BAULKHAM HILLS
HIGH
SCHOOL

2021

YEAR 12
H2: ASSESSMENT
TASK 2

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for careless or badly arranged work

Total marks: 70

Section I – 10 marks (pages 2 – 5)
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 9)
- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 Which of the following is an expression for $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x}\, dx$?

(A) $x + \frac{1}{2}\cos 2x + c$

(B) $x - \frac{1}{2}\cos 2x + c$

(C) $x + \frac{1}{2}\sin^2 x + c$

(D) $x - \frac{1}{2}\sin^2 x + c$

2 For the complex polynomial $P(z) = z^3 + az^2 + bz + c$ with real coefficients a, b and c, $P(-2) = 0$ and $P(3i) = 0$.

The values of a, b and c are;

(A) $a = -2$, $b = 9$, $c = -18$

(B) $a = -2$, $b = -9$, $c = 18$

(C) $a = 2$, $b = 9$, $c = 18$

(D) $a = 2$, $b = -9$, $c = -18$

3 Assume that a and b are positive real numbers with $a > b$.

Which of the following might be false?

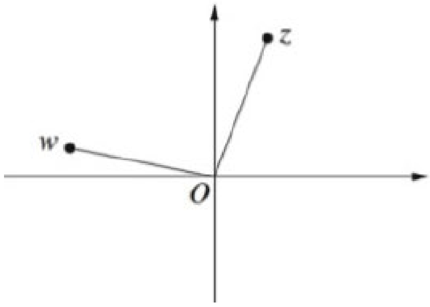
(A) $\frac{1}{a-b} > 0$

(B) $\frac{a}{b} - \frac{b}{a} > 0$

(C) $a + b > 2b$

(D) $2a > 3b$

4 The Argand diagram shows the complex numbers z and w, where z lies in the first quadrant and w lies in the second quadrant.



Which complex number could lie in the third quadrant?

(A) $-w$

(B) $2iz$

(C) $\overline{z}$

(D) $w - z$

5 Which integral is obtained when the substitution $t = \tan\dfrac{x}{2}$ is applied to $\displaystyle\int \frac{dx}{5 + 4\cos x}$?

(A) $\displaystyle\int \frac{2dt}{9 + t^2}$

(B) $\displaystyle\int \frac{2dt}{9 - 4t^2}$

(C) $\displaystyle\int \frac{1 + t^2}{9 + t^2}\,dt$

(D) $\displaystyle\int \frac{2(1 - t^2)}{(1 + t^2)(9 - t^2)}\,dt$

6 Consider the statement;

" A sufficient condition for ΔABC to be right angled, is that $a^2 + b^2 = c^2$ "

An equivalent statement is;

(A) If ΔABC is right angled then $a^2 + b^2 = c^2$

(B) If $a^2 + b^2 = c^2$ then ΔABC is right angled

(C) ΔABC is right angled iff $a^2 + b^2 = c^2$

(D) If $a^2 + b^2 \neq c^2$ then ΔABC is not right angled

7 If $f(x)$ is a non-zero odd function with period π, which of the following statements is **FALSE**?

(A) $\int_0^{2\pi} f(x)dx = 0$

(B) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)dx = 2\int_0^{\frac{\pi}{2}} f(x)dx$

(C) $\int_0^{\pi} f(x)dx = -\int_0^{\pi} f(-x)dx$

(D) $\int_{\alpha}^{\alpha+\pi} f(x)dx = \int_0^{\pi} f(x)dx \quad , \quad \forall \alpha \in \mathbb{R}$

8 Given that z is a complex number, $\dfrac{4z\overline{z}}{(z+\overline{z})^2}$ is equivalent to;

(A) $1 + \left(\dfrac{Im(z)}{Re(z)}\right)^2$

(B) $4[1 + (Re(z) + Im(z))^2]$

(C) $4([Re(z)]^2 + [Im(z)]^2)$

(D) $\dfrac{2 \times Im(z)}{[Re(z)]^2}$

9 What is the negation of the statement;

" If it is raining, you take your umbrella "

(A) If it is raining, you do not take your umbrella

(B) It is raining and you don't take your umbrella

(C) It is not raining and you don't take your umbrella

(D) It is not raining and you take your umbrella

10 Given that $(x + iy)^{14} = a + ib$, where $x, y, a, b \in \mathbb{R}$, $(y - ix)^{14}$ is equal to;

(A) $-a - ib$

(B) $b - ia$

(C) $-b - ia$

(D) $-a + ib$

END OF SECTION I

Section II

90 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question on the appropriate answer sheet. Each answer sheet must show your NESA#. Extra paper is available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (17 marks) Use the pages labelled Question 11 in the answer booklet

(a) If $z = 2i$ and $w = 1 + 3i$, express in the form $a + ib$;

(i) $z + \overline{w}$. *1*

(ii) zw. *1*

(iii) $\frac{1}{w}$. *1*

(b) (i) Express $\sqrt{3} - i$ in modulus-argument form. *1*

(ii) Hence evaluate $(\sqrt{3} - i)^6$. *2*

(c) Find $\int \frac{1 - x^{-2}}{1 - x^{-1}}\, dx$. *2*

(d) By completing the square find $\int \frac{dx}{\sqrt{x^2 - 4x + 1}}$. *2*

(e) Evaluate $\int_0^{\frac{\pi}{3}} x\sec^2 x\, dx$. *3*

(f) Evaluate $\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4 - x^2}}\, dx$. *4*

Question 12 (13 marks) Use the pages labelled Question 12 in the answer booklet

(a) In an Argand diagram, the point P represents the complex number z so that;

$$|z - 1 - i| = 1$$

(i) Sketch the locus of P. *1*

(ii) On your diagram shade the region where; *1*

$$|z - 1 - i| \le 1 \text{ and } 0 \le \arg(z - i) \le \frac{\pi}{4}.$$

(b) Consider this proposition about the positive integer n;

" If $n^3 + 5$ is odd, then n is even "

(i) Write down the contrapositive of the proposition. *1*

(ii) Prove the proposition is true by contraposition. *2*

(c) (i) By solving the equation $z^3 + 1 = 0$, find the cube roots of -1. *1*

(ii) Let λ be a non-real cube root of -1, show that $\lambda^2 = \lambda - 1$. *1*

(iii) Hence simplify $(1 - \lambda)^9$. *1*

(d) The numbers a, b and c are said to be in harmonic progression if their reciprocals $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are in arithmetic progression, b is then said to be the harmonic mean of a and c.

(i) Show that the numbers 6, 8, 12 are in harmonic progression. *1*

(ii) Show that the harmonic mean of a and c is equal to $\frac{2ac}{a + c}$. *2*

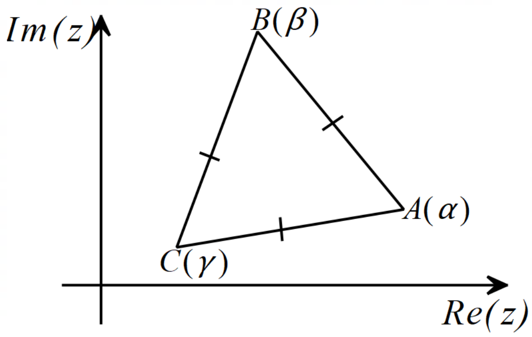
(iii) If $a > 0$ and $c > 0$, show that their geometric mean is greater than or equal to their harmonic mean. *2*

Question 13 (15 marks) Use the pages labelled Question 13 in the answer booklet

(a) The complex number $\frac{\sqrt{3}}{2} + \frac{i}{2}$ is one of the nth roots of -1. Find the least value of n for this to be so. *3*

(b) On an Argand diagram, the points A, B and C represent the complex numbers α, β and γ respectively.

ΔABC is equilateral, named with its vertices taken in anticlockwise order, as shown below.



(i) Explain why $\gamma - \alpha = (\beta - \alpha)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$. *2*

(ii) Show that $\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \alpha\gamma$. *2*

(c) (i) Show that $\frac{x}{e} > \ln x$ for $x > e$. *2*

(ii) Hence show that $e^{\pi} > \pi^{e}$. *1*

(d) (i) If $I_n = \int_0^1 (1 + x^2)^n \, dx$, $n = 0, 1, 2, \ldots$ show that; *3*

$$(2n + 1)I_n = 2^n + 2nI_{n-1}.$$

(ii) Hence find a reduction formula for; *2*

$$J_n = \int_0^{\frac{\pi}{4}} \sec^{2n} x \, dx.$$

Question 14 (15 marks) Use the pages labelled Question 14 in the answer booklet

(a) Evaluate $\sum_{n=1}^{2021} ni^n$. *2*

(b) Find the volume, V, of the solid of revolution formed when the graph of *5*

$y = 2\sqrt{\dfrac{x^2 + x + 1}{(x + 1)(x^2 + 1)}}$ is rotated about the x-axis over the interval $[0,\sqrt{3}]$.

Give your answer in the form $V = 2\pi(\ln a + b)$, where a, $b \in \mathbb{R}$.

(c) Let $S_n = \sum_{r=1}^{n} \frac{1}{\sqrt{r}}$ where $n \in \mathbb{Z}^+$.

(i) Prove by induction that $S_n \leq 2\sqrt{n} - 1$. *4*

(ii) Show that $(4k + 1)\sqrt{k + 1} > (4k + 3)\sqrt{k} \;\; \forall\; k \geq 0$. *2*

(iii) Determine the smallest number c such that; *2*

$$S_n \geq 2\sqrt{n} + \frac{1}{2\sqrt{n}} - c$$

End of paper

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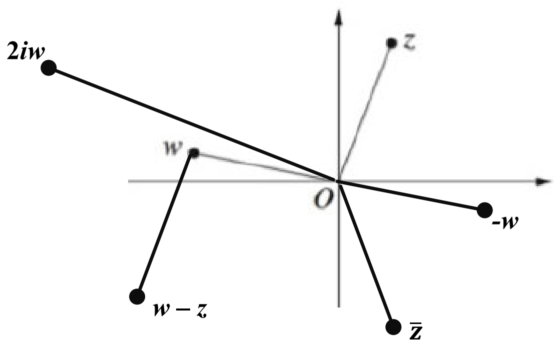
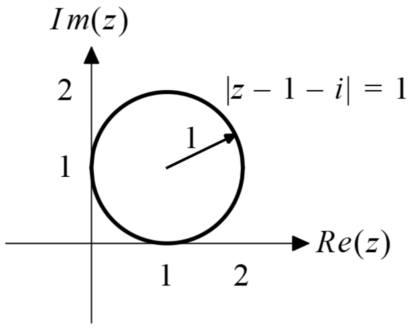
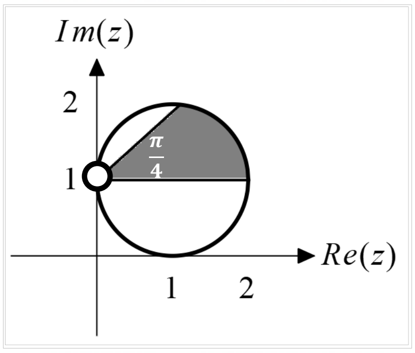
BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 2 H2: ASSESSMENT TASK 2 SOLUTIONS

Solution	Marks	Comments
SECTION I		
1. D - $\displaystyle\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x}\,dx = \int \frac{(\cos x + \sin x)(\cos^2 x - \sin x\cos x + \sin^2 x)}{\cos x + \sin x}\,dx$ $\displaystyle = \int (1 - \sin x\cos x)\,dx$ $\displaystyle = x - \frac{1}{2}\sin^2 x + c$	**1**	
2. C – as the coefficients are real, complex roots will appear in conjugate pairs $P(x) = (x+2)(x-3i)(x+3i)$ $= (x+2)(x^2+9)$ $= x^3 + 2x^2 + 9x + 18$ $\therefore a = 2,\ b = 9,\ c = 18$	**1**	
3. D – a counterexample to D would be let $a = 4$ and $b = 3$ in this case $2a < 3b$ i.e. $8 < 9$	**1**	
4. D –  	**1**	
5. A - $\displaystyle\int \frac{\frac{2dt}{1+t^2}}{5 + \frac{4(1-t^2)}{1+t^2}} = \int \frac{2dt}{5 + 5t^2 + 4 - 4t^2}$ $\displaystyle = \int \frac{2dt}{9+t^2}$	**1**	
6. B – If $P \Rightarrow Q$ means that P is a sufficient condition for Q	**1**	
7. B – As $f(x)$ is an odd function $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)dx = 0$	**1**	
8. A – $\displaystyle\frac{4z\bar{z}}{(z+\bar{z})^2} = \frac{4\left([Re(z)]^2 + [Im(z)]^2\right)}{(2Re(z))^2}$ $\displaystyle = \frac{4\left([Re(z)]^2 + [Im(z)]^2\right)}{4[Re(z)]^2}$ $\displaystyle = 1 + \left(\frac{Im(z)}{Re(z)}\right)^2$	**1**	
9. B – P: it is raining Q: you take your umbrella $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$	**1**	
10. A – $(y - ix)^{14} = (-i)^{14}(x+iy)^{14}$ $= -(a+ib)$ $= -a - ib$	**1**	

Solution	Marks	Comments
SECTION II		
QUESTION 11		
11(a) (i) $2i + 1 - 3i$ $= 1 - i$	**1**	**1 mark** • Correct answer
11(a) (ii) $2i(1 + 3i)$ $= -6 + 2i$	**1**	**1 mark** • Correct answer
11(a) (iii) $\frac{1}{w} = \frac{\overline{w}}{\lvert w\rvert^2}$ $= \frac{1 - 3i}{10}$ $= \frac{1}{10} - \frac{3}{10}i$	**1**	**1 mark** • Correct answer
11(b) (i) $\sqrt{3} - i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$	**1**	**1 mark** • Correct answer
11(b) (ii) $(\sqrt{3} - i)^6 = 2^6(\cos(-\pi) + i\sin(-\pi))$ $= -64$	**2**	**2 marks** • Correct solution **1 mark** • Attempts to use De Moivre's theorem
11(c) $\int \frac{1 - x^{-2}}{1 - x^{-1}}\,dx = \int \frac{(1 - x^{-1})(1 + x^{-1})}{1 - x^{-1}}\,dx$ $= \int \left(1 + \frac{1}{x}\right)dx$ $= x + \ln\lvert x\rvert + c$	**2**	**2 marks** • Correct solution **1 mark** • Manipulates integrand into a standard form
11(d) $\int \frac{dx}{\sqrt{x^2 - 4x + 1}} = \int \frac{dx}{\sqrt{(x-2)^2 - 3}}$ $= \ln\left\lvert x - 2 + \sqrt{x^2 - 4x + 1}\right\rvert + c$	**2**	**2 marks** • Correct solution **1 mark** • Completes the square in the denominator
11(e) $\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx$ $\quad u = x \quad v = \tan x$ $du = dx \quad dv = \sec^2 x\,dx$ $= [x\tan x]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan x\,dx$ $= \left(\frac{\pi}{3}\right)(\sqrt{3}) + [\ln\lvert\cos x\rvert]_0^{\frac{\pi}{3}}$ $= \frac{\pi\sqrt{3}}{3} + \ln\frac{1}{2}$	**3**	**3 marks** • Correct solution **2 marks** • Correctly uses parts or equivalent merit **1 mark** • Attempts to uses parts or equivalent merit

Solution	Marks	Comments
11(f) $\displaystyle\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta\cdot 2\cos\theta\,d\theta}{2\cos\theta}$ $\displaystyle = 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta\,d\theta$ $\displaystyle = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} (1-\cos 2\theta)\,d\theta$ $\displaystyle = 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$ $\displaystyle = 2\left(\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} + \frac{\pi}{6} + \frac{1}{2}\sin\left(-\frac{\pi}{3}\right)\right)$ $\displaystyle = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2}$ $= \pi - \sqrt{3}$	4	**4 marks** • Correct solution **3 marks** • Finds primitive along with the correct limits of integration **2 marks** • Simplifies integrand to a multiple of $\sin^2\theta$ **1 mark** • Transforms integrand with an appropriate trig substitution
QUESTION 12		
12(a) (i) 	1	**1 mark** • Correct diagram with location of the centre clearly identified and tangential to both axes
12(a) (ii) 	1	**1 mark** • Correct region with (0,1) excluded and angle indicated
12(b) (i) If n is not even then n^3+5 is not odd	1	**1 mark** • Correct statement
12(b) (ii) Let $n = 2k+1$ where $k \in \mathbb{Z} : k \geq 0$ (i.e. n is a positive odd number) $n^3 + 5 = (2k+1)^3 + 5$ $= 8k^3 + 12k^2 + 6k + 1 + 5$ $= 8k^3 + 12k^2 + 6k + 6$ $= 2(4k^3 + 6k^2 + 3k + 3)$ $= 2K$ where $K = 4k^3 + 6k^2 + 3k + 3 \in \mathbb{Z}$ $\therefore$ if n is odd then n^3+5 is even thus by contrapositiom, n^3+5 is odd then n is even	2	**2 marks** • Correct proof **1 mark** • Defines n • Demonstrates the technique of proof by contraposition

Solution		Marks	Comments
12(c) (i)	$z^3 + 1 = 0$ $(z+1)(z^2 - z + 1) = 0$ $z = -1 \quad , \quad z = \frac{1 \pm \sqrt{1-4}}{2}$ $= \frac{1 \pm \sqrt{3}\, i}{2}$ $\therefore$ the cube roots are -1 , $\frac{1}{2} - \frac{\sqrt{3}}{2} i$, $\frac{1}{2} + \frac{\sqrt{3}}{2} i$	**1**	**1 mark** • Correct solution
12(c) (ii)	As λ is non-real $\lambda \neq -1$, so λ must be a root of $z^2 - z + 1 = 0$ i.e. $\lambda^2 - \lambda + 1 = 0$ $\lambda^2 = \lambda - 1$	**1**	**1 mark** • Correct solution
12(c) (iii)	$(1-\lambda)^9 = (-\lambda^2)^9$ $= -\lambda^{18}$ $= -(\lambda^3)^6$ $= -(-1)^6$ $= -1$	**1**	**1 mark** • Correct solution
12(d) (i)	$\frac{1}{12} - \frac{1}{8} = -\frac{1}{24}$ $\frac{1}{8} - \frac{1}{6} = -\frac{1}{24}$ $\therefore$ 6, 8, 12 are in harmonic progression	**1**	**1 mark** • Correct solution
12(d) (ii)	$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$ $\frac{b-c}{bc} = \frac{a-b}{ab}$ $ab^2 - abc = abc - b^2 c$ $ab - ac = ac - bc$ $ab + bc = 2ac$ $b(a+c) = 2ac$ $b = \frac{2ac}{a+c}$	**2**	**2 marks** • Correct proof **1 mark** • Establishes $\frac{b-c}{bc} = \frac{a-b}{ab}$
12(d) (iii)	$\sqrt{ac} - \frac{2ac}{a+c} = \frac{\sqrt{ac}(a+c) - 2ac}{a+c}$ $= \frac{\sqrt{ac}(a - 2\sqrt{ac} + c)}{a+c}$ $= \frac{\sqrt{ac}(\sqrt{a} - \sqrt{c})^2}{a+c}$ $\geq 0 \qquad$ as $a\,,\, c > 0$ $\therefore \sqrt{ac} \geq \frac{2ac}{a+c}$	**2**	**2 marks** • Correct proof **1 mark** • Uses a valid technique
QUESTION 13			
13(a)	$\frac{\sqrt{3}}{2} + \frac{i}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = -1$ $\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} = \cos\pi + i\sin\pi$ $\frac{n\pi}{6} = \pi$ $n\pi = 6\pi$ $n = 6$	**3**	**3 marks** • Correct solution **2 marks** • Finds n to be a multiple of 6 **1 mark** • Converts both $\frac{\sqrt{3}}{2} + \frac{i}{2}$ and -1 to mod-arg form • Correctly uses De Moivre's theorem

Solution	Marks	Comments
13 (b) (i) $\angle BAC = \frac{\pi}{3}$ (vertex of an equilateral Δ) $\overrightarrow{AC} = \overrightarrow{AB}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (multiplication by a complex number represents anticlockwise rotation) $\gamma - \alpha = (\beta - \alpha)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$	2	**2 marks** • Correct explanation **1 mark** • Explains why $\angle BAC = \frac{\pi}{3}$ • Identifies the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$
13 (b) (ii) Similarly $\beta - \gamma = (\alpha - \gamma)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ Hence $\frac{\gamma - \alpha}{\beta - \alpha} = \frac{\beta - \gamma}{\alpha - \gamma} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ $(\gamma - \alpha)(\alpha - \gamma) = (\beta - \gamma)(\beta - \alpha)$ $\alpha\gamma - \gamma^2 - \alpha^2 + \alpha\gamma = \beta^2 - \alpha\beta - \beta\gamma + \alpha\gamma$ $\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \alpha\gamma$	2	**2 marks** • Correct solution **1 mark** • Finds a similar relationship to part (i)
13 (c) (i) Let $f(x) = \frac{x}{e} - \ln x$ $f'(x) = \frac{1}{e} - \frac{1}{x}$ > 0 for $x > e$ $\therefore f(x)$ is an increasing function for $x > e$ thus $f(x) > f(e)$ for $x > e$ $\frac{x}{e} - \ln x > \frac{e}{e} - \ln e$ $= 0$ $\therefore \frac{x}{e} > \ln x$ for $x > e$	2	**2 marks** • Correct solution **1 mark** • Shows that $f(x) = \frac{x}{e} - \ln x$ is increasing for $x > e$ or equivalent merit
13 (c) (ii) $\pi > e$ $\therefore \frac{\pi}{e} > \ln\pi$ $\pi > e\ln\pi$ $= \ln\pi^e$ $\therefore e^\pi > \pi^e$	1	**1 mark** • Correct solution
13(d) (i) $I_n = \int_0^1 (1 + x^2)^n\,dx$ $u = (1 + x^2)^n$ $du = 2nx(1 + x^2)^{n-1}dx$ $v = x$ $dv = dx$ $= \left[x(1 + x^2)^n\right]_0^1 - 2n\int_0^1 x^2(1 + x^2)^{n-1}dx$ $= (1)(2)^n - 0 - 2n\int_0^1 (1 + x^2 - 1)(1 + x^2)^{n-1}dx$ $= 2^n - 2n\int_0^1 (1 + x^2)^n\,dx + 2n\int_0^1 (1 + x^2)^{n-1}dx$ $= 2^n - 2nI_n + 2nI_{n-1}$ $(2n + 1)I_n = 2^n + 2nI_{n-1}$	3	**3 marks** • Correct solution **2 marks** • Rewrites x^2 as $1 + x^2 - 1$ in a valid attempt **1 mark** • Attempts to use integration by parts

Solution	Marks	Comments
13(d) (ii) $J_n = \int_0^{\frac{\pi}{4}} \sec^{2n}x\,dx$ $= \int_0^{\frac{\pi}{4}} (\sec^2 x)^n\,dx$ $= \int_0^{\frac{\pi}{4}} \sec^2 x(1 + \tan^2 x)^{n-1}dx$ $= \int_0^1 (1 + u^2)^{n-1}du$ $= I_{n-1}$ but $(2n-1)I_{n-1} = 2^{n-1} + 2(n-1)I_{n-2}$ $\therefore (2n-1)J_n = 2^{n-1} + 2(n-1)J_{n-1}$ $u = \tan x$ $du = \sec^2 x\,dx$ $x = 0,\ u = 0$ $x = \frac{\pi}{4},\ u = 1$	2	**2 marks** • Correct solution **1 mark** • Rewrites integral in terms of $u = \tan x$ • Shows $I_n = J_{n+1}$
QUESTION 14		
14 (a) $\sum_{n=1}^{2021} ni^n = 1i^1 + 2i^2 + 3i^3 + \ldots 2021i^{2021}$ $= i - 2 - 3i + 4 + 5i - 6 - 7i + 8 + \ldots + 2021i$ Now $i - 2 - 3i + 4 = 2 - 2i$ $5i - 6 - 7i + 8 = 2 - 2i$ . . $2017i - 2018 - 2019i + 2020 = 2 - 2i$ $\therefore \sum_{n=1}^{2021} ni^n = 505(2 - 2i) + 2021i$ $= 1010 - 1010i + 2021i$ $= 1010 + 1011i$	2	**2 marks** • Correct solution **1 mark** • Establishes a regular pattern within the sum
14 (b) $V = 4\pi\int_0^{\sqrt{3}} \frac{x^2 + x + 1}{(x+1)(x^2+1)}\,dx$ $= 2\pi\int_0^{\sqrt{3}} \left(\frac{1}{x+1} + \frac{x+1}{x^2+1}\right)dx$ $= 2\pi\left[\ln\lvert x+1\rvert + \frac{1}{2}\ln\lvert x^2+1\rvert + \tan^{-1}x\right]_0^{\sqrt{3}}$ $= 2\pi\left(\ln(\sqrt{3}+1) + \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3} - 0\right)$ $= 2\pi\left(\ln(2\sqrt{3}+2) + \frac{\pi}{3}\right)$ $A(x^2+1) + (Bx + C)(x+1) = x^2 + x + 1$ $x = -1$ $\quad$ $x = i$ $2A = 1$ $\quad$ $(Bi + C)(i+1) = i$ $A = \frac{1}{2}$ $\quad$ $Bi + C = \frac{i}{i+1} \times \frac{1-i}{1-i}$ $= \frac{1+i}{2}$ $B = \frac{1}{2}$ $\quad$ $C = \frac{1}{2}$	5	**5 marks** • Correct solution **4 marks** • Substitutes limits into correct primitive • Answer in correct format for primitive involving log and inverse trig **3 marks** • Finds correct primitive • Substitutes limits into an expression involving log and inverse trig **2 marks** • Decomposes correct integrand into partial fractions • Finds a primitive involving log and inverse trig **1 mark** • Obtains the correct integral • Uses the method of partial fractions

Solution	Marks	Comments
14(c) (i) When $n = 1$ LHS $= S_1 = \frac{1}{\sqrt{1}} = 1$ RHS $= 2\sqrt{1} - 1 = 2 - 1 = 1$ $\therefore$ LHS=RHS Hence the result is true for $n = 1$ Assume the result is true for $n = k$ where $k \in \mathbb{Z}^+$ i.e. $S_k \leq 2\sqrt{k} - 1$ Prove the result is true for $n = k + 1$ i.e. $S_{k+1} \leq 2\sqrt{k+1} - 1$ ***PROOF:*** $S_{k+1} = S_k + \frac{1}{\sqrt{k+1}}$ $\leq 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$ (by assumption) $= \frac{2\sqrt{k(k+1)} + 1}{\sqrt{k+1}} - 1$ $= \frac{\sqrt{4k^2 + 4k} + 1}{\sqrt{k+1}} - 1$ $< \frac{\sqrt{4k^2 + 4k + 1} + 1}{\sqrt{k+1}} - 1$ $(4k^2 + 4k < 4k^2 + 4k + 1)$ $= \frac{\sqrt{(2k+1)^2} + 1}{\sqrt{k+1}} - 1$ $= \frac{2k + 1 + 1}{\sqrt{k+1}} - 1$ $= \frac{2(k+1)}{\sqrt{k+1}} - 1$ $= 2\sqrt{k+1} - 1$ Hence the result is true for $n = k + 1$, if it is true for $n = k$ Since the result is true for $n = 1$, then it is true $\forall\ n$ where $n \in \mathbb{Z}^+$ by induction.	4	There are 5 key parts of the inequality induction; 1. Proving the result true for $n = 1$ 2. Clearly stating the assumption and what is to be proven 3. Using the assumption in the proof 4. Presenting a valid structure to the inequality proof, in particular; ○ Correct use of equality and inequality signs ○ Clearly explaining why the sins changed from equality to inequality 5. Correctly proving the required statement **4 marks** • Successfully does all of the 5 key parts **3 marks** • Successfully does 4 of the 5 key parts **2 marks** • Successfully does 3 of the 5 key parts **1 mark** • Successfully does 2 of the 5 key parts
14 (c) (ii) $(4k+1)\sqrt{k+1} - (4k+3)\sqrt{k}$ $= \sqrt{(4k+1)^2(k+1)} - \sqrt{(4k+3)^2 k}$ $= \sqrt{16k^3 + 8k^2 + k + 16k^2 + 8k + 1} - \sqrt{16k^3 + 24k^2 + 9k}$ $= \sqrt{16k^3 + 24k^2 + 9k + 1} - \sqrt{16k^3 + 24k^2 + 9k}$ > 0 (as $1 > 0$) $\therefore (4k+1)\sqrt{k+1} > (4k+3)\sqrt{k}\ \ \forall\ k > 0$	2	**2 marks** • Correct solution **1 mark** • Uses a valid technique
14 (c) (iii) $S_n \geq 2\sqrt{n} + \frac{1}{2\sqrt{n}} - c$ $c \geq 2\sqrt{n} + \frac{1}{2\sqrt{n}} - S_n$ $\geq 2\sqrt{n} + \frac{1}{2\sqrt{n}} - 2\sqrt{n} + 1$ $(S_n \leq 2\sqrt{n} - 1)$ $= \frac{1}{2\sqrt{n}} + 1$ as n increases, $\frac{1}{2\sqrt{n}}$ will decrease, thus its largest value will occur when $n = 1$ $\therefore c \geq \frac{1}{2} + 1$ $= \frac{3}{2}$	2	**2 marks** • Correct proof **1 mark** • shows $c \geq \frac{1}{2\sqrt{n}} + 1$ • shows $c \geq \frac{3}{2}$ for $n = 1$ *Note: no credit given for a bald answer*