
Student Number

Barker College

2022 YEAR 12 ASSESSMENT BLOCK
SEMESTER 1

Mathematics Extension 2

Section I – Multiple Choice

Write your student number at the top of this page.

Remove this sheet.

Select the alternative A, B C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = A 2 B 6 C 8 D 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

correct

A ● B ⊗ C ○ D ○

Start Here → 1. A○ B○ C○ D○

2. A○ B○ C○ D○

3. A○ B○ C○ D○

4. A○ B○ C○ D○

5. A○ B○ C○ D○

6. A○ B○ C○ D○

7. A○ B○ C○ D○

8. A○ B○ C○ D○

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Barker College

Student Number

2022 YEAR 12 ASSESSMENT BLOCK

Mathematics Extension 2

Staff Involved:

GDH KJL
ARM*

12:00 NOON
FRIDAY 1 APRIL

40 copies

TOPICS COVERED:
Complex Numbers I & II
Proof

General Instructions

- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale
- Write your student number at the top of this page

Total marks: 80

Section I – 8 marks (pages 4-6)
- Attempt Questions 1-8
- Allow about 15 minutes for this section

Section II – 72 marks (pages 7-12)
- Attempt Questions 9-14
- Allow about 1 hour and 45 minutes for this section

Section I

8 marks
Attempt Questions 1–8
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–8.

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1 For real x, which one of these true statements has a converse that is also true?

A. If x is positive, then x^2 is positive.

B. If $x > 3$, then $x - 3 \geq 0$.

C. If $x > 0$, then $-x < 0$.

D. If x is divisible by four then x is even.

2 What is the value of $i + i^2 + i^3 + i^4 + \ldots + i^{2022}$?

A. 0

B. i

C. $1 + i$

D. $-1 + i$

3 How many complex numbers satisfy $|z-1|=|z+1|=|z-i|$?

A. 0

B. 1

C. 2

D. Infinitely many

4 What is the area of the triangle on the Argand diagram with vertices at z, iz and $z+iz$?

A. $|z|^2$

B. $\frac{1}{2}|z|^2$

C. $\frac{\sqrt{3}}{2}|z|^2$

D. $\frac{3}{2}|z|^2$

5 Which one of the following statements is false?

A. $\forall a, b \in \mathbb{R},\ \sqrt{ab} \in \mathbb{C}$

B. $\forall a \in \mathbb{R},\ \exists b \in \mathbb{R}: \sqrt{ab} \in \mathbb{R}$

C. $\forall a \in \mathbb{R},\ \exists b \in \mathbb{R}: \frac{a}{b} \in \mathbb{C}$

D. $\forall a, b \in \mathbb{R},\ \frac{a}{b} \in \mathbb{R}$

6 What is the smallest positive integer, k, for which $(1+i)^{2k}=(1-i)^{2k}$?

A. 1

B. 2

C. 3

D. 4

7 Which expression is equivalent to $\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)\frac{\sqrt{3}+i}{i-1}$?

A. $\sqrt{2}e^{i\frac{13\pi}{12}}$

B. $\sqrt{2}e^{i\frac{11\pi}{12}}$

C. $\sqrt{2}e^{i\frac{7\pi}{12}}$

D. $\sqrt{2}e^{i\frac{5\pi}{12}}$

8 If the cube roots of unity are 1, w and w^2, what are the roots of $(z-1)^3+8=0$?

A. $z=-1, -1, -1$

B. $z=-1, -1+2w, -1-2w^2$

C. $z=-1, 1+2w, 1-2w^2$

D. $z=-1, 1-2w, 1-2w^2$

End of Section I

Section II

72 marks
Attempt Questions 9–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 9 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the complex numbers $z = 12e^{i\frac{\pi}{6}}$ and $w = 3e^{i\frac{\pi}{4}}$.

(i) Express z in the simplest form of $x + iy$. **1**

(ii) Express $\frac{z}{w}$ in the simplest form of $re^{i\theta}$. **1**

(iii) Express $\sqrt{zw}$ in the simplest form of $re^{i\theta}$. **2**

(b) (i) Find the two square roots of $-3 + 4i$. **2**

(ii) Hence, solve $z^2 - z + 1 - i = 0$. **2**

(c) (i) Express $\sqrt{3} - i$ in modulus-argument form. **1**

(ii) Express $\left(\sqrt{3} - i\right)^7$ in modulus-argument form. **1**

(iii) Hence, express $\left(\sqrt{3} - i\right)^7$ in the simplest form of $x + iy$. **2**

Question 10 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the statement, S: if $3a^2 - 4a + 5$ is even, then a is odd.

(i) State the contrapositive of S. **1**

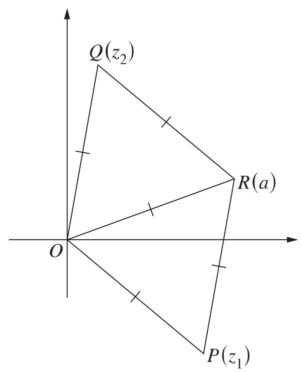
(ii) Use the contrapositive to prove S. **2**

(b) (i) Sketch the region on the Argand diagram satisfying both the inequalities **4**

$$|z - \bar{z}| < 2 \quad \text{and} \quad |z - 1| \geq 1$$

(ii) Consider the arguments of all the complex numbers in the region from (i). List all the values in $(-\pi, \pi]$ that are not included. **1**

(c)



The points P, Q and R on the Argand diagram represent the complex numbers z_1, z_2 and a, respectively.

The triangles OPR and OQR are equilateral with unit sides so that $|z_1| = |z_2| = |a| = 1$.

Let $w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

(i) Explain why $z_2 = aw$. **1**

(ii) Show that $z_1 z_2 = a^2$. **1**

(iii) Show that z_1 and z_2 are the roots of $z^2 - az + a^2 = 0$. **2**

Question 11 (12 marks) Use a SEPARATE writing booklet.

(a) The polynomial $P(z) = z^3 - 5z^2 + az + b$, where a and b are real, has one root at $z = 3 - 2\sqrt{2}i$. Find the values of a and b. **3**

(b) (i) Prove that $\frac{1}{10} - \frac{1}{11} < \frac{1}{100}$. **1**

(ii) For positive real n, prove that $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$. **2**

(c) The complex number z moves such that $\operatorname{Im}\left(\frac{1}{\bar{z} - i}\right) = 2$. **3**

Show that the locus of z is a circle and find its centre and radius.

(d) Prove by induction that $x^n - y^n$ is divisible by $x^2 - y^2$ for all positive even integers n, where x and y are integers. **3**

Question 12 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers. **1**

(ii) Hence, show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a, b and c are distinct positive real numbers. **1**

(iii) Hence, or otherwise, prove that **2**

$$\frac{a^2b^2 + b^2c^2 + c^2a^2}{a+b+c} > abc$$

where a, b and c are distinct positive real numbers.

(b) Prove that n^2 is divisible by 5 **if and only if** n is divisible by 5. **3**

(c) (i) Show that $(1 + i\tan\theta)^n - (1 - i\tan\theta)^n = \frac{2\sin n\theta}{\cos^n\theta}i$ **2**

where $\cos\theta \neq 0$ and n is a positive integer.

(ii) Use the result from (i) to solve $(1 + z)^4 - (1 - z)^4 = 0$, where $\operatorname{Re}(z) = 0$. **2**

(iii) Briefly explain why $(1 + z)^4 - (1 - z)^4 = 0$ only has three roots, not four. **1**

Question 13 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the equation $z^5 + 1 = 0$.

(i) Draw a clearly labelled sketch of the roots of $z^5 + 1 = 0$ on an Argand diagram. **1**

(ii) Factorise $z^5 + 1$ into a product of linear and quadratic factors with real coefficients. **2**

(iii) Show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = \frac{-1}{4}$. **3**

(iv) Write a quadratic equation with integer coefficients that has roots **2**

$\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$.

Hence find the value of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ as surds.

(b) Prove by induction for integers, $n \geq 2$ that **4**

$$1^3 + 2^3 + \ldots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \ldots + n^3$$

Question 14 (12 marks) Use a SEPARATE writing booklet.

(a) Use proof by contradiction to prove the following statement: **3**

If a is rational and b is irrational, then $a + b$ is irrational.

(b) By proving the function $f(x) = x - \ln\left(1 + x + \frac{1}{2}x^2\right)$ is increasing for $x \neq 0$, **4**

show that $e^x < 1 + x + \frac{1}{2}x^2$ for $x < 0$.

(c) (i) If $z = \cos\theta + i\sin\theta$, express $z^n + z^{-n}$ in terms of θ. **1**

(ii) Consider the following function where k is real: **3**

$$f(\theta) = 1 + k\cos\theta + k^2\cos 2\theta + k^3\cos 3\theta + \ldots + k^n\cos n\theta + \ldots$$

Using the result in (i) and by expressing $2f(\theta)$ as the sum of two geometric series, prove that

$$f(\theta) = \frac{1 - k\cos\theta}{1 - 2k\cos\theta + k^2} \quad \text{for } |k| < 1$$

You may assume that complex numbers can be used in geometric series calculations, just as real numbers can.

(iii) Verify the result in (ii) for $|k| < 1$ and $\theta = \frac{\pi}{2}$. **1**

End of paper

Extension 2 Mathematics

Solutions to 1st April, 2022.

Q1. The converse of $p \Rightarrow q$ is $q \Rightarrow p$.

If $-x < 0$ then $x > 0$ is true.

Hence (c)

Q2. Now $i + i^2 + i^3 + i^4$

$= i - 1 - i + 1$

$= 0$

So every 4 terms sums to 0.

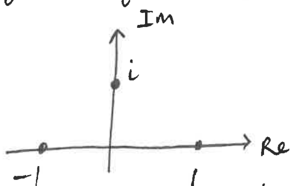
And $2022 \div 4 = 505$ remainder 2.

So sum $= 0 + i + i^2$

$= -1 + i$

Hence (d)

Q3. $|z-1| = |z+1| = |z-i|$



The only complex number equidistant from -1, 1 and i is the origin.

Hence (b)

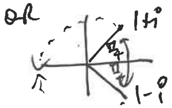
OR: $(1+i)^2 = 2i$ $\quad (1-i)^2 = -2i$

$\therefore (2i)^k = (-2i)^k$

$(2i)^k = (2i)^k(-1)^k$

$\therefore (-1)^k = 1 \quad \therefore k = 2$

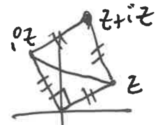
OR



$(1+i)^4 = (1-i)^4 \quad \left(= \sqrt{2}^4 \operatorname{cis} \pi = -4\right)$

$\therefore k = 2$

OR



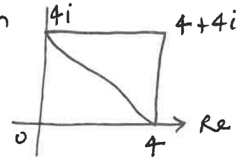
$A =$ half square $= \dfrac{|z|^2}{2}$

Q4. Let $z = 4$

$iz = 4i$

$z + iz = 4 + 4i$

Then



Area of $\Delta = 8$ units^2

$8 = \frac{1}{2}\left(\sqrt{16}\right)^2$

$A = \frac{1}{2}|z|^2$

Hence (b)

Q5. The statement

$\forall a, b \in \mathbb{R}, \ \frac{a}{b} \in \mathbb{R}$

is false when $b = 0$.

Hence (d)

Q6. When $k = 2$,

$(1+i)^4 = 1 + 4i + 6i^2 + 4i^3 + i^4 = -4$

$(1-i)^4 = 1 - 4i + 6i^2 - 4i^3 + i^4 = -4$

Hence (b)

Q7. $\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) \times \dfrac{\sqrt{3}+i}{-1+i}$

$= \operatorname{cis}\left(\frac{-\pi}{3}\right) \times \left(\dfrac{2\operatorname{cis}\frac{\pi}{6}}{\sqrt{2}\operatorname{cis}\frac{3\pi}{4}}\right)$

$= \operatorname{cis}\left(\frac{-\pi}{3}\right) \times \sqrt{2}\operatorname{cis}\left(\frac{-7\pi}{12}\right)$

$= \sqrt{2}\operatorname{cis}\left(-\frac{11\pi}{12}\right)$

$= \sqrt{2}\operatorname{cis}\left(\frac{13\pi}{12}\right)$

Hence (A)

Q8. Test roots by substitution.

When $z = 1 - 2w$

$(z-1)^3 + 8 = 0$ becomes

$(1 - 2w - 1)^3 + 8 = 0$

$-8w^3 + 8 = 0$

but $w^3 = 1$

$\therefore -8 + 8 = 0$

Hence $z = 1 - 2w$ is a root

When $z = 1 - 2w^2$ is substituted we get

$(1 - 2w^2 - 1)^3 + 8$

$= -8w^6 + 8$

$= -8 + 8$

$= 0$

Hence $z = 1 - 2w^2$ is also a root. Hence (d)

OR Roots of unity:

$w = \operatorname{cis}\frac{2\pi}{3}, \quad w^2 = \operatorname{cis}\frac{-2\pi}{3}$

So $2w = 2\operatorname{cis}\frac{2\pi}{3}$

$1 - 2w = -2\operatorname{cis}\frac{2\pi}{3} = 2\operatorname{cis}\frac{-\pi}{3}$

$2w^2 = 2\operatorname{cis}\frac{-2\pi}{3} = -2\operatorname{cis}\frac{\pi}{3}$

$1 - 2w^2 = -2\operatorname{cis}\frac{-2\pi}{3} = 2\operatorname{cis}\frac{\pi}{3}$

So for $(z-1)^3 = -8$

$(z-1)^3 = 8\operatorname{cis}(\pi + 2k\pi)$

$z - 1 = 2\operatorname{cis}\frac{\pi}{3}, \ 2\operatorname{cis}\pi, \ 2\operatorname{cis}\frac{-\pi}{3}$

$= -2w^2, \ -2, \ -2w$

$\therefore z = 1 - 2w^2, \ -1, \ 1 - 2w$

Q9

(a) $z = 12 e^{i\frac{\pi}{6}}$

$w = 3 e^{i\frac{\pi}{4}}$

(i) $\frac{z}{w} = \frac{12 e^{i\frac{\pi}{6}}}{3 e^{i\pi/4}}$

$= 4 e^{i\frac{-\pi}{12}}$

(ii) $\sqrt{zw}$

$= \sqrt{36 e^{i\,5\pi/12}}$

$= \pm 6 e^{i\frac{5\pi}{24}}$

(b) (i) Let $(x+iy)^2 = -3+4i$

then $x^2 - y^2 = -3$ and $2xyi = 4i$

$y = \frac{2}{x}$

$x^2 - \left(\frac{2}{x}\right)^2 = -3$

$x^4 + 3x^2 - 4 = 0$

$(x^2+4)(x^2-1) = 0$

$x = \pm 1$

Hence $\pm(1+2i)$

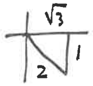
(ii) $z^2 - z + 1 - i = 0$

$z = \frac{1 \pm \sqrt{-3+4i}}{2}$

$z = \frac{1 \pm (1+2i)}{2}$

$z = -i, \ 1+i$

(c) (i) $\sqrt{3} - i = 2 \operatorname{cis}\left(\frac{-\pi}{6}\right)$



(ii) $(\sqrt{3}-i)^7$

$= \left(2 \operatorname{cis} \frac{-\pi}{6}\right)^7$

$= 128 \operatorname{cis}\left(\frac{-7\pi}{6}\right)$

$= 128 \operatorname{cis}\left(5\pi/6\right)$

(iii) $(\sqrt{3}-i)^7$

$= 128\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 128\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$= -64\sqrt{3} + 64i$

Q10 (a) S: if $3a^2 - 4a + 5$ is even, then a is odd.

(i) The contrapositive of S is:

"If a is even, then $3a^2 - 4a + 5$ is odd."

(ii) Suppose that a is even.

Let $a = 2j$, j an integer.

Then $3a^2 - 4a + 5$

$= 3(2j)^2 - 4(2j) + 5$

$= 12j^2 - 8j + 5$

$= \underbrace{2[6j^2 - 4j]}_{} + 5$ — ①

which is an even number.

In ① an even number added to an odd number makes an odd number.

Hence, if a is even, then $3a^2 - 4a + 5$ is odd.

Therefore, if $3a^2 - 4a + 5$ is even, then a is odd.

Q10 (b)

(i) $|z-\bar{z}| < 2$

$|x+iy-(x-iy)| < 2$

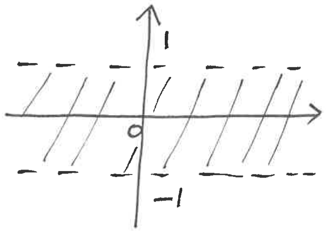
$|2iy| < 2$

$\sqrt{4y^2} < 2$

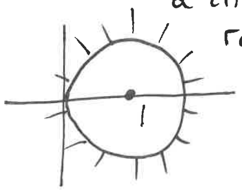
$\pm 2y < 2$

$-1 < y < 1$

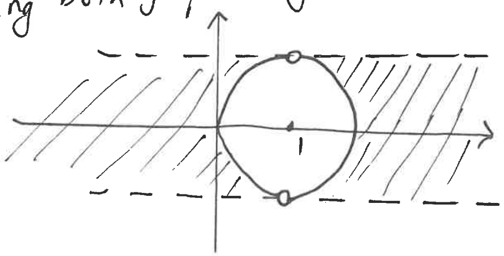
gives



and $|z-1| \geq 1$ gives a circle, centre $(1,0)$ radius 1 unit



Putting both graphs together gives:



(ii) $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ are the only two values of the argument of z not representing any complex number in the shaded region. These values correspond to the "holes" in the circle.

(c) (i) Now $\angle QOR = \frac{\pi}{3}$

We can rotate OR anticlockwise about O by multiplying a by $\text{cis}\left(\frac{\pi}{3}\right)$.

Hence $z_2 = a \times \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= aw$

as required.

(ii) Now OP is OR rotated $\frac{\pi}{3}$ clockwise. This can be done by dividing a by $\text{cis}\,\frac{\pi}{3}$ or multiplying by $\text{cis}\left(-\frac{\pi}{3}\right)$.

So $z_1 = a\,\text{cis}\left(-\frac{\pi}{3}\right)$

$\therefore z_1 z_2 = a\,\text{cis}\left(-\frac{\pi}{3}\right) \times a\,\text{cis}\left(\frac{\pi}{3}\right)$

$= a^2\,\text{cis}(0)$

$= a^2$

as required.

(iii) Now $z_1 = a\,\text{cis}\left(-\frac{\pi}{3}\right)$

$= a\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

and $z_2 = a\,\text{cis}\left(\frac{\pi}{3}\right)$

$= a\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

Now sum of roots above is

$\frac{1}{2}a + \frac{1}{2}a = a$

same!

And sum of roots of equation is

$\frac{-b}{a} = \frac{-(-a)}{1} = a$

The product of the roots above is

$a^2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= a^2(1)$

$= a^2$

same!

And product of roots of equation is

$\frac{c}{a} = \frac{a^2}{1} = a^2$

$\therefore z_1$ and z_2 are the roots of

$z^2 - az + a^2 = 0$

Q11 (a) Since $3-2\sqrt{2}i$ is a root, and all the coefficients of the equation are real, then $3+2\sqrt{2}i$ is also a root.

Then the sum of the roots is

$$(3-2\sqrt{2}i)+(3+2\sqrt{2}i)+\gamma=5$$

$$\gamma=-1$$

The product of the roots is

$$(-1)\times(3-2\sqrt{2}i)\times(3+2\sqrt{2}i)$$

$$=-17$$

Hence $-b=-17 \Rightarrow b=17$

$\therefore$ $P(z)=z^3-5z^2+az+17$

but $P(-1)=(-1)-5(-1)^2-a+17=0$

$0=11-a \Rightarrow a=11$

(b) (i) $\frac{1}{10}-\frac{1}{11}$

$=\frac{11}{110}-\frac{10}{110}$

$=\frac{1}{110}$

$<\frac{1}{100}$

(ii) To prove that $\frac{1}{n}-\frac{1}{n+1}<\frac{1}{n^2}$:

Consider RHS − LHS

$=\frac{1}{n^2}-\frac{1}{n}+\frac{1}{n+1}$

$=\frac{1}{n^2}-\frac{1}{n(n+1)}$

$=\frac{(n+1)-n}{n^2(n+1)}$

$=\frac{1}{n^2(n+1)}>0$

$\therefore$ RHS > LHS

$\therefore$ $\frac{1}{n}-\frac{1}{n+1}<\frac{1}{n^2}$ as required.

Q11 (c) $\text{Im}\left(\frac{1}{\bar{z}-i}\right)=2$

Let $z=x+iy$ then

$$\text{Im}\left(\frac{1}{x-iy-i}\right)=2$$

$$\text{Im}\left(\frac{1}{x-i(y+1)}\times\frac{x+i(y+1)}{x+i(y+1)}\right)=2$$

$$\text{Im}\left(\frac{x+i(y+1)}{x^2+(y+1)^2}\right)=2$$

$$\frac{y+1}{x^2+(y+1)^2}=2$$

$$y+1=2x^2+2y^2+4y+2$$

$$0=2x^2+2y^2+3y+1$$

$$0=x^2+y^2+\frac{3}{2}y+\frac{1}{2}$$

$$0=x^2+y^2+\frac{3}{2}y+\frac{9}{16}+\frac{1}{2}-\frac{9}{16}$$

$$0=x^2+\left(y+\frac{3}{4}\right)^2-\frac{1}{16}$$

$$\frac{1}{16}=x^2+\left(y+\frac{3}{4}\right)^2$$

which is a circle, centre $(0,-\frac{3}{4})$ and radius $\frac{1}{4}$ units, with a hole at $(0,-1)$.

Q11 (d)

When $n=2$, $x^n - y^n = x^2 - y^2$, which is divisible by x^2-y^2.

Hence the statement is true when $n=2$.

Assume that the statement is true when $n=2k$, k an integer.

Assume that $x^{2k} - y^{2k}$ is div. by x^2-y^2, ie, assume that

$$x^{2k} - y^{2k} = M.(x^2-y^2) \text{ where } m \text{ is an integer.}$$

Next, try to prove that the statement is true when $n=2k+2$

Now $x^{2k+2} - y^{2k+2}$

$$= x^{2k}.x^2 - y^{2k+2}$$

$$= (y^{2k} + M(x^2-y^2)).x^2 - y^{2k+2} \quad \text{(by assumption)}$$

$$= x^2y^{2k} + Mx^2(x^2-y^2) - y^{2k+2}$$

$$= y^{2k}(x^2-y^2) + Mx^2(x^2-y^2)$$

$$= (x^2-y^2)[Mx^2 + y^{2k}]$$

which is divisible by (x^2-y^2)

as $[Mx^2+y^{2k}]$ is an integer.

Hence the statement is true by mathematical induction.

Q12 (a)

(i) Now $(a-b)^2 > 0$ if $a \neq b$

$$a^2 - 2ab + b^2 > 0$$

Hence $a^2 + b^2 > 2ab$ — ①

where a and b are distinct positive real numbers.

(ii) Similarly, $a^2+c^2 > 2ac$ — ②

and $b^2+c^2 > 2bc$ — ③

Adding ①, ② and ③ gives

$$a^2+b^2+a^2+c^2+b^2+c^2 > 2ab+2ac+2bc$$

$$2(a^2+b^2+c^2) > 2(ab+ac+bc)$$

Hence $a^2+b^2+c^2 > ab+ac+bc$, as required.

(iii) Hence $a^2b^2+b^2c^2 > 2ab^2c$

and $b^2c^2+a^2c^2 > 2abc^2$

and $a^2b^2+a^2c^2 > 2a^2bc$

add together to get

$$2(a^2b^2+b^2c^2+a^2c^2) > 2(a^2bc+ab^2c+abc^2)$$

so $a^2b^2+b^2c^2+a^2c^2 > abc(a+b+c)$

Hence $\dfrac{a^2b^2+b^2c^2+a^2c^2}{a+b+c} > abc$, as required.

Q12// (b)

To prove that n^2 is divisible by 5 if and only if n is divisible by 5.

We have to show

(i) That if n is divisible by 5, then n^2 is divisible by 5.

Suppose that n is divisible by 5. Let $n = 5a$.

Then $n^2 = (5a)^2 = 25a^2 = 5(5a^2)$, which is div. by 5.

(ii) That if n^2 is div by 5, then n is div. by 5.

We can prove (ii) by contrapositive.

I will show that if n is not div. by 5, then n^2 is not div. by 5.

Suppose that n is not div. by 5.

Then n leaves a remainder of $\{1, 2, 3, 4\}$ upon division by 5.

In that case, n^2 will leave a remainder of $\{1, 4, 4, 1\}$.

Hence n^2 is not divisible by 5.

So, if n is not divisible by 5, then n^2 is not div. by 5.

So, if n^2 is div. by 5, then n is div. by 5.

(c) (i) To show that

$$(1+i\tan\theta)^n - (1-i\tan\theta)^n = \frac{2\sin n\theta}{\cos^n\theta}\, i$$

where n is an integer and $\cos\theta \neq 0$.

$$LHS = \left(1 + \frac{i\sin\theta}{\cos\theta}\right)^n - \left(1 - \frac{i\sin\theta}{\cos\theta}\right)^n$$

$$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n - \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n = \frac{(\cos\theta + i\sin\theta)^n}{\cos^n\theta} - \frac{(\cos(-\theta) + i\sin(-\theta))^n}{\cos^n\theta}$$

$$= \frac{\cos n\theta + i\sin n\theta - (\cos(n\theta) + i\sin(-n\theta))}{\cos^n\theta}$$

$$= \frac{\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta}{\cos^n\theta}$$

$$= \frac{2i\sin n\theta}{\cos^n\theta} = \frac{2\sin n\theta}{\cos^n\theta}\, i = RHS$$

(ii) Now part (i) says that

$$(1+i\tan\theta)^n - (1-i\tan\theta)^n = \frac{2\sin n\theta}{\cos^n\theta}\, i$$

Let $z = i\tan\theta$, then

$$(1+z)^n - (1-z)^n = \frac{2\sin n\theta}{\cos^n\theta}\, i$$

And if $n = 4$, we have

$$(1+z)^4 - (1-z)^4 = \frac{2\sin 4\theta}{\cos^4\theta}\, i$$

Hence the solutions of

$$(1+z)^4 - (1-z^4) = 0$$

are the solutions of $\dfrac{2\sin 4\theta}{\cos^4\theta} = 0$

So $2\sin 4\theta = 0$

$\sin 4\theta = 0$

$4\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

$i\tan\theta = 0, i, 0, -i, 0$

$\therefore z = 0, \pm i$

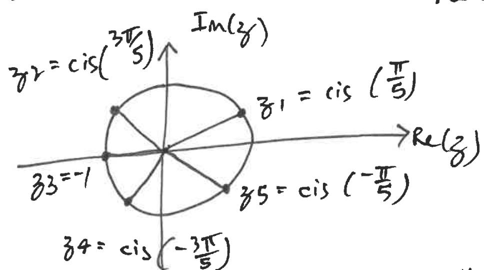
(iii) Now $(1+z)^4 - (1-z)^4 = ((1+z)^2 + (1-z)^2)((1+z)^2 - (1-z)^2)$

$= 2(1+z^2)(4z)$

$= 8z(1+z^2)$ which is of degree 3, hence only 3 solutions.

Q13 (a)

(i)



The 5 roots of $z^5 + 1 = 0$ are equally spaced around the unit circle, separated by $\frac{2\pi}{5}$ radians.

The roots are $z_1 = \text{cis}(\frac{\pi}{5})$

$z_2 = \text{cis}(\frac{3\pi}{5})$

$z_3 = \text{cis}(\pi) = -1$

$z_4 = \text{cis}(\frac{-3\pi}{5})$

$z_5 = \text{cis}(\frac{-\pi}{5})$

(ii) $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$

$= (z+1)(z - \text{cis}\frac{\pi}{5})(z - \text{cis}(\frac{-\pi}{5}))(z - \text{cis}(\frac{3\pi}{5}))(z - \text{cis}(\frac{-3\pi}{5}))$

$= (z+1)(z^2 - 2\cos\frac{\pi}{5}z + 1)(z^2 - 2\cos\frac{3\pi}{5}z + 1)$

(iii) Now the sum of the roots is $\frac{-b}{a} = 0$.

Hence $\text{cis}(\frac{\pi}{5}) + \text{cis}(\frac{3\pi}{5}) + (-1) + \text{cis}(\frac{-3\pi}{5}) + \text{cis}(\frac{-\pi}{5}) = 0$

$\cos(\frac{\pi}{5}) + i\sin(\frac{\pi}{5}) + \cos(\frac{-\pi}{5}) + i\sin(\frac{-\pi}{5}) +$

$\cos(\frac{3\pi}{5}) + \cancel{i\sin(\frac{3\pi}{5})} + \cos(\frac{-3\pi}{5}) + \cancel{i\sin(\frac{-3\pi}{5})} = 1$

So $2\cos(\frac{\pi}{5}) + 2\cos(\frac{3\pi}{5}) = 1$

Hence $\cos(\frac{\pi}{5}) + \cos(\frac{3\pi}{5}) = \frac{1}{2}$, as required.

By equating the coefficients of z^2 we have

$0 = -2\cos\frac{\pi}{5} - 2\cos\frac{3\pi}{5} + 1 + 1 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$

$0 = -2(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}) + 2 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$

$0 = -2 \times \frac{1}{2} + 2 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$

$-1 = 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$

Hence $\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = \frac{-1}{4}$, as required.

OR For $z^4 - z^3 + z^2 - z + 1 = (z^2 - 2\cos\frac{\pi}{5}z + 1)(z^2 - 2\cos\frac{3\pi}{5}z + 1)$

equating coefficients of z^2: $1 = 1 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} + 1$

$-1 = 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$

$\vdots$

(iv) A quadratic equation that has roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ is

$$\left(x-\cos\frac{\pi}{5}\right)\left(x-\cos\frac{3\pi}{5}\right)=0$$

$$x^2-x\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}\right)+\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=0$$

$$x^2-\frac{1}{2}x-\frac{1}{4}=0\text{ , using the results from (iii)}$$

$$x=\frac{\frac{1}{2}\pm\sqrt{\frac{1}{4}-4\left(\frac{-1}{4}\right)}}{2}$$

$$x=\frac{\frac{1}{2}\pm\frac{\sqrt{5}}{2}}{2}$$

$$x=\frac{1\pm\sqrt{5}}{4}$$

Hence $\cos\left(\frac{3\pi}{5}\right)=\frac{1-\sqrt{5}}{4}$ because the angle is in the 2nd quadrant making $\cos\theta$ negative

and $\cos\left(\frac{\pi}{5}\right)=\frac{1+\sqrt{5}}{4}$ because the angle is in the 1st quadrant making $\cos\theta$ positive.

(b)

Prove by induction for integers $n\geq 2$ that:

$$1^3+2^3+\dots+(n-1)^3<\frac{n^4}{4}<1^3+2^3+\dots+n^3$$

when $n=2$,

$$(2-1)^3=1^3=1$$

$$\frac{2^4}{4}=\frac{16}{4}=4$$

$$1^3+2^3=8+1=9$$

and $1<4<9$.

So the statement is true when $n=2$.

Assume that the statement is true when $n=k$, ie, assume that

$$1^3+2^3+\dots+(k-1)^3<\frac{k^4}{4}<1^3+2^3+\dots+k^3$$

Then try to prove that

$$1^3+2^3+\dots+(k-1)^3+(k)^3<\frac{(k+1)^4}{4}<1^3+2^3+\dots+k^3+(k+1)^3$$

Consider $\underbrace{1^3+2^3+\dots+k^3}+(k+1)^3-\frac{(k+1)^4}{4}$

$$>\frac{k^4}{4}+(k+1)^3-\frac{(k+1)^4}{4}\text{ , by assumption}$$

$$=\frac{k^4}{4}+(k+1)^3-\left(\frac{k^4+4k^3+6k^2+4k+1}{4}\right)$$

$$=(k+1)^3-\left(k^3+\frac{3}{2}k^2+k+\frac{1}{4}\right)$$

$$=k^3+3k^2+3k+1-k^3-\frac{3}{2}k^2-k-\frac{1}{4}$$

$$=\frac{3}{2}k^2+2k+\frac{3}{4}$$

$$>0$$

Hence $\frac{(k+1)^4}{4} < 1^3 + 2^3 + \dots + k^3 + (k+1)^3$.

Now consider the sign of

$$\frac{(k+1)^4}{4} - \left[1^3 + 2^3 + \dots + (k-1)^3 + k^3\right]$$

$$> \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{4} - \left[\frac{k^4}{4} + k^3\right] \text{ by the assumption}$$

$$= k^3 + \frac{3}{2}k^2 + k + \frac{1}{4} - k^3$$

$$= \frac{3}{2}k^2 + k + \frac{1}{4}, \quad \text{which is positive.}$$

Hence $1^3 + 2^3 + \dots + (k-1)^3 + k^3 < \frac{(k+1)^4}{4}$

and $\frac{(k+1)^4}{4} < 1^3 + 2^3 + \dots + k^3 + (k+1)^3$.

Hence $1^3 + 2^3 + \dots + k^3 < \frac{(k+1)^4}{4} < 1^3 + 2^3 + \dots + (k+1)^3$,

as required.

Q14/ (a)

S : "If a is rational and b is irrational, then a + b is irrational."

Proof by Contradiction:

Assume that a is rational and
b is irrational and that
a + b is rational.

Since a and a + b are rational, we can let

$a = \frac{p}{q}$ and $a + b = \frac{R}{s}$ where

p and q are integers with no common factors and
R and s are integers with no common factors.

Now $a + b = \frac{R}{s}$

so $b = \frac{R}{s} - \frac{p}{q}$

$$b = \frac{Rq - ps}{sq}$$

Contradiction!

(This is a contradiction because $Rq - ps$ is an integer and sq is an integer, hence b is rational. But we assumed that b is irrational.)

Hence our assumption was false.

Hence if a is rational and b is irrational, then a + b is irrational too.

(b) $f(x) = x - \ln(1+x+\frac{1}{2}x^2)$

$$f'(x) = 1 - \frac{1}{1+x+\frac{1}{2}x^2} \times (1+x)$$

$$f'(x) = \frac{1+x+\frac{1}{2}x^2 - 1 - x}{1+x+\frac{1}{2}x^2}$$

$$f'(x) = \frac{\frac{1}{2}x^2}{1+x+\frac{1}{2}x^2} = \frac{x^2}{2+2x+x^2} = \frac{x^2}{(1+x)^2+1}$$

The numerator is positive if $x \neq 0$.

The denominator is always positive.

Hence $f'(x) > 0$ for $x < 0$.

Hence $y = f(x)$ is an increasing function for $x < 0$.

Now to show that $e^x < 1+x+\frac{1}{2}x^2$ for $x<0$.

Notice that $f(0) = 0 - \ln(1+0+0) = 0$

and $f(x)$ is increasing for $x<0$, so

$f(x) < 0$ for $x < 0$.

So $x - \ln(1+x+\frac{1}{2}x^2) < 0$ for $x<0$

$x < \ln(1+x+\frac{1}{2}x^2)$

$\therefore$ $e^x < 1+x+\frac{1}{2}x^2$ for $x<0$

as required.

(c) (i)

$z = \cos\theta + i\sin\theta$

$z^n = \cos(n\theta) + i\sin(n\theta)$ — ①

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$z^{-n} = \cos n\theta - i\sin n\theta$ — ②

① + ② gives $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

(ii) $f(\theta) = 1 + k\cos\theta + k^2\cos 2\theta + k^3\cos 3\theta + \dots + k^n\cos n\theta + \dots$

$2f(\theta) = 2 + k(2\cos\theta) + k^2(2\cos 2\theta) + k^3(2\cos 3\theta) + \dots$

$= 2 + k(z+z^{-1}) + k^2(z^2+z^{-2}) + k^3(z^3+z^{-3}) + \dots$

$= [1 + kz + k^2z^2 + k^3z^3 + \dots] + [1 + kz^{-1} + k^2z^{-2} + k^3z^{-3} + \dots]$

$$= \frac{1}{1-kz} + \frac{1}{1-kz^{-1}}$$

$$= \frac{1}{1-kz} + \frac{z}{z-k}$$

$$= \frac{1}{(1-k\cos\theta) - ki\sin\theta} + \frac{\cos\theta + i\sin\theta}{(\cos\theta - k) + i\sin\theta}$$

$$= \frac{1-k\cos\theta + ki\sin\theta}{(1-k\cos\theta)^2 + k^2\sin^2\theta} + \frac{(\cos\theta + i\sin\theta)(\cos\theta - k - i\sin\theta)}{(\cos\theta - k)^2 + \sin^2\theta}$$

$$= \frac{1-k\cos\theta + ki\sin\theta}{1-2k\cos\theta + k^2} + \frac{1-k\cos\theta - ki\sin\theta}{1-2k^2\cos\theta + k^2}$$

$$2f(\theta) = \frac{2 - 2k\cos\theta}{1 - 2k\cos\theta + k^2}$$

$$\therefore\ f(\theta) = \frac{1 - k\cos\theta}{1 - 2k\cos\theta + k^2}, \quad \text{as required.}$$

(iii) To verify this result, let $\theta = \frac{\pi}{2}$ then from

$$f(\theta) = 1 + k\cos\theta + k^2\cos 2\theta + k^3\cos 3\theta + \dots$$

$$f(\tfrac{\pi}{2}) = 1 + 0 - k^2 + 0 + k^4 + 0 - k^6 + 0 + k^8 + \dots$$

$$= \frac{1}{1-(-k^2)}$$

$$= \frac{1}{1+k^2}$$

And when $\theta = \frac{\pi}{2}$,

$$f(\theta) = \frac{1 - k\cos\theta}{1 - 2k\cos\theta + k^2}$$

$$f(\tfrac{\pi}{2}) = \frac{1 - k(0)}{1 - 2k(0) + k^2}$$

$$= \frac{1}{1+k^2}$$

Since these results are equal, the result in (ii) has been verified.

END OF SOLUTIONS