Barker College

Student Number

2023 YEAR 12
ASSESSMENT TASK 3

Mathematics Extension 2

Staff Involved:

DZP* Mr Peattie
BHC Mr Carruthers

8:30AM
WEDNESDAY 31 MAY

33 copies

TOPICS COVERED
- Vectors / 25
- Integration / 25

General Instructions:
- Working time – 55 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- Write all answers in the spaces provided
- Show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale

Total marks: 50

Question 1 **Marks**

(a) Find the vector parallel to $\underset{\sim}{v} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ which has a magnitude of 6. **2**

(b) (i) Find, in exact form, the angle between the position vectors: **2**

$$\overrightarrow{OA} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k} \text{ and}$$
$$\overrightarrow{OB} = -\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}.$$

(ii) Hence find the exact area of triangle OAB. **1**

(c) Consider the points $A\ (5,\ 6,\ 7)$ and $B\ (2,\ -1,\ 0)$.

(i) Find a vector equation of the line passing through A and B **2**

(ii) Prove that the point $C\ (11,\ 20,\ 21)$ lies on the line passing through A and B. **1**

(iii) Using (i), find a vector equation for the **interval** from A to B. **1**

Question 2

(a) The points $A\ (9,\ 2,\ 5)$, $B\ (8,\ 3,\ W)$ and $C\ (10,\ 10,\ 9)$ form a right angled triangle, right angled at B.
Find the possible values of W. **3**

(b) (i) What is the angle between the two vectors

$$\underset{\sim}{v} = -\underset{\sim}{i} + \sqrt{2}\,\underset{\sim}{j} + \underset{\sim}{k} \text{ and}$$

$$\underset{\sim}{u} = \underset{\sim}{i} - \sqrt{2}\,\underset{\sim}{j} - \underset{\sim}{k}\ ?$$ **1**

(ii) What is the geometrical significance of the result in (i)? **1**

(c) Given $\underset{\sim}{u} = 3\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{v} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

$\underset{\sim}{w} = -\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$

(i) Show that $\underset{\sim}{u} \circ \underset{\sim}{v} = -\underset{\sim}{u} \circ \underset{\sim}{w}$ **1**

(ii) Rearrange $\underset{\sim}{u} \circ \underset{\sim}{v} = -\underset{\sim}{u} \circ \underset{\sim}{w}$ so that the expression equals zero. Therefore which two vectors are perpendicular? **2**

(d) A parallelogram has all sides equal in length.
Show, using vectors, that its diagonals are perpendicular. **3**

Question 3

The equations $\frac{x}{2} = y - 2 = z - 2$ determine a straight line.

(i) Write the equation of the line in vector form. **1**

(ii) This line intersects the sphere $x^2 + y^2 + z^2 + 2x + 2z = 4$ in two points.

Find the coordinates of these points. **4**

Student Number

Question 4

(a) Find $\int \frac{4x+3}{x^2+4x+5} dx$ **3**

(b) Use the substitution $x = 2\cos\theta$ to find $\int \frac{dx}{\sqrt{(4-x^2)^3}}$ **4**

(c) Find $\int e^{2x} \sin x \, dx$ **3**

(d) Evaluate $\int_0^1 \frac{\sqrt{x}}{1+x} dx$ using an appropriate trigonometric substitution. **3**

Question 5

Find $\int \frac{x-2}{(x^2+1)(x+1)} dx$ **3**

Question 6

Let $A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$, where n is an integer ≥ 0.

Show that $nA_n = \frac{2n-1}{2} A_{n-1}$ for $n \geq 1$. **4**

Question 7

Let $I_n = \int_0^{\frac{\pi}{4}} sec^n x \, dx$, where n is an integer ≥ 0.

(i) Show that $I_n = \frac{1}{n-1}[(\sqrt{2})^{n-2} + (n-2)I_{n-2}]$ for $n \geq 2$. **4**

(ii) Hence evaluate I_4. **1**

END OF PAPER

Question 1 **Marks**

(a) Find the vector parallel to $\underset{\sim}{v} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ which has a magnitude of 6. **2**

$|\underset{\sim}{v}| = \sqrt{4+9+25} = \sqrt{38}$

$\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{38}}\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$

The required vector is $\frac{6}{\sqrt{38}}\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$

(b) (i) Find, in exact form, the angle between the position vectors: ~~1~~ 2

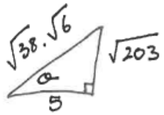
$\overrightarrow{OA} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ and

$\overrightarrow{OB} = -\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$.

$\cos\theta = \frac{-2-3+10}{\sqrt{38}\cdot\sqrt{6}} = \frac{5}{\sqrt{38}\cdot\sqrt{6}}$

$\theta = \cos^{-1}\left(\frac{5}{\sqrt{38}\cdot\sqrt{6}}\right)$

(ii) Hence find the exact area of triangle OAB. ~~2~~ 1



Area of ΔOAB

$= \frac{1}{2} \times \sqrt{38} \times \sqrt{6} \times \frac{\sqrt{203}}{\sqrt{38}\cdot\sqrt{6}}$

$= \frac{1}{2}\sqrt{203}$ units^2

(c) Consider the points $A\ (5,\ 6,\ 7)$ and $B\ (2,\ -1,\ 0)$.

(i) Find a vector equation of the line passing through A and B **2**

$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -7 \\ -7 \end{pmatrix}$

$\underset{\sim}{r} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -7 \\ -7 \end{pmatrix}$ OR $\underset{\sim}{r} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 7 \end{pmatrix}$

OR $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -7 \\ -7 \end{pmatrix}$ OR $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 7 \end{pmatrix}$

(ii) Prove that the point $C\ (11,\ 20,\ 21)$ lies on the line passing through A and B. **1**

Does $\begin{pmatrix} 11 \\ 20 \\ 21 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -7 \\ -7 \end{pmatrix}$?

Yes, when $\lambda = -2$ (or equivalent).

Hence the point C lies on the line.

(iii) Using (i), find a vector equation for the **interval** from A to B. **1**

The interval from A to B is

$\underset{\sim}{r} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -7 \\ -7 \end{pmatrix}$ for $0 \le \lambda \le 1$

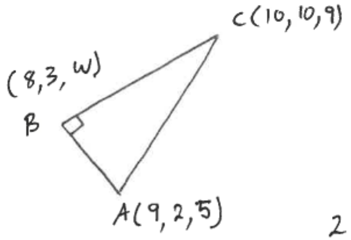
OR $\underset{\sim}{r} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 7 \end{pmatrix}$ for $-1 \le \lambda \le 0$

OR $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -7 \\ -7 \end{pmatrix}$ for $-1 \le \lambda \le 0$

OR $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 7 \end{pmatrix}$ for $0 \le \lambda \le 1$

Question 2

(a) The points $A\ (9,\ 2,\ 5)$, $B\ (8,\ 3,\ W)$ and $C\ (10,\ 10,\ 9)$ form a right angled triangle, right angled at B.
Find the possible values of W. **3**



Now $\overrightarrow{BA} \circ \overrightarrow{BC} = 0$ so

$\begin{pmatrix} 1 \\ -1 \\ 5-W \end{pmatrix} \circ \begin{pmatrix} 2 \\ 7 \\ 9-W \end{pmatrix} = 0$

$2 - 7 + (5-W)(9-W) = 0$

$-5 + 45 - 5W - 9W + W^2 = 0$

$W^2 - 14W + 40 = 0$

$(W-10)(W-4) = 0$

$W = 4$ or 10.

(b) What is the angle between the two vectors
$\underset{\sim}{v} = -\underset{\sim}{i} + \sqrt{2}\,\underset{\sim}{j} + \underset{\sim}{k}$ and
$\underset{\sim}{u} = \underset{\sim}{i} - \sqrt{2}\,\underset{\sim}{j} - \underset{\sim}{k}$?
And what is the geometrical significance of this result? **2**

$\cos\theta = \dfrac{\begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ -\sqrt{2} \\ -1 \end{pmatrix}}{2 \times 2}$

$= \dfrac{-1-2-1}{4}$

$= -1$ $\quad \therefore \theta = 180°$

This means that the vectors point in opposite directions.

(c) Given $\underset{\sim}{u} = 3\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{v} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

$\underset{\sim}{w} = -\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$

(i) Show that $\underset{\sim}{u} \circ \underset{\sim}{v} = -\underset{\sim}{u} \circ \underset{\sim}{w}$ **1**

$$\left.\begin{array}{l} u \circ v = 9+6-3 = 12 \\ -u \circ w = 3+3+6 = 12 \end{array}\right\} \text{Hence } u \circ v = -u \circ w$$

(ii) Rearrange $\underset{\sim}{u} \circ \underset{\sim}{v} = -\underset{\sim}{u} \circ \underset{\sim}{w}$ so that the expression equals zero. Therefore which two vectors are perpendicular? **2**

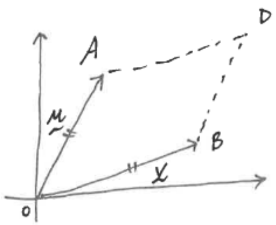
$$u \circ v = -u \circ w$$

so $u \circ v + u \circ w = 0$

$$u \circ (v + w) = 0$$

u is perpendicular to $(v + w)$

(d) A parallelogram has all sides equal in length.
Show, using vectors, that its diagonals are perpendicular. **3**



Let $\overrightarrow{OA} = u$ and $\overrightarrow{OB} = v$.

Then $\overrightarrow{OD} = u + v$ and

$\overrightarrow{BA} = u - v$.

Consider $\overrightarrow{OD} \circ \overrightarrow{BA}$

$$= (u+v) \circ (u - v)$$

$$= u \circ u + u \circ v - u \circ v - v \circ v$$

$$= u \circ u - v \circ v$$

$$= |u|^2 - |v|^2$$

$= 0$ since $OA = OB$ (data)

Hence the diagonals are perpendicular.

Question 3

The equations $\frac{x}{2} = y - 2 = z - 2$ determine a straight line.

(i) Write the equation of the line in vertor form. **1**

$\frac{x}{2} = \lambda$ $\quad y - 2 = \lambda$ $\quad z - 2 = \lambda$

$x = 0 + 2\lambda$ $\quad y = 2 + \lambda$ $\quad z = 2 + \lambda$

$$r = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

(ii) This line intersects the sphere $x^2 + y^2 + z^2 + 2x + 2z = 4$ in two points.

Find the coordinates of these points. **4**

The Cartesian equation of the sphere is

$(x+1)^2 + y^2 + (z+1)^2 = 6$,

with radius $\sqrt{6}$ units and centre at $(-1, 0, -1)$.

Hence the vector equation of the sphere is

$$\left| r - \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \right| = \sqrt{6}.$$

We need to solve

$$\left| \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \right| = \sqrt{6} \text{ ie, } \left\| \begin{pmatrix} 1+2\lambda \\ 2+\lambda \\ 3+\lambda \end{pmatrix} \right\| = \sqrt{6}$$

$$\sqrt{1 + 4\lambda + 4\lambda^2 + 4 + 4\lambda + \lambda^2 + 9 + 6\lambda + \lambda^2} = \sqrt{6}$$

$$6\lambda^2 + 14\lambda + 14 = 6$$

$$3\lambda^2 + 7\lambda + 4 = 0$$

$$(3\lambda + 4)(\lambda + 1) = 0$$

$$\lambda = -1 \text{ or } -\tfrac{4}{3}$$

Hence the points of intersection are $\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - \frac{4}{3}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -8/3 \\ 2/3 \\ 2/3 \end{pmatrix}$

Question 4 **Student Number**

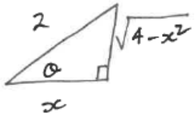
(a) Find $\displaystyle\int \frac{4x+3}{x^2+4x+5}\,dx$ **3**

$$= \int \frac{2(2x+4)-5}{x^2+4x+5}\,dx$$

$$= 2\int \frac{2x+4}{x^2+4x+5}\,dx - \int \frac{5}{1+(x+2)^2}\,dx$$

$$= 2\ln\left|x^2+4x+5\right| - 5\tan^{-1}(x+2) + C$$

$\cos\theta = \frac{x}{2}$



(b) Use the substitution $x = 2\cos\theta$ to find $\displaystyle\int \frac{dx}{\sqrt{(4-x^2)^3}}$ **4**

Let $x = 2\cos\theta$

$$\frac{dx}{d\theta} = -2\sin\theta$$

$$I = \int \frac{-2\sin\theta\,d\theta}{\sqrt{(4-4\cos^2\theta)^3}}$$

$$= -\frac{2}{8}\int \frac{\sin\theta\,d\theta}{\sqrt{\sin^6\theta}}$$

$$= -\frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$

$$= -\frac{1}{4}\int \operatorname{cosec}^2\theta\,d\theta$$

$$= -\frac{1}{4}\left[-\cot\theta\right]$$

$$= \frac{1}{4}\cot\theta$$

$$= \frac{1}{4}\times\frac{x}{\sqrt{4-x^2}}$$

$$= \frac{x}{4\sqrt{4-x^2}} + C$$

(c) Find $\int e^{2x}\sin x\,dx$ **3**

$$= \frac{1}{2}e^{2x}\sin x - \int \frac{1}{2}e^{2x}\cos x\,dx$$

$$= \frac{1}{2}e^{2x}\sin x - \left[\frac{1}{4}e^{2x}\cos x - \int \frac{1}{4}e^{2x}(-\sin x)\,dx\right]$$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\,dx$$

$$\therefore\ \frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\therefore \int e^{2x}\sin x\,dx = \frac{4}{5}\left(\frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x\right)$$

$$= \frac{e^{2x}}{5}\left(2\sin x - \cos x\right) + C$$

(d) Evaluate $\displaystyle\int_0^1 \frac{\sqrt{x}}{1+x}\,dx$ using an appropriate trigonometric substitution. **3**

Let $x = \tan^2\theta$ then $\frac{dx}{d\theta} = 2\tan\theta\sec^2\theta$

when $x = 1$, $\theta = \frac{\pi}{4}$ and when $x = 0$, $\theta = 0$. Hence

$$I = \int_0^{\frac{\pi}{4}} \frac{\sqrt{\tan^2\theta}\times 2\tan\theta\sec^2\theta\,d\theta}{1+\tan^2\theta}$$

$$= \int_0^{\frac{\pi}{4}} 2\tan^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\,d\theta$$

$$= 2\left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[1 - \frac{\pi}{4} - 0\right] = 2 - \frac{\pi}{2}$$

Question 5

Find $\int \frac{x-2}{(x^2+1)(x+1)}\,dx$ 3

Let $\dfrac{x-2}{(x^2+1)(x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

then $x-2 = A(x^2+1) + (Bx+C)(x+1)$

when $x=-1$: $-3 = 2A$ $\quad A = \frac{-3}{2}$

when $x=0$: $-2 = A + C$

$-2 = \frac{-3}{2} + C \quad C = \frac{-1}{2}$

when $x=1$: $-1 = 2A + (B+C)\times 2$

$-1 = -3 + 2B - 1$ $\quad B = \frac{3}{2}$

$3 = 2B$

$$I = \int \frac{-3/2}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}\,dx$$

$$= -\frac{3}{2}\ln|x+1| + \frac{3}{4}\int\frac{2x}{x^2+1}\,dx - \frac{1}{2}\int\frac{dx}{1+x^2}$$

$$= -\frac{3}{2}\ln|x+1| + \frac{3}{4}\ln|1+x^2| - \frac{1}{2}\tan^{-1}(x) + C$$

Question 6

Let $A_n = \int_0^{\frac{\pi}{2}} \cos^{2n}x\,dx$, where n is a positive integer.

Show that $nA_n = \frac{2n-1}{2}A_{n-1}$, for $n \geq 1$. 4

$$A_n = \int_0^{\pi/2} \cos x \cdot \cos^{2n-1}x\,dx$$

$$= \left[\sin x \cdot \cos^{2n-1}x\right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \sin x \cdot (2n-1)\cos^{2n-2}x \cdot (-\sin x)\,dx$$

$$= 0 + (2n-1)\int_0^{\pi/2} \sin^2 x \cos^{2n-2}x\,dx$$

$$= (2n-1)\int_0^{\pi/2} (1-\cos^2 x)\cos^{2n-2}x\,dx$$

$$= (2n-1)\int_0^{\pi/2} (\cos^{2n-2}x\,dx - \cos^{2n}x)\,dx$$

Must show this!

$$= (2n-1)\int_0^{\pi/2} \cos^{2(n-1)}x\,dx - (2n-1)\int_0^{\frac{\pi}{2}} \cos^{2n}x\,dx$$

$$= (2n-1)A_{n-1} - (2n-1)A_n$$

$$A_n + (2n-1)A_n = (2n-1)A_{n-1}$$

$$2n\,A_n = (2n-1)A_{n-1}$$

$$n A_n = \frac{2n-1}{2}A_{n-1}$$

Question 7

Let $I_n = \int_0^{\frac{\pi}{4}} sec^n x\, dx$, where n is a positive integer.

(i) Show that $I_n = \frac{1}{n-1}[(\sqrt{2})^{n-2} + (n-2)I_{n-2}]$ for $n > 1$. **4**

$$I_n = \int_0^{\frac{\pi}{4}} \sec^2 x . \sec^{n-2} x \, dx$$

$$= \left[\tan x . \sec^{n-2} x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x . (n-2) \sec^{n-3} x . (\sec x \tan x) \, dx$$

$$= (\sqrt{2})^{n-2} - 0 - \int_0^{\frac{\pi}{4}} (n-2) \tan^2 x \sec^{n-2} x \, dx$$

$$= (\sqrt{2})^{n-2} - (n-2) \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) . \sec^{n-2} x \, dx$$

$$= (\sqrt{2})^{n-2} - (n-2) \int (\sec^n x - \sec^{n-2} x) \, dx$$

$$= (\sqrt{2})^{n-2} - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$$

$$(n-1) I_n = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$$

$$\therefore I_n = \frac{1}{n-1}\left[(\sqrt{2})^{n-2} + (n-2) I_{n-2}\right], \text{ as required.}$$

(ii) Hence evaluate I_4. **1**

$$I_4 = \frac{1}{3}\left[(\sqrt{2})^2 + 2 I_2\right]$$

$$I_2 = 1\left[(\sqrt{2})^0 + 0 \times I_0\right] = 1$$

$$\therefore I_4 = \frac{1}{3}(2 + 2 \times 1) = \frac{4}{3}$$

END OF PAPER