Baulkham Hills High School

2022 **YEAR 12 ASSESSMENT TASK 2**

Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 2 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • For questions in Section II, show relevant mathematical reasoning and/or calculations
Total marks: 70	**Section I – 10 marks** (pages 2 – 5) • Attempt Questions 1 – 10 • Allow about 15 minutes for this section **Section II – 60 marks** (pages 6 – 9) • Attempt Questions 11 – 14 • Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer page in the writing booklet for Questions 1 – 10.

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1 Given that ω is a non-real cube root of unity, which of the following identities is **CORRECT**?

A. $1-\omega-\omega^2=0$

B. $1-\omega+\omega^2=0$

C. $1+\omega-\omega^2=0$

D. $1+\omega+\omega^2=0$

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2 A polynomial $P(z)$ with real coefficients has a complex root of $3+2i$. Which of the following quadratic expressions **MUST** be a factor of $P(z)$?

A. $z^2+6z+13$

B. $z^2-6z+13$

C. $z^2+6z-13$

D. $z^2-6z-13$

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3 Which of the following expressions is equal to $\int x^2 \sin x\,dx$?

A. $-x^2\cos x+\int 2x\cos x\,dx$

B. $-x^2\cos x-\int 2x\cos x\,dx$

C. $-2x\cos x+\int x^2\cos x\,dx$

D. $-2x\cos x-\int x^2\cos x\,dx$

4 Consider the following statement about a positive integer N:

"If N is a square number, then N is not prime."

Which of the following is logically equivalent to the given statement?

A. If N is not a square number, then N is prime.

B. If N is not a square number, then N is not prime.

C. If N is not prime, then N is a square number.

D. If N is prime, then N is not a square number.

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5 Which of the following statements is **FALSE**?

A. $\forall a, b \in \mathbb{R}, \quad a < 0 < b \Rightarrow e^a < e^b.$

B. $\forall a, b \in \mathbb{R}, \quad a < 0 < b \Rightarrow \frac{1}{a} < \frac{1}{b}.$

C. $\forall a, b \in \mathbb{R}, \quad a < 0 < b \Rightarrow a^2 < b^2.$

D. $\forall a, b \in \mathbb{R}, \quad a < 0 < b \Rightarrow a^3 < b^3.$

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6 Which of the following statements is **NOT ALWAYS TRUE**?

A. $((p \Rightarrow q) \wedge p) \Rightarrow p$

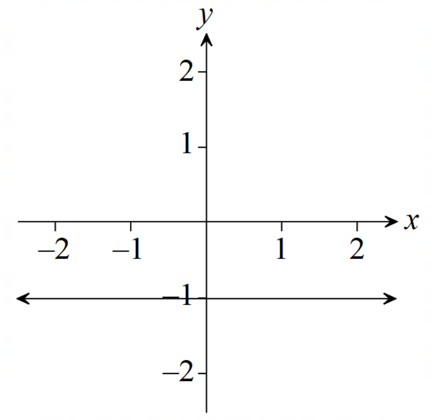
B. $((p \Rightarrow q) \wedge p) \Rightarrow q$

C. $((p \Rightarrow q) \wedge \sim p) \Rightarrow \sim q$

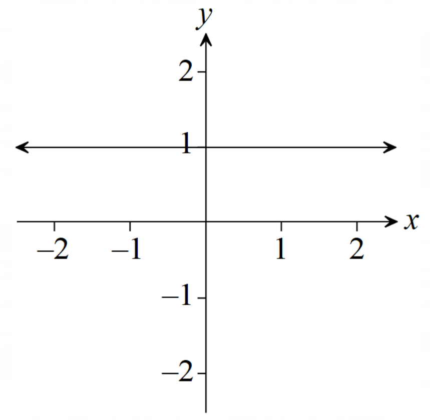
D. $((p \Rightarrow q) \wedge \sim q) \Rightarrow \sim p$

7 Which of the following diagrams shows the subset of the complex plane satisfied by the relation $iz - i\overline{z} = 2$?

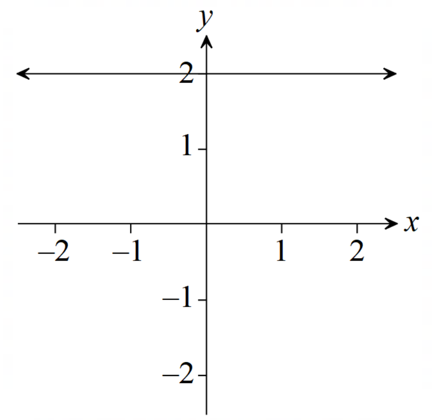
A.



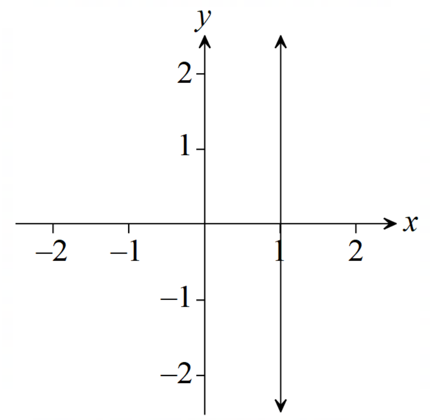
B.



C.



D.



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8 A complex number z satisfies the condition $\left|\dfrac{i-z}{i+z}\right| = 1$, on which of the following must the complex number z lie?

A. The y-axis

B. The x-axis

C. The line $x + y = 1$

D. The circle $x^2 + y^2 = 1$

9 Which of the following definite integrals takes on the smallest value?

A. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$

B. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$

C. $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$

D. $\displaystyle\int_0^{\frac{\pi}{4}} \sin(2x)\,dx$

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10 Suppose $f(x)$ is a continuous, differentiable function such that the inequality below holds true for all real values of a and b.

$$\frac{f(a)+f(b)}{2} < f\left(\frac{a+b}{2}\right)$$

Which of the following statements is always true?

A. $f'(1) \geq 0$

B. $f'(1) \leq 0$

C. $f''(1) \geq 0$

D. $f''(1) \leq 0$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate section of the writing booklet. Extra writing paper is available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 section of the writing booklet.

(a) Find $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$. **2**

(b) (i) Find the square roots of $16+30i$. **2**

(ii) Hence solve the quadratic equation $z^2-2z-(15+30i)=0$. **2**

(c) Find $\displaystyle\int x\tan^{-1}x\,dx$. **3**

(d) Find $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x\cos^2 x\,dx$. **3**

(e) Prove using mathematical induction that $3^n > 5\times 2^n$ for all integers $n\geq 5$. **3**

End of Question 11

Question 12 (15 marks) Use the Question 12 section of the writing booklet.

(a) Prove by contradiction that $\log_6 8$ is irrational. **3**

(b) (i) Prove that the following inequality holds for all positive real values of a and b. **2**

$$\frac{4a}{b}+\frac{b}{a}\geq 4$$

(ii) Show using a counterexample that the same inequality would not always hold true for all real values of a and b. **1**

(c) The terms of a sequence a_n is defined recursively by the formula $a_{n+1}=2a_n-n+1$ for positive integers n, with $a_1=3$. Use the principle of mathematical induction to show that for all integers $n\geq 1$, $a_n=2^n+n$. **3**

(d) (i) Using the substitution $t=\tan\frac{x}{2}$, prove that $\int \sec x\,dx=\ln\left|\sec x+\tan x\right|+c$. **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{6}} \sec^3 x\,dx$, express your answer in exact form. **3**

End of Question 12

Question 13 (15 marks) Use the Question 13 section of the writing booklet.

(a) For a pair of integers, a and b, let P be the proposition:

"If $a+b \geq 15$, then $a \geq 8$ or $b \geq 8$."

(i) Write down the contrapositive of the proposition P. **1**

(ii) Hence prove proposition P by contraposition. **2**

(iii) Write down the converse of proposition P and determine if the converse is true. **2**

(b) The complex number z simultaneously satisfied the inequalities given below. On an Argand diagram, sketch the region in which z lies. **3**

$$\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{2} \quad \text{and} \quad |z-1-i| < \sqrt{2}$$

(c) (i) Find the values of A, B and C such that **2**

$$\frac{4}{(1-x)^2(1+x)} \equiv \frac{A}{(1-x)^2} + \frac{B}{1-x} + \frac{C}{1+x}.$$

(ii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{4}{(1-x)^2(1+x)}\,dx$. **2**

(d) By using a suitable substitution, or otherwise, find the value of $\displaystyle\int_0^1 \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 section of the writing booklet.

(a) (i) Given $I_n = \int_0^{\frac{\pi}{4}} \tan^{2n+1} x \, dx$, show that $I_n + I_{n-1} = \dfrac{1}{2n}$ for integers $n \geq 1$. **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^7 x \, dx$. **3**

(b) Given that $x + y = m$ with $x > 0$ and $y > 0$.

(i) Show that $(x+y)^2 \geq 4xy$. **1**

(ii) Hence prove that $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{m}$. **1**

(iii) Using the result from part (ii), prove that $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{8}{m^2}$. **2**

(c) Consider the integral $I = \int_0^{\frac{\pi}{2}} \dfrac{\sin^2 x}{\sin x + \cos x} dx$.

(i) By substituting $x = \dfrac{\pi}{2} - u$, show that the integral can also be expressed as **1**

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} dx.$$

(ii) Hence, or otherwise, show that $I = \dfrac{\ln\left(1+\sqrt{2}\right)}{\sqrt{2}}$. **4**

End of Paper

YEAR 12 ASSESSMENT TASK 2 2022
MATHEMATICS EXTENSION 2
MARKING GUIDELINES

Section I

Multiple-choice Answer Key

Question	Answer
1	D
2	B
3	A
4	D
5	C

Question	Answer
6	C
7	A
8	B
9	C
10	D

Questions 1 – 10

Sample solution	
1.	$\omega^3 = 1$ $\omega^3 - 1 = 0$ $(\omega - 1)(\omega^2 + \omega + 1) = 0$ $1 + \omega + \omega^2 = 0 \quad (\text{since } \omega \neq 1 \text{ as } \omega \text{ is non-real})$
2.	Since the coefficients of $P(z)$ are real, if $3+2i$ is a root, then as is the conjugate $3-2i$. Therefore, $[z-(3+2i)][z-(3-2i)]$ must be a quadratic factor of $P(z)$. $[z-(3+2i)][z-(3-2i)] = (z-3-2i)(z-3+2i)$ $= (z-3)^2 - (2i)^2$ $= z^2 - 6z + 9 + 4$ $= z^2 - 6z + 13$
3.	Using integration by part: $\int x^2 \sin x\,dx = -x^2 \cos x - \int (-\cos x) \times 2x\,dx \qquad \begin{pmatrix} \text{Let} & u = x^2 & v = -\cos x \\ & du = 2x\,dx & dv = \sin x\,dx \end{pmatrix}$ $= -x^2 \cos x + \int 2x \cos x\,dx$
4.	Let $p: N$ is a square number, $q: N$ is not a prime number The given statement can be written as $p \Rightarrow q$, the statement $\sim q \Rightarrow \sim p$ would be logically equivalent, which in words would be "If *N* is a prime number, then *N* is not a square number.", therefore option D is correct. The other three statements (A, B and C) can be written as $\sim p \Rightarrow \sim q$, $\sim p \Rightarrow q$ and $\sim q \Rightarrow p$ respectively. A truth table for each of these statements would verify that these are not logically equivalent to $p \Rightarrow q$.
5.	$a = -2$ and $b = 1$ is a counterexample of statement C.
6.	Statement A reads as: "If *p* implies *q* and given *p*, then *p*." This is always true, the "If *p* implies *q*" is irrelevant. Statement B reads as: "If *p* implies *q* and given *p*, then *q*." This is always true. Statement D reads as: "If *p* implies *q* and given not *q*, then not *p*." This is always true by contraposition.
7.	$iz - i\overline{z} = 2$ $z - \overline{z} = -2i$ $2\,\text{Im}(z)i = -2i$ $y = -1 \quad (\text{Let } z = x + iy)$

Questions 1 – 10 (continued)

Sample solution

8.

$$\left|\frac{i-z}{i+z}\right| = 1 \quad (\text{noting } z \neq \pm i)$$

$$\frac{|i-x-iy|}{|i+x+iy|} = 1 \quad (\text{letting } z = x+iy)$$

$$|-x+(1-y)i| = |x+(1+y)i|$$

$$|-x+(1-y)i|^2 = |x+(1+y)i|^2$$

$$(-x)^2+(1-y)^2 = x^2+(1+y)^2$$

$$\cancel{x^2}+1-2y+\cancel{y^2} = \cancel{x^2}+1+2y+\cancel{y^2}$$

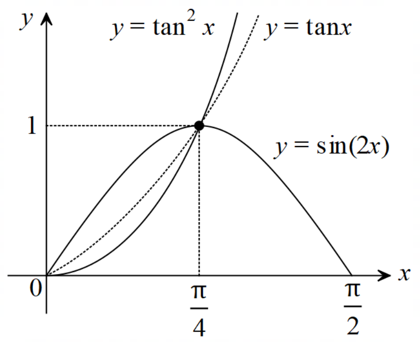
$$0 = 4y$$

$$y = 0 \quad (y \neq \pm 1)$$

9.

Both $y = \tan^2 x$ and $y = \sin(2x)$ pass through the origin and $\left(\frac{\pi}{4}, 1\right)$.

Using graphing techniques, $y = \tan^2 x$ is concave up and $y = \sin^2 x$ is concave down for $0 \le x \le \frac{\pi}{4}$, $\therefore \int_0^{\frac{\pi}{4}} \tan^2 x\,dx \le \int_0^{\frac{\pi}{4}} \sin(2x)\,dx$.



$$\sec^2 x = \tan^2 x + 1$$
$$> \tan^2 x$$

$$\int_0^{\frac{\pi}{4}} \sec^2 x\,dx > \int_0^{\frac{\pi}{4}} \tan^2 x\,dx$$

$$\cos^2 x \le 1 \quad (\text{since } 0 \le \cos x \le 1)$$

$$\frac{1}{\cos^2 x} \ge 1 \quad (\text{since } \cos^2 x \text{ and 1 are both positive})$$

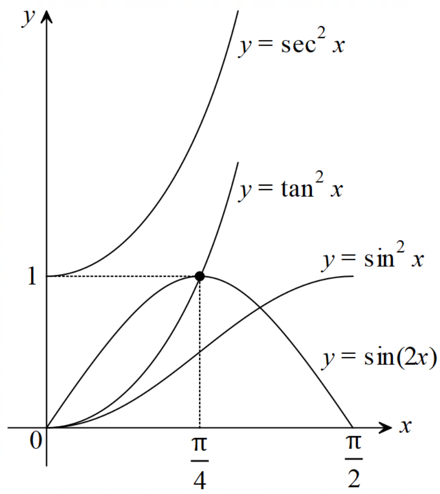
$$\frac{\sin^2 x}{\cos^2 x} \ge \sin^2 x$$

$$\tan^2 x \ge \sin^2 x$$

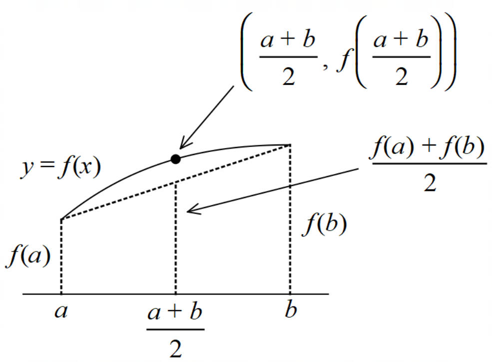
$$\int_0^{\frac{\pi}{4}} \tan^2 x\,dx \ge \int_0^{\frac{\pi}{4}} \sin^2 x\,dx$$

Therefore, $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$ takes on the smallest value.

For an entirely graphical approach:



10.



From the diagram, if the inequality is to hold true for all values of a and b, the function must be a concave down function, therefore the only statement that is always true is statement D.

Section II

Question 11

	Sample solution	Suggested marking criteria
(a)	$\displaystyle\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+9}\,dx$ $\displaystyle = \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+c$	• 2 – correct answer • 1 – correctly completes the square in the denominator
(b)	(i) Let $\sqrt{16+30i} \equiv x+iy$; $x, y \in \mathbb{R}$: $16+30i=(x+iy)^2$ $16+30i=(x^2-y^2)+(2xy)i$ Comparing real and imaginary parts: $x^2-y^2=16$ $2xy=30$ Solving simultaneously: $x=5, y=3$ and $x=-5, y=-3$ $\therefore \sqrt{16+30i}=\pm(5+3i)$	• 2 – correct answer • 1 – correct equations in x and y, or equivalent merit
	(ii) $z^2-2z-(15+30i)=0$ $\Delta=(-2)^2-4\times1\times[-(15+30i)]$ $=4+60+120i$ $=64+120i$ $\displaystyle z=\frac{-(-2)\pm\sqrt{64+120i}}{2}$ $\displaystyle =\frac{2\pm2\sqrt{16+30i}}{2}$ $=1\pm(5+3i)$ $z=6+3i, -4-3i$	• 2 – correct answer • 1 – Correct discriminant (separately or as part of the quadratic formula)
(c)	$\displaystyle\int x\tan^{-1}x\,dx=\frac{x^2}{2}\tan^{-1}x-\int\frac{x^2}{2}\times\frac{dx}{1+x^2}$ $\left(\text{Let } u=\tan^{-1}x \quad v=\frac{x^2}{2} \quad du=\frac{dx}{1+x^2} \quad dv=x\,dx\right)$ $\displaystyle =\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{1+x^2-1}{1+x^2}\,dx$ $\displaystyle =\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int 1-\frac{1}{1+x^2}\,dx$ $\displaystyle =\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+c$	• 3 – correct solution • 2 – uses algebraic manipulations to split $\displaystyle\int\frac{x^2\,dx}{1+x^2}$ into standard forms • 1 – correctly uses integration by parts

Question 11 (continued)

	Sample solution	Suggested marking criteria
(d)	$\int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x \, dx = \int_0^{\frac{\pi}{3}} \sin^2 x \cos^2 x \times \sin x \, dx$ $= \int_0^{\frac{\pi}{3}} \left(1-\cos^2 x\right)\cos^2 x \times \sin x \, dx$ $= -\int_1^{\frac{1}{2}} \left(1-u^2\right)u^2 \, du \quad \begin{pmatrix} \text{let } u = \cos x \text{ , } du = -\sin x \, dx \\ x = 0 \Rightarrow u = 1 \text{ , } x = \frac{\pi}{3} \Rightarrow u = \frac{1}{2} \end{pmatrix}$ $= \int_{\frac{1}{2}}^{1} u^2 - u^4 \, du$ $= \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_{\frac{1}{2}}^{1}$ $= \left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \left(\frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^5}{5}\right)$ $= \frac{47}{480}$	• 3 – correct solution • 2 – correctly integrates the integral in term of the substituted variable • 1 – rewrites the integral into an appropriate form using a Pythagorean identity
(e)	Let $S(n)$ be the statement that $3^n > 5 \times 2^n$. Show $S(5)$ is true: $\text{LHS} = 3^5 \qquad \text{RHS} = 5 \times 2^5$ $\quad = 243 \qquad \quad = 160$ Since LHS > RHS , $S(5)$ is true. Assume $S(k)$ is true, i.e.: $3^k > 5 \times 2^k$ Show $S(k+1)$ is true, i.e.: $3^{k+1} > 5 \times 2^{k+1}$ $\text{LHS} = 3^{k+1}$ $= 3 \times 3^k$ $> 3 \times 5 \times 2^k$ $> 2 \times 5 \times 2^k$ $= 5 \times 2^{k+1}$ $= \text{RHS}$ $\therefore S(k+1)$ is true if $S(k)$ is assumed true. Since $S(5)$ is shown true, by the principle of mathematical induction, $S(n)$ is true for all integers $n \geq 5$.	• 3 – correct solution • 2 – attempts to use the assumption to prove the required result • 1 – correctly proves the base case

Question 12

<table>
<tr><th colspan="2">Sample solution</th><th>Suggested marking criteria</th></tr>
<tr><td>(a)</td><td>Proof by contradiction: Assume $\log_6 8$ is rational, i.e. $\log_6 8 = \frac{p}{q}$, where p and q are positive integers that are coprime.

$$\begin{aligned} \log_6 8 &= \frac{p}{q} \\ 8 &= 6^{\frac{p}{q}} \\ 8^q &= 6^p \\ 2^{3q} &= 2^p \times 3^p \end{aligned}$$

Since p and q are integers, the RHS is divisible by 3 and the LHS is not, therefore no such integers p and q could make $2^{3q} = 2^p \times 3^p$. By contradiction, $\log_6 8$ must be irrational.</td><td>• 3 – correct proof
• 2 – derives the identity $2^{3q} = 2^p \times 3^p$
• 1 – correctly sets up the proof by contradiction with correct definitions
(Note: Conclusion must be logically correct and sound. For example, using "LHS is even and RHS is odd" would be incorrect, while concluding that the LHS and RHS are different from the $8^q = 6^p$ step would be insufficient reasoning.)</td></tr>
<tr><td>(b)</td><td>(i) For positive values of a and b:

$$\begin{aligned} \left(\frac{2\sqrt{a}}{\sqrt{b}} - \frac{\sqrt{b}}{\sqrt{a}}\right)^2 &\geq 0 \\ \frac{4a}{b} - 4 + \frac{b}{a} &\geq 0 \\ \frac{4a}{b} + \frac{b}{a} &\geq 4 \end{aligned}$$

Alternatively:

$$\frac{\frac{4a}{b} + \frac{b}{a}}{2} \geq \sqrt{\frac{4a}{b} \times \frac{b}{a}} \quad \text{(AM-GM inequality)}$$
$$\frac{4a}{b} + \frac{b}{a} \geq 2 \times \sqrt{4}$$
$$\frac{4a}{b} + \frac{b}{a} \geq 4$$</td><td>• 2 – correct proof
• 1 – attempts to prove the required result using appropriate algebraic techniques</td></tr>
<tr><td></td><td>(ii) For $a = -1$, $b = 1$:

$$\begin{aligned} \frac{4a}{b} + \frac{b}{a} &= \frac{4 \times (-1)}{1} + \frac{1}{(-1)} \\ &= -5 \\ &\ngeq 4 \end{aligned}$$</td><td>• 1 – correct counterexample</td></tr>
<tr><td>(c)</td><td>(i)

$$\begin{aligned} \int \sec x \, dx &= \int \frac{1+t^2}{1-t^2} \times \frac{2dt}{1+t^2} \\ &= \int \frac{2dt}{1-t^2} \\ &= \int \frac{2dt}{(1-t)(1+t)} \\ &= \int \frac{1}{1-t} + \frac{1}{1+t} \, dt \\ &= -\ln|1-t| + \ln|1+t| + c \\ &= \ln\left|\frac{1+t}{1-t}\right| + c \\ &= \ln\left|\frac{(1+t)^2}{(1-t)(1+t)}\right| + c \\ &= \ln\left|\frac{1+2t+t^2}{1-t^2}\right| + c \\ &= \ln\left|\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right| + c \\ &= \ln|\sec x + \tan x| + c \end{aligned}$$

$$t = \tan\frac{x}{2} \Rightarrow \frac{2dt}{1+t^2} = dx$$

Let $\frac{2}{(1-t)(1+t)} \equiv \frac{A}{1-t} + \frac{B}{1+t}$

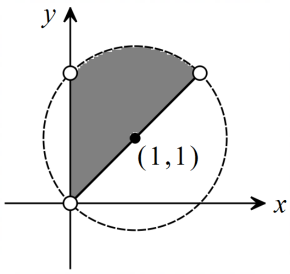
$$2 \equiv A(1+t) + B(1-t)$$

$t = 1 \Rightarrow A = 1$
$t = -1 \Rightarrow B = 1$</td><td>• 3 – correct proof
• 2 – correctly finds $\int \frac{2dt}{1-t^2}$
• 1 – converts the integral using t-formulas</td></tr>
</table>

Question 12 (continued)

Sample solution		Suggested marking criteria
(c)	(ii) Let $u = \sec x$ $\quad v = \tan x$ $du = \sec x \tan x \, dx \quad dv = \sec^2 x \, dx$ $\int_0^{\frac{\pi}{6}} \sec^3 x \, dx = \left[\sec x \tan x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sec x \tan^2 x \, dx$ $= \left[\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right) - \sec 0 \tan 0\right] - \int_0^{\frac{\pi}{6}} \sec x\left(\sec^2 x - 1\right) dx$ $= \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}} - \int_0^{\frac{\pi}{6}} \sec^3 x - \sec x \, dx$ $= \frac{2}{3} - \int_0^{\frac{\pi}{6}} \sec^3 x \, dx + \int_0^{\frac{\pi}{6}} \sec x \, dx$ $2\int_0^{\frac{\pi}{6}} \sec^3 x \, dx = \frac{2}{3} + \int_0^{\frac{\pi}{6}} \sec x \, dx$ $\int_0^{\frac{\pi}{6}} \sec^3 x \, dx = \frac{1}{3} + \frac{1}{2}\left[\ln\left\|\sec x + \tan x\right\|\right]_0^{\frac{\pi}{6}}$ $= \frac{1}{3} + \frac{1}{2}\left[\ln\left\|\sec\frac{\pi}{6} + \tan\frac{\pi}{6}\right\| - \ln\left\|\sec 0 + \tan 0\right\|\right]$ $= \frac{1}{3} + \frac{1}{2}\left[\ln\left\|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right\| - \ln\left\|1 + 0\right\|\right]$ $= \frac{1}{3} + \frac{1}{2}\ln\left(\frac{3}{\sqrt{3}}\right)$ $= \frac{1}{3} + \frac{1}{4}\ln 3$	• 3 – correct answer • 2 – correctly uses integration by parts and the result in part (i) • 1 – attempts to use integration by parts
(d)	Let $S(n)$ be the statement that $a_n = 2^n + n$. <u>Show $S(1)$ is true:</u> $\text{LHS} = a_1 \quad \text{RHS} = 2^1 + 1$ $= 3 \qquad\qquad = 3$ $\therefore S(1)$ is true. <u>Assume $S(k)$ is true, i.e.:</u> $a_k = 2^k + k$ <u>Show $S(k+1)$ is true, i.e.:</u> $a_{k+1} = 2^{k+1} + k + 1$ $\text{LHS} = a_{k+1}$ $= 2a_k - k + 1$ $= 2\left(2^k + k\right) - k + 1$ $= 2^{k+1} + 2k - k + 1$ $= 2^{k+1} + k + 1$ $= \text{RHS}$ $\therefore S(k+1)$ is true if $S(k)$ is assumed true. Since $S(1)$ is shown true, by the principle of mathematical induction, $S(n)$ is true for all positive integers n.	• 3 – correct solution • 2 – attempts to use the assumption to prove the required result • 1 – correctly proves the base case

Question 13

Sample solution			Suggested marking criteria
(a)	(i)	If $a<8$ and $b<8$, then $a+b<15$	• 1 – correct statement
	(ii)	Since a and b are integers $a<8 \Rightarrow a \le 7$ and $b<8 \Rightarrow b \le 7$. Therefore: $a+b \le 7+7$ $=14$ <15	• 2 – correct proof • 1 – recognises that $a \le 7$ (or $b \le 7$)
	(iii)	The converse of proposition P is "If $a \ge 8$ or $b \ge 8$, then $a+b \ge 15$." This statement is false, consider the counterexample where $a=8, b=1$. $a+b=8+1 \ngeq 15$	• 2 – correct conclusion • 1 – correct converse statement
(b)			To be awarded the full 3 marks, the following crucial features were considered (1 mark each): • Shape of the boundaries are correct, e.g. a circle centred at (1,1) passing through (0,0), or labelling an angle of $\frac{\pi}{4}$ • Recognising the correct region to shade in • Correct inclusion / exclusion of boundaries and points of intersection of these boundaries
(c)	(i)	$\dfrac{4}{(1-x)^2(1+x)} \equiv \dfrac{A}{(1-x)^2}+\dfrac{B}{1-x}+\dfrac{C}{1+x}$ $4 \equiv A(1+x)+B(1-x)(1+x)+C(1-x)^2$ $x=1 \Rightarrow A=2$, $x=-1 \Rightarrow C=1$ When $x=0$: $4=A+B+C$ $4=2+B+1$ $B=1$ $\therefore \dfrac{4}{(1-x)^2(1+x)} \equiv \dfrac{2}{(1-x)^2}+\dfrac{1}{1-x}+\dfrac{1}{1+x}$	• 2 – correct values for A, B and C • 1 – finds a correct value for A, B or C
	(ii)	$\displaystyle\int_0^{\frac{1}{2}} \frac{4}{(1-x)^2(1+x)}\,du$ $=\displaystyle\int_0^{\frac{1}{2}} \frac{2}{(1-x)^2}+\frac{1}{1-x}+\frac{1}{1+x}\,du$ $=\left[\dfrac{2}{1-x}-\ln\lvert 1-x\rvert+\ln\lvert 1+x\rvert\right]_0^{\frac{1}{2}}$ $=\left(\dfrac{2}{1-\frac{1}{2}}-\ln\left\lvert 1-\dfrac{1}{2}\right\rvert+\ln\left\lvert 1+\dfrac{1}{2}\right\rvert\right)-\left(\dfrac{2}{1-0}-\ln\lvert 1-0\rvert+\ln\lvert 1+0\rvert\right)$ $=4-\ln\dfrac{1}{2}+\ln\dfrac{3}{2}-2$ $=2+\ln 3$	• 2 – correct solution • 1 – correctly integrates the partial fractions – correctly substitutes into an incorrect primitive

Question 13 (continued)

	Sample solution	Suggested marking criteria
(d)	Let $x = \sin\theta$, then $\theta = \sin^{-1} x$ and $dx = \cos\theta\, d\theta$. When $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{\pi}{2}$. $\int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} \frac{\theta \sin\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta\, d\theta$ $= \int_0^{\frac{\pi}{2}} \frac{\theta \sin\theta}{\cos\theta} \times \cos\theta\, d\theta$ $= \int_0^{\frac{\pi}{2}} \theta \sin\theta\, d\theta$ Alternatively, let $u = \sin^{-1} x$, then $x = \sin u$ and $du = \frac{dx}{\sqrt{1-x^2}}$. When $x = 0, u = 0$ and when $x = 1, u = \frac{\pi}{2}$. $\int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} u \sin u\, du \left(= \int_0^{\frac{\pi}{2}} \theta \sin\theta\, d\theta \right)$ $\int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} \theta \sin\theta\, d\theta \quad \left(\begin{array}{lll} \text{Let} & u = \theta & v = -\cos\theta \\ & du = d\theta & dv = \sin\theta\, d\theta \end{array} \right)$ $= \left[-\theta\cos\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos\theta\, d\theta$ $= \left[\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + 0\cos 0 \right) + \left[\sin\theta\right]_0^{\frac{\pi}{2}} \right]$ $= \sin\frac{\pi}{2} - \sin 0$ $= 1$	• 3 – correct solution • 2 – correctly uses integration by parts on the integral that is in terms of θ • 1 – rewrites the integral in terms of θ

Question 14

<table>
<tr><th colspan="3">Sample solution</th><th>Suggested marking criteria</th></tr>
<tr><td>(a)</td><td colspan="2">(i)

$$\begin{aligned}
I_n + I_{n-1} &= \int_0^{\frac{\pi}{4}} \tan^{2n+1} x\,dx + \int_0^{\frac{\pi}{4}} \tan^{2(n-1)+1} x\,dx \\
&= \int_0^{\frac{\pi}{4}} \tan^{2n+1} x + \tan^{2n-1} x\,dx \\
&= \int_0^{\frac{\pi}{4}} \tan^{2n-1} x\left(1+\tan^2 x\right)dx \\
&= \int_0^{\frac{\pi}{4}} \tan^{2n-1} x \sec^2 x\,dx \\
&= \int_0^{1} u^{2n-1}\,du \quad \left(\text{let } u = \tan x, du = \sec^2 x dx\right) \\
&= \left[\frac{u^{2n}}{2n}\right]_0^1 \\
&= \frac{1^{2n}}{2n} - \frac{0^{2n}}{2n} \\
&= \frac{1}{2n}
\end{aligned}$$

</td><td>

• 3 – correct proof

• 2 – correctly integrates the integral in terms of u, or equivalent merit

• 1 – attempts to simplify the integral to a point where substitution would be appropriate

– uses integration by parts in a valid manner

</td></tr>
<tr><td></td><td>(ii)

$$\int_0^{\frac{\pi}{4}} \tan^7 x\,dx = I_3$$

$$\begin{aligned}
I_0 &= \int_0^{\frac{\pi}{4}} \tan x\,dx \\
&= \left[-\ln\left|\cos x\right|\right]_0^{\frac{\pi}{4}} \\
&= -\ln\left|\cos\tfrac{\pi}{4}\right| + \ln\left|\cos 0\right| \\
&= -\ln\left|\frac{1}{\sqrt{2}}\right| \\
&= -\ln\left(2^{-\frac{1}{2}}\right) \\
&= \frac{\ln 2}{2}
\end{aligned}$$

</td><td>

$$\begin{aligned}
I_1 + I_0 &= \frac{1}{2} \\
I_1 &= \frac{1}{2} - I_0 \\
&= \frac{1}{2} - \frac{\ln 2}{2}
\end{aligned}$$

$$\begin{aligned}
I_2 + I_1 &= \frac{1}{4} \\
I_2 &= \frac{1}{4} - I_1 \\
&= \frac{1}{4} - \left(\frac{1}{2} - \frac{\ln 2}{2}\right) \\
&= -\frac{1}{4} + \frac{\ln 2}{2}
\end{aligned}$$

$$\begin{aligned}
I_3 + I_2 &= \frac{1}{6} \\
I_3 &= \frac{1}{6} - I_2 \\
&= \frac{1}{6} - \left(-\frac{1}{4} + \frac{\ln 2}{2}\right) \\
&= \frac{5}{12} - \frac{\ln 2}{2}
\end{aligned}$$

</td><td>

• 3 – correct value for the given integral

• 2 – attempts to use the recursion to find the value of the given integral

• 1 – correctly finds the value of I_0

– correctly uses the recursion formula in an attempt to find I_7 (as opposed to I_3)

</td></tr>
</table>

Question 14 (continued)

Sample solution			Suggested marking criteria
(b)	(i)	$(x-y)^2 \geq 0$ $x^2 - 2xy + y^2 \geq 0$ $x^2 + 2xy + y^2 \geq 4xy$ $(x+y)^2 \geq 4xy$	• 1 – correct proof
	(ii)	Since x and y are positive, then as are $x+y$ and xy: $(x+y)^2 \geq 4xy$ $\dfrac{x+y}{xy} \geq \dfrac{4}{x+y}$ $\dfrac{x}{xy} + \dfrac{y}{xy} \geq \dfrac{4}{m}$ $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{m}$	• 1 – correct proof
	(iii)	$\dfrac{\frac{1}{x^2}+\frac{1}{y^2}}{2} \geq \sqrt{\dfrac{1}{x^2} \times \dfrac{1}{y^2}}$ (AM-GM inequality) $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq 2\sqrt{\dfrac{1}{x^2 y^2}}$ $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{2}{xy}$ (since $xy > 0$) $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{2}{xy} = \left(\dfrac{1}{x} + \dfrac{1}{y}\right)^2$ $\geq \dfrac{16}{m^2}$ $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{16}{m^2}$ $2\left(\dfrac{1}{x^2} + \dfrac{1}{y^2}\right) \geq \dfrac{16}{m^2}$ $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{8}{m^2}$ Alternatively, $(x+y)^2 \geq 4xy$ (from (i)), and: $\dfrac{x^2+y^2}{2} \geq \sqrt{x^2 y^2}$ (AM-GM inequality) $x^2 + y^2 \geq 2xy$ $\left(\text{since } xy > 0, \sqrt{x^2 y^2} = \lvert xy \rvert = xy\right)$ As $x^2 + y^2, (x+y)^2, 2xy, 4xy > 0$, it follows from multiplying that: $(x+y)^2 \left(x^2 + y^2\right) \geq 4xy \times 2xy$ $m^2\left(x^2 + y^2\right) \geq 8x^2 y^2$ $\dfrac{x^2+y^2}{x^2 y^2} \geq \dfrac{8}{m^2}$ (since $x^2 y^2, m^2 > 0$) $\dfrac{x^2}{x^2 y^2} + \dfrac{y^2}{x^2 y^2} \geq \dfrac{8}{m^2}$ $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{8}{m^2}$	• 2 – correct proof • 1 – shows that $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{2}{xy}$

Question 14 (continued)

Sample solution		Suggested marking criteria
(c)	(i) $x=\frac{\pi}{2}-u \Rightarrow dx=-du$ $I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\sin x+\cos x}dx$ $=\int_{\frac{\pi}{2}}^{0}\frac{\cos^2\left(\frac{\pi}{2}-u\right)}{\sin\left(\frac{\pi}{2}-u\right)+\cos\left(\frac{\pi}{2}-u\right)}\times(-du)$ $=\int_0^{\frac{\pi}{2}}\frac{\sin^2 u}{\cos u+\sin u}du$ $=\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\cos x+\sin x}dx$	• 1 – correct proof
(c)	(ii) $I+I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\sin x+\cos x}dx+\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\cos x+\sin x}dx$ $2I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x+\sin^2 x}{\sin x+\cos x}dx$ $=\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+\cos x}dx$ $I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}dx$ $=\frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\sec\left(x-\frac{\pi}{4}\right)dx$ $=\frac{1}{2\sqrt{2}}\left[\ln\left\|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right\|\right]_0^{\frac{\pi}{2}}$ $=\frac{1}{2\sqrt{2}}\left[\ln\left\|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right\|-\ln\left\|\sec\left(-\frac{\pi}{4}\right)+\tan\left(-\frac{\pi}{4}\right)\right\|\right]$ $=\frac{1}{2\sqrt{2}}\left(\ln\left\|\sqrt{2}+1\right\|-\ln\left\|\sqrt{2}-1\right\|\right)$ $=\frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$ $=\frac{1}{2\sqrt{2}}\ln\left[\frac{\left(\sqrt{2}+1\right)^2}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}\right]$ $=\frac{1}{2\sqrt{2}}\ln\left(\sqrt{2}+1\right)^2$ $=\frac{1}{\sqrt{2}}\ln\left(\sqrt{2}+1\right)$	• 4 – correct solution • 3 – correctly integrates the final integral • 2 – correctly uses an appropriate method to convert the integral $\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+\cos x}dx$ • 1 – sums the given integral and the alternate form in part (i)