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Student Number

Barker College

2023 YEAR 12 ASSESSMENT BLOCK
SEMESTER 1

# Mathematics Extension 2

## Section I – Multiple Choice

Write your student number at the top of this page.

Remove this sheet.

Select the alternative A, B C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = A 2 B 6 C 8 D 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

*correct*

A ● B ~~●~~ C ○ D ○

Start Here →

1. A○ B○ C○ D○
2. A○ B○ C○ D○
3. A○ B○ C○ D○
4. A○ B○ C○ D○
5. A○ B○ C○ D○
6. A○ B○ C○ D○
7. A○ B○ C○ D○
8. A○ B○ C○ D○

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Barker College

Student Number

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2023 YEAR 12 ASSESSMENT BLOCK

# Mathematics Extension 2

**Staff Involved:**

**BHC (Mr Carruthers)**
**DZP (Mr Peattie)**
**ARM* (Mr Mallam)**

**12:00 NOON**
**FRIDAY 31 MARCH**

**35 copies**

| TOPICS COVERED: |
|---|
| Complex Numbers I & II |
| Proof |

**General Instructions**

- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale
- Write your student number at the top of this page

**Total marks: 80**

**Section I – 8 marks** (pages 4-6)
- Attempt Questions 1-8
- Allow about 15 minutes for this section

**Section II – 72 marks** (pages 7-12)
- Attempt Questions 9-14
- Allow about 1 hour and 45 minutes for this section

## Section I

**8 marks**
**Attempt Questions 1–8**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–8.

**1** What is the negation of the statement: “If it is raining, then I carry an umbrella”?

A. If it is not raining, then I do not carry an umbrella.

B. If I am carrying an umbrella, then it is raining.

C. It is not raining and I am carrying an umbrella.

D. It is raining and I am not carrying an umbrella.

**2** What is the value of $(4 - 3i)^2$ ?

A. $7 - 24i$

B. $7 + 24i$

C. $25 - 24i$

D. $25 + 24i$

**3** If $n$ is a positive integer, for how many values of $n$ is $9n^2 - 4$ a prime number?

A. 0

B. 1

C. 2

D. More than 2

**4** Which of the following best describes the locus of points in the Argand plane given by

$$|z-2| = 2\,|z+1| \ ?$$

A. A vertical line through (0, 0).

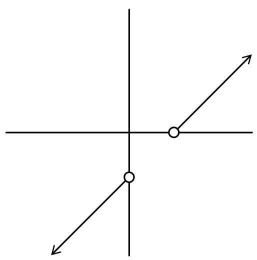
B. A vertical line through (1, 0).

C. A circle with centre (0, 0).

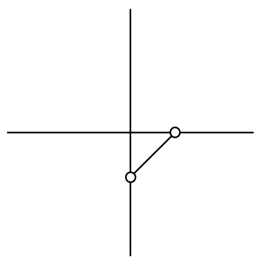
D. A circle with centre (–2, 0).

**5** Which diagram best represents the solutions to the equation $\operatorname{Arg}(z-2) = \operatorname{Arg}(z+2i)$?

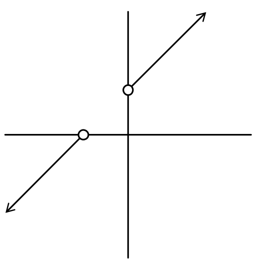
A.



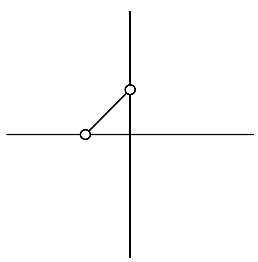
B.



C.



D.





**6** For complex $z$, if $\operatorname{Im}(z) > 0$, what is the value of $\operatorname{Arg}\left(\dfrac{z\bar{z}}{z-\bar{z}}\right)$?

A. $-\dfrac{\pi}{2}$

B. $0$

C. $\dfrac{\pi}{2}$

D. $\pi$

**7** For integers $x$ and $y$, which one of the following statements is equivalent to the statement: "There is no integer which is the largest integer"?

A. $\exists y\ (\forall x,\ y > x)$

B. $\forall x\ (\exists y,\ y > x)$

C. $\exists x\ (\forall y,\ y > x)$

D. $\forall y\ (\exists x,\ y > x)$

**8** Let $(x+iy)^{18} = a + ib$ for $a, b, x, y \in \mathbb{R}$. Which expression is equivalent to $(y - ix)^{18}$ ?

A. $-a - ib$

B. $b - ia$

C. $-b + ia$

D. $-a + ib$

**End of Section I**

## Section II

**72 marks**
**Attempt Questions 9–14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 9** (12 marks) Use a SEPARATE writing booklet.

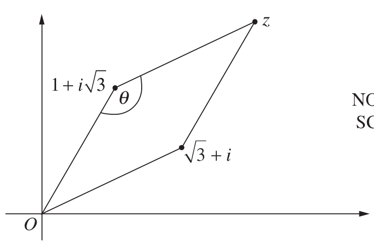
(a) Consider the complex numbers $w = 2 - 3i$ and $z = 3 + 4i$.

(i) Find $\overline{w} + z$. **1**

(ii) Find $|w|$. **1**

(iii) Express $\dfrac{w}{z}$ in the simplest form of $x + iy$ where $x$ and $y$ are real. **2**

(b) On the Argand diagram below, the complex numbers 0, $1 + i\sqrt{3}$, $\sqrt{3} + i$ and $z$ form a rhombus.




NOT TO SCALE

(i) Find $z$ in the form $x + iy$. **1**

(ii) An interior angle of the rhombus, $\theta$, is marked on the diagram. Find the value of $\theta$. **2**

(c) Use proof by contradiction to show that $\log_2 7$ is irrational. **2**

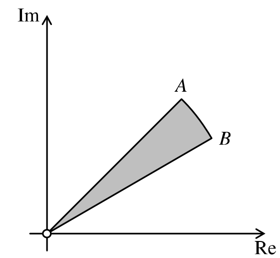
(d) Prove that the sum of two consecutive positive even powers of 3 is always a multiple of 90. **3**



**Question 10** (12 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows the region on the complex plane given by:

$$|z| \le 4\sqrt{2} \text{ and } \frac{\pi}{6} \le \operatorname{Arg}(z) \le \frac{\pi}{4}.$$




(i) Find the complex number represented by point $A$ in modulus-argument form. **1**

(ii) Find the complex number represented by point $B$ in the form $x + iy$. **1**

(iii) Determine whether $5 + 2i$ lies within the region. **2**

(b) (i) Given that $\cos\theta + i\sin\theta = e^{i\theta}$, prove de Moivre's theorem: **1**

for positive integers $n$, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

(ii) Using de Moivre's theorem, or otherwise, show that, for positive integers $n$, **2**

$$(1+i)^n + (1-i)^n = 2\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}.$$

(c) Two of the zeros of the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ are $a + ib$ and $a + 2ib$, where $a$ and $b$ are real and $b > 0$.

(i) Find the values of $a$ and $b$. **3**

(ii) Hence, or otherwise, express $P(x)$ as a product of quadratic factors with real coefficients. **2**

**Question 11** (12 marks) Use a SEPARATE writing booklet.

(a) Consider the following statement for integers $m$ and $n$.

$$P\text{: If } \frac{mn}{2} \text{ is an integer then } m \text{ or } n \text{ must be even.}$$

(i) Write down the contrapositive of $P$. **1**

(ii) Prove $P$ by proving the contrapositive. **3**

(b) Use mathematical induction to prove the following statement for integers $n \geq 1$: **4**

$$\sum_{r=1}^{n} (r-1)^3 \leq \frac{n^4}{4}.$$

(c) On the complex plane, the line given by $|z+1| = |z-2+3i|$ cuts the circle given by $|z+1| = 3$ into two segments. Find the exact area of the major segment. **4**



**Question 12** (12 marks) Use a SEPARATE writing booklet.

(a) For any integer $n$, prove that $n^2 + n$ is even. **2**

(b) (i) Expand and simplify $(\cos\theta + i\sin\theta)^5$ using the binomial theorem. **2**

(ii) Hence, using de Moivre's theorem, show that **2**

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

(iii) Show that $x = \sin\frac{\pi}{10}$ is one of the solutions to $16x^5 - 20x^3 + 5x - 1 = 0$. **1**

(iv) Write down the polynomial $P(x)$ such that $(x-1)P(x) = 16x^5 - 20x^3 + 5x - 1$. **1**

(v) Find the value of $a$ such that $P(x) = (4x^2 + ax - 1)^2$. **1**

(vi) Hence find the exact value of $\sin\frac{\pi}{10}$. **1**

(vii) Sketch the function $f(x) = 16x^5 - 20x^3 + 5x - 1$, showing all intercepts. **2**

**Question 13** (12 marks) Use a SEPARATE writing booklet.

(a) $z = 2e^{\frac{\pi}{12}i}$ is a root of the equation $z^4 = a\left(1+\sqrt{3}i\right)$, where $a$ is real.

(i) Express $1+\sqrt{3}i$ in the form $re^{i\theta}$. **1**

(ii) Find the value of $a$. **1**

(iii) Find the other three roots of the equation in the form $re^{i\theta}$. **2**

(b) The point $P$ on an Argand diagram represents the complex number $z$, which satisfies **4**

$$\frac{1}{z}+\frac{1}{\bar{z}}=1\,.$$

Give a geometrical description of the locus of $P$ as $z$ varies.

(c) For real numbers $a, b > 0$ where $ab \geq 1$, prove that **4**

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}\geq\frac{2}{ab+1}\,.$$



**Question 14** (12 marks) Use a SEPARATE writing booklet.

(a) If $z$ is one of the fifth roots of unity other than 1, show that **3**

$$z-z^2+z^3-z^4=2i\left(\sin\frac{2\pi}{5}-\sin\frac{\pi}{5}\right).$$

(b) (i) For any integer $a \geq 2$, show that $\frac{1}{a^2}<\frac{1}{a-1}-\frac{1}{a}$. **2**

(ii) Hence, for any positive integer $n$, show that **3**

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}<2\,.$$

*Suggestion: do not use induction.*

(c) (i) By considering $f'(x)$ where $f(x)=e^x-x$, show that $e^x > x$ for $x \geq 0$. **1**

(ii) Hence, by considering areas under the curves $y=e^x$ and $y=x$, prove that $\sqrt{e}>\frac{9}{8}$. **1**

(iii) Use mathematical induction and the result in (i) to show that, for $x \geq 0$, $e^x>\frac{x^n}{n!}$ for all integers $n \geq 1$. **2**

**End of paper**

Yr 12 Ext 2
2023
Assessment 2
Solutions
31/3/2023

Q1. $\sim(p \Rightarrow q) \equiv p \wedge \sim q$

(D)

Q2. $(4-3i)^2$

$= 16 - 24i - 9$

$= 7 - 24i$ (A)

Q3. $9n^2 - 4 = (3n-2)(3n+2)$

when $n = 1$,

RHS $= 1 \times 5 = 5$ (prime).

For all other $n$,

RHS $= a \times b$ where $a, b > 1$

$\therefore$ no more primes

(B)

Q4. $|z-2| = 2|z+1|$

$(x-2)^2 + y^2 = 4(x+1)^2 + 4y^2$

$x^2 - 4x + 4 + y^2 = 4x^2 + 8x + 4 + 4y^2$

$0 = 3x^2 + 12x + 3y^2$

$0 = x^2 + 4x + y^2$

$4 = (x+2)^2 + y^2$

(D)

Q5. (A)

Q6. Let $z = x + iy$ then

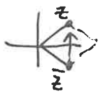
$\text{Arg}(z\bar{z}) - \text{Arg}(z - \bar{z})$

$= \text{Arg}(x^2+y^2) - \text{Arg}(2iy)$

$= 0 - \frac{\pi}{2}$

$= -\frac{\pi}{2}$ (A)

OR $\text{Arg}\, z + \text{Arg}\, \bar{z} = 0$




$\text{Arg}(z - \bar{z})$ is $\frac{\pi}{2}$ using vectors.

Q7. (B)

Q8. $(x+iy)^{18} = a + ib$

Now $(y - ix)^{18}$

$= \left[i^3\left(x + \frac{y}{i^3}\right)\right]^{18}$

$= i^{54}\left(x + \frac{y}{-i}\right)^{18}$

$= i^{54}(x + iy)^{18}$

$= (i^4)^{13} \times i^2 \times (a+ib)$

$= 1 \times -1 \times (a+ib)$

$= -a - ib$

(A)

OR $(y-ix)^{18} = \left[-i(x+iy)\right]^{18} = i^{18}(x+iy)^{18}$

$= (i^4)^4 i^2 (x+iy)^{18}$

$= (1)(-1)(x+iy)^{18}$

$= -(x+iy)^{18}$

$= -(a+ib)$

$= -a-ib$

Q9. $w = 2 - 3i$

$z = 3 + 4i$

(a) (i) $\bar{w} + z$

$= 2 + 3i + 3 + 4i$

$= 5 + 7i$

(ii) $|w| = \sqrt{4+9}$

$= \sqrt{13}$

(iii) $\frac{w}{z}$

$= \frac{2-3i}{3+4i} \times \frac{3-4i}{3-4i}$

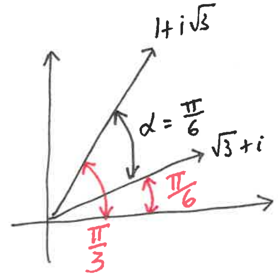
$= \frac{6-8i-9i-12}{9+16}$

$= \frac{-6}{25} - \frac{17i}{25}$

(b) (i) $z = (1+\sqrt{3}i) + (\sqrt{3}+i)$

$= (\sqrt{3}+1) + i(1+\sqrt{3})$

(ii)




$\theta = \pi - \frac{\pi}{6}$ (cointerior angles are supplementary if parallel lines)

$\theta = \frac{5\pi}{6}$

(c) Proof by Contradiction.

Suppose that $\log_2 7$ is rational.

Let $\log_2 7 = \frac{p}{q}$ where $p$ and $q$ are integers.

Then $7 = 2^{\frac{p}{q}}$

so $7^q = 2^p$.

Contradiction!

The LHS is odd while the RHS is even.

$\therefore \log_2 7$ is irrational.

(d) Let the consecutive even powers of 3 be $3^{2n}$ and $3^{2n+2}$, $n$ integer $\geq 1$

Then $3^{2n} + 3^{2n+2}$

$= 3^{2n}(1 + 3^2)$

$= 10 \times 3^{2n}$

$= 10 \times 9 \times 3^{2n-2}$

$= 90 \times 3^{2n-2}$

which is a multiple of 90.

if $n$ integer $\geq 1$

Q10// (a)

(i) $A = 4\sqrt{2}\ \text{cis}(\frac{\pi}{4})$

(ii) $B = 4\sqrt{2}\ \text{cis}(\frac{\pi}{6})$

$= 4\sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$= 4\sqrt{2}(\frac{\sqrt{3}}{2} + i\frac{1}{2})$

$= 2\sqrt{6} + 2\sqrt{2}i$

(iii) Now $|5+2i| = \sqrt{29}$

and $\sqrt{29} < 4\sqrt{2}$ $(=\sqrt{32})$ so

it's close enough and a maybe!

But $\arg(5+2i) = 21°48'5''$ so $< 30°$

$(5+2i)$ does not lie in the shaded region because its argument is too small.

(b) (i) Given $\cos\theta + i\sin\theta = e^{i\theta}$

prove that $(\cos\theta + i\sin\theta)^n$

$= \cos(n\theta) + i\sin(n\theta)$

Now $(\cos\theta + i\sin\theta)^n$

$= (e^{i\theta})^n$

$= e^{in\theta} = e^{i(n\theta)}$

$= \cos(n\theta) + i\sin(n\theta)$

as required.

b)(ii) Required to show that

$(1+i)^n + (1-i)^n = 2(\sqrt{2})^n \cos\frac{n\pi}{4}$.

Now $(1+i)^n + (1-i)^n$

$= (\sqrt{2}\,e^{i\frac{\pi}{4}})^n + (\sqrt{2}\,e^{-i\frac{\pi}{4}})^n$

$= (\sqrt{2})^n(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}) +$

$(\sqrt{2})^n(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4})$

$= (\sqrt{2})^n\cos\frac{n\pi}{4} + (\sqrt{2})^n\cos\frac{n\pi}{4}$

$= 2(\sqrt{2})^n\cos\frac{n\pi}{4}$, as required.

(c) All the coefficients are real, so roots occur in conjugate pairs.

(i) The roots are $a+ib$, $a-ib$, where $a, b$ real

$a+2ib$, $a-2ib$.

$\Sigma(\alpha) = -\frac{b}{a} = 4 = 4a \quad \therefore a = 1$

$\Pi(\alpha) = (a^2+b^2)(a^2+4b^2) = \frac{e}{a} = 10$

$\therefore (1+b^2)(1+4b^2) = 10$

$1 + 5b^2 + 4b^4 = 10$. Let $b^2 = m$.

$4m^2 + 5m - 9 = 0$

$(4m+9)(m-1) = 0$

$m = -\frac{9}{4}$     $m = 1$

$\therefore b^2 = -\frac{9}{4}$     $\therefore b^2 = 1$

But $b$ real $\therefore$ No soln     $\therefore b = \pm 1$

$a = 1$ and $b = \pm 1$

(ii) $\therefore P(x) = (x-(1+i))(x-(1-i))(x-(1+2i))(x-(1-2i))$

$= (x^2-2x+2)(x^2-2x+5)$

or zeroes: $1+i, 1-i$ | $1+2i, 1-2i$

Sum $= 2$, Product $= 2$ | Sum $= 2$, Product $= 5$

Q11// The statement P is of the form

(i) ($\frac{mn}{2}$ is an integer) $\Rightarrow$ ($m$ or $n$ is even)

The contrapositive is of the form

$\sim$($m$ or $n$ is even) $\Rightarrow$ $\sim$($\frac{mn}{2}$ is an integer).

In words, the contrapositive is:

"If neither $m$ nor $n$ is even, then $\frac{mn}{2}$ is not an integer."

Or: "If $m$ and $n$ are both odd, then $\frac{mn}{2}$ is not an integer."

(ii) Let $m$ and $n$ both be odd.

Let $m = 2j+1$ and $n = 2k+1$.

Then $\frac{mn}{2} = \frac{(2j+1)(2k+1)}{2}$

$= \frac{4jk + 2j + 2k + 1}{2}$

$= 2jk + j + k + \frac{1}{2}$

Since $j$ and $k$ are integers,

$2jk + j + k$ is an integer.

Hence $2jk + j + k + \frac{1}{2}$ is not an integer.

If both $m$ and $n$ are odd, then $\frac{mn}{2}$ is not an integer.

$\therefore$ If $\frac{mn}{2}$ is an integer, then $m$ or $n$ must be even.

(b) Prove that $\sum_{r=1}^{n}(r-1)^3 \leq \frac{n^4}{4}$ for integers $n \geq 1$.

when $n=1$, $LHS = \sum_{r=1}^{1}(1-1)^3$ and $RHS = \frac{1^4}{4}$

$= 0$ $\qquad = \frac{1}{4}$

Hence $LHS \leq RHS$ when $n=1$.

The statement is true when $n=1$.

Let the statement be true when ~~$n=1$~~, ie, let

$$\sum_{r=1}^{k}(r-1)^3 \leq \frac{k^4}{4}$$

and then try to prove that

$$\sum_{r=1}^{k+1}(r-1)^3 \leq \frac{(k+1)^4}{4}.$$

Now $LHS = \sum_{r=1}^{k+1}(r-1)^3$

$$= \sum_{r=1}^{k}(r-1)^3 + [(k+1)-1]^3$$

$$\leq \frac{k^4}{4} + k^3, \text{ by our assumption}$$

$$= \frac{k^4+4k^3}{4}$$

$$\leq \frac{k^4+4k^3+6k^2+4k+1}{4}$$

$$= \frac{(k+1)^4}{4}$$

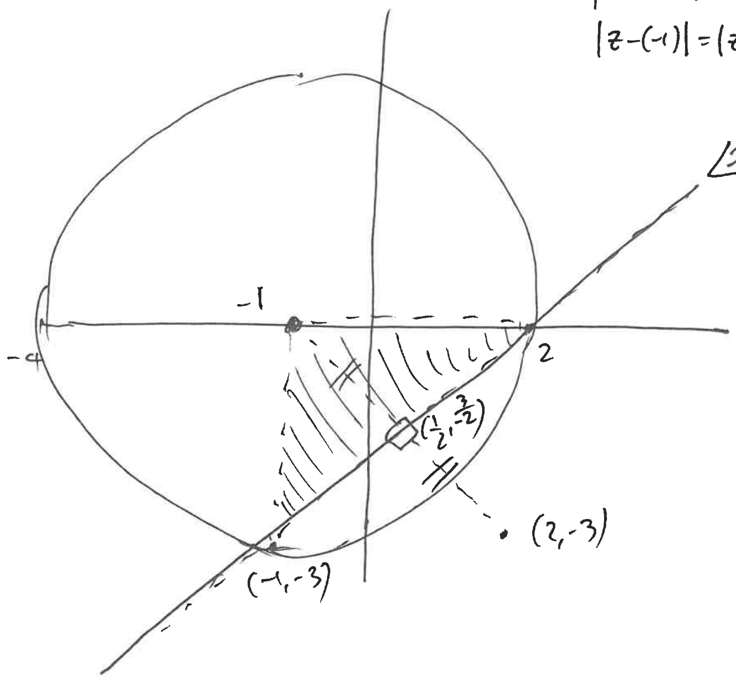
Hence $\sum_{r=1}^{k+1}(r-1)^3 \leq \frac{(k+1)^4}{4}$, as required.

Hence the statement is true by mathematical induction.

11 (c)

$|z+1| = |z-(2-3i)|$

$|z-(-1)| = |z-(2-3i)|$




So area of major segment $= \frac{3}{4}$ of circle + shaded triangle

$= \frac{3}{4} \times \pi 3^2 + \frac{1}{2} \times 3 \times 3$

$= \frac{27\pi}{4} + \frac{9}{2}\ u^2$

Q1) (a) To prove that $n^2+n$ is even for any integer $n$.

Case 1 Suppose $n$ is even.

Let $n = 2j$.

Then $n^2+n = (2j)^2+2j$

$= 4j^2+2j$

$= 2(2j^2+j)$, which is even.

or $n^2+n = n(n+1)$

If $n$ even, $n+1$ odd

If $n$ odd, $n+1$ even

Either way, even $\times$ odd $=$ even

$\therefore n^2+n$ even.

Case 2 Suppose $n$ is odd.

Let $n = 2k+1$.

Then $n^2+n = (2k+1)^2+(2k+1)$

$= 4k^2+4k+1+2k+1$

$= 4k^2+6k+2$

$= 2(2k^2+3k+1)$, which is even.

$\therefore n^2+n$ is even for any integer $n$.

(b) i) $(\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i[5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta]$

(ii) But $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

So $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

by equating Real parts

and $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

by equating imaginary parts

$\therefore \sin 5\theta = 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta).\sin^3\theta + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$, as required.

(iii) The solutions of $16x^5 - 20x^3 + 5x - 1 = 0$

lie amongst the solutions to

$\sin 5\theta = 1$. if we let $x = \sin\theta$

Solving $\sin 5\theta = 1$

$5\theta = (\frac{\pi}{2} + 2k\pi)$, $k = 0, 1, 2, \ldots$

$5\theta = (\frac{4k\pi+\pi}{2})$

$\theta = \frac{4k\pi+\pi}{10}$

when $k=0$, $\theta = \frac{\pi}{10}$.

Hence $x = \sin\frac{\pi}{10}$ is one of the solutions of

$16x^5 - 20\sin^3\theta + 5\sin\theta - 1 = 0$.

(iv)

$$\begin{array}{r}
16x^4 + 16x^3 - 4x^2 - 4x + 1 \\
(x-1)\overline{)\,16x^5 + 0x^4 - 20x^3 + 0x^2 + 5x - 1} \\
16x^5 - 16x^4 \qquad\qquad\qquad\qquad \\
16x^4 - 20x^3 \qquad\qquad\quad \\
16x^4 - 16x^3 \qquad\qquad\quad \\
-4x^3 + 0x^2 \qquad\quad \\
-4x^3 + 4x^2 \qquad\quad \\
-4x^2 + 5x \quad \\
-4x^2 + 4x \quad \\
x - 1 \\
x - 1 \\
0
\end{array}$$

$\therefore P(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1$

(v) $(4x^2 + ax - 1)(4x^2 + ax - 1)$

The coefficient of the term in $x^3$ is 16.

$\therefore 4a + 4a = 16$

$a = 2$

(vi) Clearly $\sin\frac{\pi}{10}$ is a solution to $P(x)=0$ as $\sin\frac{\pi}{10}\neq 1$.

Now $P(x) = (4x^2+2x-1)^2$

$\therefore$ Solve $4x^2+2x-1=0$

$$x = \frac{-2\pm\sqrt{4+16}}{8} = \frac{-2\pm\sqrt{20}}{8} = \frac{-2\pm 2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{4}$$

But $\sin\frac{\pi}{10}>0$, $\therefore \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$.

(vii) From (iii), other solutions than $\sin\frac{\pi}{10}$ were:

$k=1$: $\sin\frac{5\pi}{10} = \sin\frac{\pi}{2} = 1$ which is the $(x-1)$ factor

$k=2$: $\sin\frac{9\pi}{10} = \sin\frac{\pi}{10}$ × duplicate

$k=3$: $\sin\frac{13\pi}{10} = -\sin\frac{3\pi}{10} = \frac{-\sqrt{5}-1}{4}$ (from (vi))

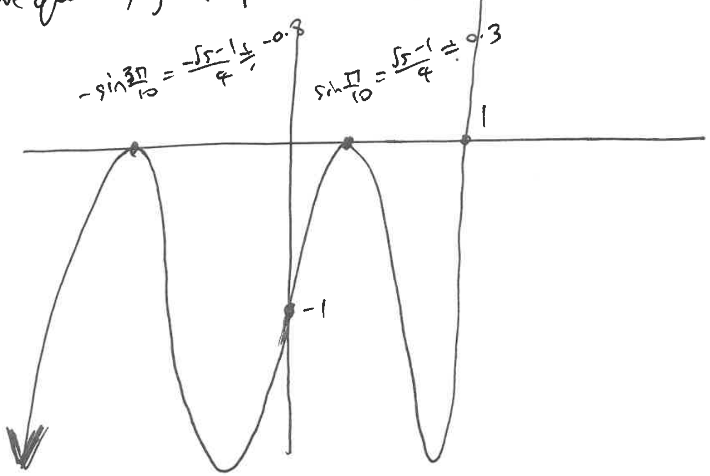
$k=4$: $\sin\frac{17\pi}{10} = -\sin\frac{3\pi}{10}$ × duplicate

$k=5$: $\sin\frac{21\pi}{10} = \sin\frac{\pi}{10}$ × duplicate and around we go again.

From (v) we have 2 double roots.

$\therefore$ roots are: $\frac{\sqrt{5}-1}{4}, \frac{\sqrt{5}-1}{4}, \frac{-\sqrt{5}-1}{4}, \frac{-\sqrt{5}-1}{4}, 1$

Positive quintic, y-intercept $-1$




Q13 (a)

(i) $1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}$

(ii) Now $z^4 = a(1+\sqrt{3}i)$

$\therefore \left(2e^{i\frac{\pi}{12}}\right)^4 = a\left(2e^{i\frac{\pi}{3}}\right)$

$16e^{i\frac{\pi}{3}} = 2a\,e^{i\frac{\pi}{3}}$

$\therefore a = 8$

(iii) To find the other 3 roots:

$z^4 = 8\left(2e^{i\left(\frac{\pi}{3}+2k\pi\right)}\right)$

$z^4 = \sqrt[4]{16}\,e^{i\left(\frac{6k\pi+\pi}{12}\right)}$

when $k=-1$,

$z_1 = 2e^{-i\frac{5\pi}{12}}$

when $k=1$,

$z_2 = 2e^{i\frac{7\pi}{12}}$

when $k=2$,

$z_3 = 2e^{i\frac{13\pi}{12}}$

ie, $z_3 = 2e^{-i\frac{11\pi}{12}}$

(b) Let $z = x+iy$ then

$$\frac{1}{x+iy} + \frac{1}{x-iy} = 1$$

$$\frac{2x}{x^2+y^2} = 1$$

$2x = x^2+y^2$

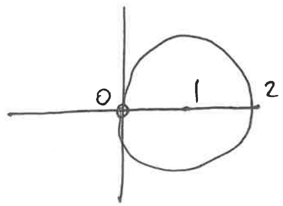
$0 = x^2-2x+y^2$

$0 = (x-1)^2+y^2-1$

$1 = (x-1)^2+y^2$

The locus of P is a circle, centre $(1,0)$ and radius $=1$.

but a hole at $z=0$ as $\frac{1}{z}$ & $\frac{1}{\bar{z}}$ are undefined when $z=0$



(c) Prove that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} \geq \frac{2}{ab+1} \quad \text{where } a, b > 0 \text{ and } ab \geq 1.$$

Consider LHS − RHS

$$= \frac{1}{a^2+1} + \frac{1}{b^2+1} - \frac{2}{ab+1}$$

$$= \frac{(a^2+b^2+2)(ab+1) - 2(a^2+1)(b^2+1)}{(a^2+1)(b^2+1)(ab+1)}$$

$$= \frac{a^3b + ab^3 + 2ab + a^2 + b^2 + 2 - 2a^2b^2 - 2a^2 - 2b^2 - 2}{(a^2+1)(b^2+1)(ab+1)}$$

$$= \frac{a^3b + ab^3 - a^2 - b^2 + 2ab - 2a^2b^2}{(a^2+1)(b^2+1)(ab+1)}$$

$$= \frac{ab(a^2+b^2) - 1(a^2+b^2) + 2ab(1-ab)}{(a^2+1)(b^2+1)(ab+1)}$$

$$= \frac{(ab-1)(a^2+b^2) - 2ab(ab-1)}{(a^2+1)(b^2+1)(ab+1)}$$

$$= \frac{(ab-1)(a^2-2ab+b^2)}{(a^2+1)(b^2+1)(ab+1)}$$

$$= \frac{(ab-1)(a-b)^2}{(a^2+1)(b^2+1)(ab+1)}$$

which equals 0 when $a = b$, but is otherwize positive since $ab \geq 1$.

Hence LHS $\geq$ RHS.

Q14 (a) $z$ is one of the fifth roots of unity other than 1. So $z$ is one of

$\text{cis}\left(\frac{2\pi}{5}\right)$, $\text{cis}\left(\frac{4\pi}{5}\right)$, $\text{cis}\left(\frac{-2\pi}{5}\right)$, $\text{cis}\left(\frac{-4\pi}{5}\right)$.

Let $z = \text{cis}\left(\frac{2\pi}{5}\right)$ then

$$z - z^2 + z^3 - z^4$$

$$= \text{cis}\left(\frac{2\pi}{5}\right) - \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(\frac{6\pi}{5}\right) - \text{cis}\left(\frac{8\pi}{5}\right)$$

$$= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) - \left[-\cos\left(\frac{\pi}{5}\right) + i\sin\frac{\pi}{5}\right] + \left(-\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right) - \left[\cos\left(\frac{-2\pi}{5}\right) + i\sin\left(\frac{-2\pi}{5}\right)\right]$$

$$= \cancel{\cos\left(\frac{2\pi}{5}\right)} + i\sin\left(\frac{2\pi}{5}\right) + \cancel{\cos\left(\frac{\pi}{5}\right)} - i\sin\left(\frac{\pi}{5}\right) - \cancel{\cos\frac{\pi}{5}} - i\sin\frac{\pi}{5} - \cancel{\cos\left(\frac{2\pi}{5}\right)} + i\sin\left(\frac{2\pi}{5}\right)$$

$$= 2i\sin\left(\frac{2\pi}{5}\right) - 2i\sin\left(\frac{\pi}{5}\right)$$

$$= 2i\left(\sin\frac{2\pi}{5} - \sin\frac{\pi}{5}\right), \quad \text{as required.}$$

(b) (i) Consider RHS − LHS

$$= \frac{1}{a-1} - \frac{1}{a} - \frac{1}{a^2}$$

$$= \frac{a^2 - a(a-1) - (a-1)}{a^2(a-1)}$$

$$= \frac{a^2 - a^2 + a - a + 1}{a^2(a-1)}$$

$$= \frac{1}{a^2(a-1)}, \text{ which is a positive expression as } a \geq 2.$$

Hence RHS > LHS.

$$\therefore \frac{1}{a^2} < \frac{1}{a-1} - \frac{1}{a}$$

(ii) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2}$

$< \frac{1}{1^2}+\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{n-2}-\frac{1}{n-1}+\frac{1}{n-1}-\frac{1}{n}$

$= 2-\frac{1}{n}$

$<2$ for positive integers $n$

Hence $\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}<2$

(c) (i) $f(x) = e^x - x$

$f'(x) = e^x - 1$

$f'(0) = 0$

$\therefore (0,1)$ is a stationary point

$f''(x) = e^x$

$f''(0) = 1 > 0$

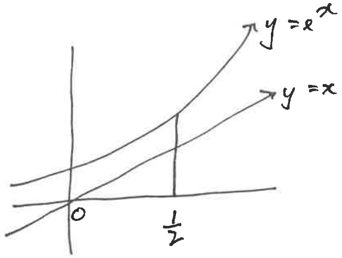
$\therefore (0,1)$ is a minimum turning point.

$\therefore f(x) = e^x - x$ is increasing for all $x>0$

$\therefore$ for $x>0$, $f(x)>1$ $\quad \therefore e^x - x > 1$ $\quad \therefore e^x > x+1$

$\therefore e^x > x$ for all $x \geq 0$, noting that $e^0 > 0$ also

(ii)




noting from (i) that $e^x > x$ for $x \geq 0$

Now $\int_0^{\frac{1}{2}} e^x\,dx > \int_0^{\frac{1}{2}} x\,dx$

$\left[e^x\right]_0^{\frac{1}{2}} > \left[\frac{x^2}{2}\right]_0^{\frac{1}{2}}$

$\sqrt{e}-1 > \frac{1}{8}$

$\therefore \sqrt{e} > 1\frac{1}{8}$

$\therefore \sqrt{e} > \frac{9}{8}$ as required.

(iii) To prove that $e^x > \frac{x^n}{n!}$ for all integers $n \geq 1$, $x \geq 0$

when $n=1$, LHS $= e^x$

RHS $= \frac{x^1}{1!} = x$

Since $e^x > x$ from part (i) then LHS > RHS.

So the statement is true when $n=1$.

Let the statement be true when $n=k$, ie, let

$e^x > \frac{x^k}{k!}$ for $x \geq 0$

and then try to prove the statement is true for $n=k+1$, ie,

try to prove that $e^x > \frac{x^{k+1}}{(k+1)!}$ for $x \geq 0$

Given $e^x > \frac{x^k}{k!}$

then $\int_0^t e^x\,dx > \int_0^t \frac{x^k}{k!}\,dx$

$\left[e^x\right]_0^t > \left[\frac{x^{k+1}}{(k+1)!}\right]_0^t$

$e^t - 1 > \frac{t^{k+1}}{(k+1)!} - 0$

$\therefore e^t > 1 + \frac{t^{k+1}}{(k+1)!}$

$\therefore e^t > \frac{t^{k+1}}{(k+1)!}$

$\therefore e^x > \frac{x^{k+1}}{(k+1)!}$ as required. #