Student Number

2022 YEAR 12 ASSESSMENT TASK 3

# Mathematics Extension 2

**Staff Involved:**

KJL* GDH

35 copies

8:30AM
WEDNESDAY 1 JUNE

| TOPICS COVERED |
| --- |
| • Vectors / 23 |
| • Integration / 24 |

**General Instructions:**

- Working time – 55 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- Write all answers in the spaces provided
- Show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale

**Total marks:** 47

**Question 1** **Marks**

If $p = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$ and $q = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$, find the value of:

(i) $p \bullet q$ (ii) $|p - 2q|$ **3**

**Question 2**

The coordinates of three points are $P = (3, 5, 2)$, $Q = (-1, 2, 4)$ and $R = (1, 0, 5)$

(i) Using vectors, find the point $S$ such that $PQRS$ is a parallelogram. **3**

(ii) Prove that $PQRS$ is in fact a rectangle. **2**

**Question 3**

Find the parametric vector equation of the **interval** $AB$ where $A=(0, 4, 5)$ and $B=(-3, 4, 1)$. **2**

**Question 4**

Consider two lines, $\underset{\sim}{L_1}$ and $\underset{\sim}{L_2}$ given by the equations below:

$$\underset{\sim}{L_1}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix} \quad \text{and} \quad \underset{\sim}{L_2}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

Show that $\underset{\sim}{L_1}$ and $\underset{\sim}{L_2}$ are skew lines. **3**



**Question 5**

Let $A=(-1, 3, 4)$ and $B=(1, 2, 2)$.

(i) Find the angle between the two position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in exact form. **2**

(ii) Hence find the exact value of the area of $\Delta OAB$ in simplest surd form. **3**

**Question 6**

Consider the sphere given by the Cartesian equation $x^2 + y^2 + z^2 + 2x - 4z = 4$.

(i) Show that the vector equation of the sphere is $\left| \underset{\sim}{r} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right| = 3$ **2**

(ii) Consider the line $\underset{\sim}{L} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ which intersects with the sphere at two points.

Find the Cartesian coordinates of these two points. **3**



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Student Number

**Question 7**

Find $\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$ **2**

**Question 8**

Find $\int \sec^4 \theta \, d\theta$ **2**



**Question 9** **Marks**

Find $\int \frac{2-x}{(x+1)(x^2+1)}\,dx$ **4**

**Question 10**

(i) If $t = \tan\frac{x}{2}$, show that $dx = \frac{2dt}{1+t^2}$ **1**

(ii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} \frac{2}{1+\sin x}\,dx$ **3**



**Question 11**

Using the method of integration by parts, find $\int \sin(\ln x)\,dx$ **3**

**Question 12**

Using an appropriate trigonometric substitution, find $\int \frac{1}{2x\sqrt{16-x^2}}\,dx$ **4**



**Question 13**

Let $I_n = \int_0^1 \frac{x^{2n}}{x^2+1}\,dx$ for integers $n \geq 0$

Show that $I_n + I_{n-1} = \frac{1}{2n-1}$ for integers $n \geq 1$ **3**

**Question 14**

It is known that for a certain integral, $I = \int_0^{36} [f(x)]^2 dx$.

Find the value of $\int_0^{18} [4f(2x)]^2 dx$, giving your answer in terms of $I$. **2**

**END OF PAPER**



**Question 1** **Marks**

If $p = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$ and $q = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$, find the value of:

(i) $p \cdot q$ (ii) $|p - 2q|$ **3**

$= (-3)(0) + (1)(2) + (5)(-2)$

$= 0 + 2 - 10$

$= -8$

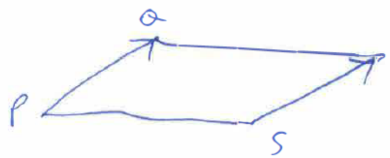
$= \left| \begin{pmatrix} -3 - 2(0) \\ 1 - 2(2) \\ 5 - 2(-2) \end{pmatrix} \right| = \left| \begin{pmatrix} -3 \\ -3 \\ 9 \end{pmatrix} \right|$

$= \sqrt{9 + 9 + 81} = \sqrt{99} = 3\sqrt{11}$

**Question 2**

The coordinates of three points are $P = (3, 5, 2)$, $Q = (-1, 2, 4)$ and $R = (1, 0, 5)$

(i) Using vectors, find the point $S$ such that $PQRS$ is a parallelogram. **3**




$\vec{PQ} = \vec{SR}$ (opp sides equal & parallel in parallelogram)

$\vec{PQ} = \begin{pmatrix} -1-3 \\ 2-5 \\ 4-2 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$ $\therefore \vec{SR} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$

$\vec{OS} = \vec{OR} + \vec{RS}$

$= \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$ $\therefore S = (5, 3, 3)$

(ii) Prove that $PQRS$ is in fact a rectangle. **2**

$\vec{QR} = \begin{pmatrix} 1 - -1 \\ 0 - 2 \\ 5 - 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\vec{QR} \cdot \vec{SR} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix} = -8 + 6 + 2 = 0$

$\therefore \angle QRS = 90°$

$\therefore$ PQRS is a rectangle (parallelogram with right angle)

**Question 3**

Find the parametric vector equation of the **interval** $AB$ where $A=(0,4,5)$ and $B=(-3,4,1)$. **2**

$\overrightarrow{AB} = \begin{pmatrix}-3-0\\4-4\\1-5\end{pmatrix} = \begin{pmatrix}-3\\0\\-4\end{pmatrix}$

$\therefore \underset{\sim}{r_1} = \begin{pmatrix}0\\4\\5\end{pmatrix} + \lambda\begin{pmatrix}-3\\0\\-4\end{pmatrix}$ is the line AB.

For the interval AB, $0 \le \lambda \le 1$

$\therefore \underset{\sim}{r_2} = \begin{pmatrix}0\\4\\5\end{pmatrix} + \lambda\begin{pmatrix}-3\\0\\-4\end{pmatrix}$ for $0 \le \lambda \le 1$

**Question 4**

Consider two lines, $\underset{\sim}{L_1}$ and $\underset{\sim}{L_2}$ given by the equations below:

$$\underset{\sim}{L_1} = \begin{pmatrix}1\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\1\end{pmatrix} \quad \text{and} \quad \underset{\sim}{L_2} = \begin{pmatrix}2\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

Show that $\underset{\sim}{L_1}$ and $\underset{\sim}{L_2}$ are skew lines. **3**

Note that direction vectors $\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$ and $\mu\begin{pmatrix}1\\2\\-1\end{pmatrix}$ cannot be parallel as no value of $\lambda$ or $\mu$ satisfies $\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix} = \mu\begin{pmatrix}1\\2\\-1\end{pmatrix}$ [ignoring the trivial case of the zero vector]

Do they intersect?

Solve: $1-\lambda = 2+\mu$, $3+\lambda = -1+2\mu$, $2+\lambda = 3-\mu$

Adding vertically, $3=5$. Inconsistent equation

$\therefore$ Lines don't intersect

$\therefore$ Lines skew, since not parallel & don't intersect.

or: eqn 2 & 3:

$1 = -4+3\mu$

$3\mu = 5$ $\therefore \mu = \frac{5}{3}$ $\therefore \lambda = -\frac{2}{3}$

Substitute into eqn 1, does $1 - -\frac{2}{3} = 2 + \frac{5}{3}$?

Does $\frac{5}{3} = \frac{11}{3}$? No.

$\therefore$ Inconsistent equation etc.

or: eqn 1 & 2

$4 = 1+3\mu$ $\therefore \mu = 1$, $\lambda = -2$

Substitute into eqn 3, does $2+-2 = 3-1$, $0 = 2$? No

Inconsistent equation etc.



**Question 5**

Let $A=(-1,3,4)$ and $B=(1,2,2)$.

(i) Find the angle between the two position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in exact form. **2**

$\overrightarrow{OA} \cdot \overrightarrow{OB} = |\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta$

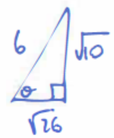
$-1+6+8 = \sqrt{1+9+16}\sqrt{1+4+4}\cos\theta$

$13 = 3\sqrt{26}\cos\theta$

$\therefore \cos\theta = \frac{13}{3\sqrt{26}} = \frac{13\sqrt{26}}{3\times26} = \frac{\sqrt{26}}{6}$

$\therefore \theta = \cos^{-1}\left(\frac{\sqrt{26}}{6}\right)$

(ii) Hence find the exact value of the area of $\Delta OAB$ in simplest surd form. **3**




$\frac{1}{2}|\overrightarrow{OA}||\overrightarrow{OB}|\sin\angle AOB$

$= \frac{1}{2}\sqrt{26}\,3\,\frac{\sqrt{10}}{6}$

$= \frac{1}{4}\sqrt{260}$

$= \frac{1}{4}\sqrt{4\times65}$

$= \frac{\sqrt{65}}{2}\ u^2$

**Question 6**

Consider the sphere given by the Cartesian equation $x^2 + y^2 + z^2 + 2x - 4z = 4$.

(i) Show that the vector equation of the sphere is $\left| \underset{\sim}{r} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right| = 3$ **2**

$$x^2+2x+1+y^2+z^2-4z+4 = 4+1+4$$

$$(x+1)^2+y^2+(z-2)^2 = 9.$$

$\therefore$ sphere has centre $(-1,0,2)$, radius 3

ie $\left| \underset{\sim}{r} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right| = 3$

(ii) Consider the line $\underset{\sim}{L} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ which intersects with the sphere at two points.

Find the Cartesian coordinates of these two points. **3**

$$\left| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \lambda \\ \lambda \\ 2\lambda \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right| = 3$$

$$\left| \begin{matrix} \lambda+1 \\ \lambda+1 \\ 2\lambda-1 \end{matrix} \right| = 3$$

$$\therefore \sqrt{(\lambda+1)^2+(\lambda+1)^2+(2\lambda-1)^2} = 3$$

$$\therefore 6\lambda^2+3 = 9$$

$$6\lambda^2 = 6$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$\therefore$ Points are $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

Check that both satisfy

$\left| \underset{\sim}{r} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right| = 3$ ✓✓ Both



**Question 7** **Student Number**

Find $\int \frac{1}{\sqrt{7-6x-x^2}} dx$ **2**

$$= \int \frac{dx}{\sqrt{-(x^2+6x-7)}} = \int \frac{dx}{\sqrt{-((x+3)^2-16)}} \quad \int \frac{dx}{\sqrt{16-(x+3)^2}}$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right) + C$$

**Question 8**

Find $\int \sec^4 \theta \, d\theta$ **2**

$$= \int \sec^2\theta \sec^2\theta \, d\theta$$

$$= \int \sec^2\theta (1+\tan^2\theta) \, d\theta$$

$$= \int \sec^2\theta + \sec^2\theta \tan^2\theta \, d\theta$$

$$= \tan\theta + \frac{\tan^3\theta}{3} + C$$

**Question 9**

**Marks**

Find $\int \frac{2-x}{(x+1)(x^2+1)}\,dx$ **4**

let $\frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{2-x}{(x+1)(x^2+1)}$

$\therefore A(x^2+1) + (Bx+C)(x+1) = 2-x$

let $x=-1$ $\therefore 2A = 3$ $\therefore A = \frac{3}{2}$

Coeff $x^2$ : $A+B=0$ $\therefore B = -\frac{3}{2}$

Constant : $A+C=2$ $\therefore C = \frac{1}{2}$

$$\therefore \int \frac{\frac{3}{2}}{x+1} + \frac{\frac{1}{2}-\frac{3}{2}x}{x^2+1}\,dx$$

$$= 3\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx - \frac{3}{4}\int \frac{2x}{x^2+1}\,dx$$

$$= \frac{3}{2}\ln|x+1| + \frac{1}{2}\tan^{-1}x - \frac{3}{4}\ln|x^2+1| + C$$



**Question 10**

(i) If $t = \tan\frac{x}{2}$, show that $dx = \frac{2dt}{1+t^2}$ **1**

$\tan^{-1}t = \frac{x}{2}$

$\therefore x = 2\tan^{-1}t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$\therefore dx = \frac{2dt}{1+t^2}$

or $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$\therefore 2dt = dx(1+t^2)$

$\therefore dx = \frac{2dt}{1+t^2}$

(ii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} \frac{2}{1+\sin x}\,dx$ **3**

let $t = \tan\frac{x}{2}$


$x=0 : t=0$

$x=\frac{\pi}{2} : t = \tan\frac{\pi}{4} = 1$

$$\therefore = \int_0^1 \frac{2}{1+\frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= 4\int_0^1 \frac{dt}{1+t^2+2t}$$

$$= 4\int_0^1 \frac{dt}{(t+1)^2}$$

$$= 4\int_0^1 (t+1)^{-2}\,dt$$

$$= 4\left[\frac{(t+1)^{-1}}{-1}\right]_0^1$$

$$= -4\left[\frac{1}{1+t}\right]_0^1$$

$$= -4\left[\frac{1}{2}-1\right]$$

$$= 2$$

**Question 11**

Using the method of integration by parts, find $\int \sin(\ln x)\,dx$ **3**

let $u' = 1$, $v = \sin(\ln x)$

$\therefore u = x$, $v' = \frac{\cos(\ln x)}{x}$

$\therefore I = \int \sin(\ln x)\,dx = x\sin(\ln x) - \int x\frac{\cos(\ln x)}{x}\,dx$

$I = x\sin(\ln x) - \underbrace{\int \cos(\ln x)\,dx}$

Let $u' = 1$, $v = \cos(\ln x)$

$\therefore u = x$, $v' = \frac{-\sin(\ln x)}{x}$

$\therefore I = x\sin(\ln x) - \left[x\cos(\ln x) - \int x\left(\frac{-\sin(\ln x)}{x}\right)dx\right]$

$I = x\sin(\ln x) - x\cos(\ln x) + -\int \sin(\ln x)\,dx$

$\therefore 2I = x(\sin(\ln x) - \cos(\ln x))$

$\therefore I = \frac{x(\sin(\ln x) - \cos(\ln x))}{2}$

**Question 12**

Using an appropriate trigonometric substitution, find $\int \frac{1}{2x\sqrt{16 - x^2}}\,dx$ **4**

let $x = 4\cos\theta$ $\therefore \frac{dx}{d\theta} = -4\sin\theta$ $\therefore dx = -4\sin\theta\,d\theta$

$\therefore = \int \frac{-4\sin\theta\,d\theta}{2(4\cos\theta)\sqrt{16 - 16\cos^2\theta}}$

$= -\frac{1}{2}\int \frac{\sin\theta\,d\theta}{\cos\theta\,4\sqrt{1-\cos^2\theta}}$

$= -\frac{1}{8}\int \frac{\sin\theta\,d\theta}{\cos\theta\sin\theta}$

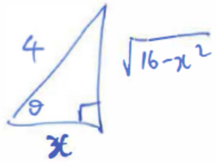
$= -\frac{1}{8}\int \sec\theta\,d\theta$

$= -\frac{1}{8}\int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\,d\theta$

$= -\frac{1}{8}\ln|\sec\theta + \tan\theta|$

$= -\frac{1}{8}\ln\left|\frac{4}{x} + \frac{\sqrt{16-x^2}}{x}\right|$

$= -\frac{1}{8}\ln\left|\frac{4 + \sqrt{16-x^2}}{x}\right| + C$



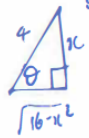

or let $x = 4\sin\theta$

$= \frac{1}{8}\int \operatorname{cosec}\theta\,d\theta$

$= -\frac{1}{8}\int \frac{\operatorname{cosec}\theta(\cot\theta + \operatorname{cosec}\theta)}{\cot\theta + \operatorname{cosec}\theta}\,d\theta$

$= -\frac{1}{8}\ln|\cot\theta + \operatorname{cosec}\theta|$

or $\frac{1}{8}\ln|\operatorname{cosec}\theta - \cot\theta|$



**Question 13**

Let $I_n = \int_0^1 \frac{x^{2n}}{x^2+1}\,dx$ for integers $n \geq 0$

Show that $I_n + I_{n-1} = \frac{1}{2n-1}$ for integers $n \geq 1$ **3**

$$I_n = \int_0^1 \frac{x^2 x^{2n-2}}{x^2+1}\,dx$$

$$= \int_0^1 \frac{(x^2+1-1)x^{2n-2}}{x^2+1}\,dx$$

$$= \int_0^1 \left(\frac{x^2+1}{x^2+1}\right)x^{2n-2}\,dx - \int_0^1 \frac{x^{2n-2}}{x^2+1}\,dx$$

$$= \int_0^1 x^{2n-2}\,dx - \int_0^1 \frac{x^{2(n-1)}}{x^2+1}\,dx$$

$$= \left[\frac{x^{2n-1}}{2n-1}\right]_0^1 - I_{n-1}$$

$$I_n = \left(\frac{1}{2n-1} - 0\right) - I_{n-1}$$

$$I_n + I_{n-1} = \frac{1}{2n-1}$$

OR

$$I_n + I_{n-1}$$

$$= \int_0^1 \frac{x^{2n}}{x^2+1}\,dx + \int_0^1 \frac{x^{2n-2}}{x^2+1}\,dx$$

$$= \int_0^1 \frac{x^{2n} + x^{2n-2}}{x^2+1}\,dx$$

$$= \int_0^1 \frac{x^{2n-2}(x^2+1)}{x^2+1}\,dx$$

$$= \int_0^1 x^{2n-2}\,dx$$

$$= \left[\frac{x^{2n-1}}{2n-1}\right]_0^1 = \frac{1}{2n-1} - 0 = \frac{1}{2n-1}$$



**Question 14**

It is known that for a certain integral, $I = \int_0^{36} [f(x)]^2 dx$.

Find the value of $\int_0^{18} [4f(2x)]^2 dx$, giving your answer in terms of $I$. **2**

$$= \int_0^{18} 16[f(2x)]^2\,dx$$

$$= 16\int_0^{18} [f(2x)]^2\,dx$$

Let $u = 2x$

$\frac{du}{dx} = 2$

$\therefore dx = \frac{du}{2}$

$x = 0;\ u = 0$

$x = 18;\ u = 36$

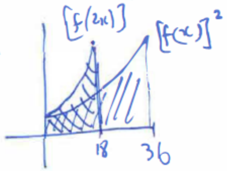
$$\therefore = 16\int_0^{36} [f(u)]^2 \frac{du}{2}$$

$$= 8\int_0^{36} [f(u)]^2\,du$$

$$= 8\int_0^{36} [f(x)]^2\,dx \quad \text{with change of variable}$$

$$= 8I$$

Graphically:




In a sense, $\int_0^{36} [f(x)]^2 dx = 2\int_0^{18} [f(2x)]^2 dx$

since horizontally everything is dilated

$\therefore 8\int_0^{36} [f(x)]^2 dx = 16\int_0^{18} [f(2x)]^2 dx$

$\therefore 8I = \int_0^{18} [4f(2x)]^2 dx$

**END OF PAPER**