \{2+\sqrt{3}+2-\sqrt{3}\}/2$$

$$= 2$$

Hence $\theta = -i\ln(2+\sqrt{3})$ is a solution to $\cos\theta = 2$.

(iii) To solve $\dfrac{e^{i\theta}+e^{-i\theta}}{2} = 2$ sub $u = e^{i\theta}$ to get

$$\frac{u+\frac{1}{u}}{2} = 2$$

$$u^2 - 4u + 1 = 0$$

$$u = \frac{4 \pm \sqrt{12}}{2}$$

$$u = 2 \pm \sqrt{3}$$

$$\therefore e^{i\theta} = 2 \pm \sqrt{3}$$

$$i\theta = \ln(2 \pm \sqrt{3})$$

$$\therefore \theta = \frac{1}{i}\ln(2\pm\sqrt{3})$$

$$\therefore \theta = -i\ln(2\pm\sqrt{3})$$

$\theta = -i\ln(2+\sqrt{3})$ $\qquad$ $\theta = -i\ln(2-\sqrt{3})$

$\therefore$ Another solution is $\theta = -i\ln(2-\sqrt{3})$

(c) (i) $f(x) = \dfrac{\ln x}{x}$

$f'(x) = \dfrac{x\left(\frac{1}{x}\right) - \ln x}{x^2}$

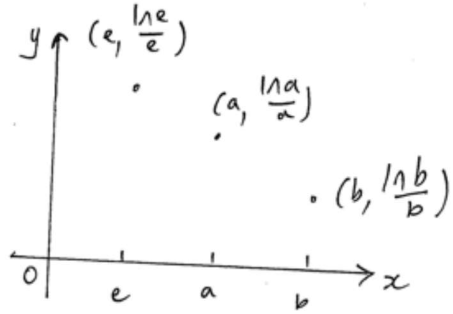
$f'(x) = \dfrac{1 - \ln x}{x^2}$

The denominator is always positive.

The numerator is negative if $x > e$.

Hence $f(x)$ is a decreasing function for $x > e$.

(ii) Graphically, we have




From this graph, clearly

$\dfrac{\ln a}{a} > \dfrac{\ln b}{b}$

for $e < a < b$.

(iii) Now $\dfrac{\ln a}{a} > \dfrac{\ln b}{b}$

$b \ln a > a \ln b$

$\ln a^b > \ln b^a$

$\therefore a^b > b^a$ — (2)

when $e < a < b$.

Hence $200^{201} > 201^{200}$ is true

using the result called (2).

Question 14:

(a) (i) $LHS = (x+1)(x^2 - x + 1)$

$= x^3 - x^2 + x + x^2 - x + 1$

$= x^3 + 1$

Hence $(x+1)(x^2 - x + 1) = x^3 + 1$, as required

(ii) If $x^2 - x + 1 = 0$ then $x^3 + 1 = 0$.

So $x$ is either $-1$ or a complex cube root of $-1$.

But $x = -1$ is not a solution of $x^2 - x + 1 = 0$, so

$x$ must be a cube root of $-1$.

Hence $x^{2022} + \dfrac{1}{x^{2022}}$

$= (x^3)^{674} + \dfrac{1}{(x^3)^{674}}$

$= (-1)^{674} + \dfrac{1}{(-1)^{674}}$

$= 1 + \dfrac{1}{1}$

$= 2$.

(b) (i)

$$(e^{i\theta}-e^{-i\theta})^3 = (e^{i\theta})^3 - 3(e^{i\theta})^2(e^{-i\theta}) + 3(e^{i\theta})^1(e^{-i\theta})^2 - (e^{-i\theta})^3$$

$$= e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}$$

$$= e^{i3\theta} - e^{-i3\theta} - 3(e^{i\theta} - e^{-i\theta})$$

$$= 2i\sin 3\theta - 3(2i\sin\theta)$$

$$= 2i\sin 3\theta - 6i\sin\theta$$

but $(e^{i\theta}-e^{-i\theta})^3 = (2i\sin\theta)^3$

$$= -8i\sin^3\theta$$

so $-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$$4\sin^3\theta = 3\sin\theta - \sin 3\theta$$

$\therefore \sin 3\theta = 3\sin\theta - 4\sin^3\theta$, as Required

(ii) Now $8x^3 - 6x + 1 = 0$

$$1 = 6x - 8x^3$$

$$\tfrac{1}{2} = 3x - 4x^3$$

Let $x = \sin\theta$ and solve $\sin 3\theta = \frac{1}{2}$

$$3\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \ldots$$

$$\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \ldots$$

$\therefore x = \sin\left(\frac{\pi}{18}\right), \sin\left(\frac{5\pi}{18}\right), \sin\left(\frac{25\pi}{18}\right)$.

(c) RTP that

if $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$ then $x > 0$.

Proof by contrapositive:

Suppose $x \leq 0$.

Consider

$$x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4$$

$$= (\underset{(-)}{x^5} + \underset{(-)}{3x^3} + \underset{(-)}{3x}) - (\underset{(+)}{4x^4} + \underset{(+)}{x^2} + \underset{(+)}{4})$$

The expression above is a negative (or zero) number minus a positive number.

That will definitely give a negative result.

Hence $(x^5 + 3x^3 + 3x) - (4x^4 + x^2 + 4) < 0$

So if $x < 0$ then $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$.

So therefore if

$$x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0,$$

then $x > 0$.