Baulkham Hills High School

2023 **YEAR 12 ASSESSMENT TASK 2**

Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 2 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • For questions in Section II, show relevant mathematical reasoning and/or calculations
Total marks: 70	**Section I – 10 marks** (pages 2 – 5) • Attempt Questions 1 – 10 • Allow about 15 minutes for this section **Section II – 60 marks** (pages 6 – 9) • Attempt Questions 11 – 14 • Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer page in the writing booklet for Questions 1 – 10.

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1 Which of the following is the negation of the statement “$\forall z \in \mathbb{C}$, $z+\overline{z} \in \mathbb{R}$.” ?

A. $\forall z \in \mathbb{C}$, $z+\overline{z} \notin \mathbb{R}$.

B. $\forall z \notin \mathbb{C}$, $z+\overline{z} \in \mathbb{R}$.

C. $\exists z \notin \mathbb{C} : z+\overline{z} \in \mathbb{R}$.

D. $\exists z \in \mathbb{C} : z+\overline{z} \notin \mathbb{R}$.

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2 If $z^2 = 4e^{i\frac{4\pi}{3}}$, which of the following pair of complex numbers represent z?

A. $1-\sqrt{3}i$, $-1+\sqrt{3}i$

B. $3+i$, $-3-i$

C. $-1+\sqrt{3}i$, $-1-\sqrt{3}i$

D. $3-i$, $-3+i$

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3 What is the value of $\int_0^1 xe^{-x^2}\,dx$?

A. $\dfrac{1-e}{2e}$

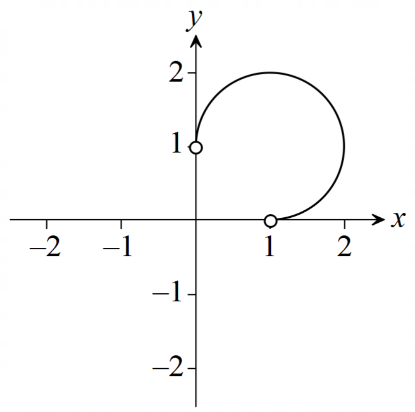
B. $\dfrac{e-1}{2e}$

C. $\dfrac{2e-1}{e}$

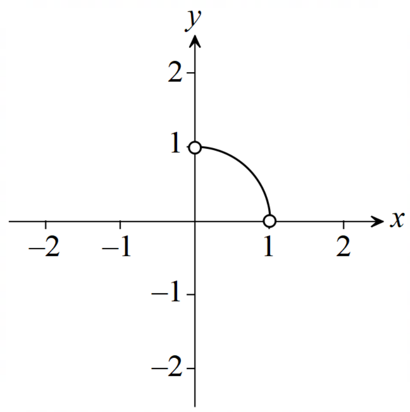
D. $\dfrac{1-2e}{e}$

4 Which of the following diagrams shows the subset of the complex plane satisfied by the relation $\arg\left(\dfrac{z+1}{z+i}\right)=\dfrac{\pi}{4}$?

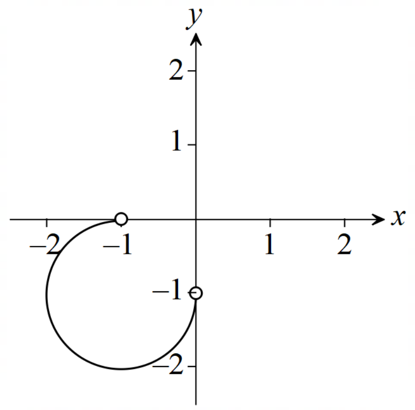
A.



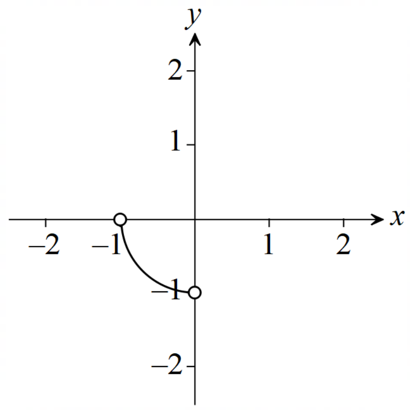
B.



C.



D.



5 In the following statement, m and n are two positive integers.

"If m^2 is odd and n^2 is even, then mn is even."

Which of the following is correct?

A. The given statement and its converse are both true.

B. The given statement is true, but its converse is false.

C. The given statement is false, but its converse is true.

D. The given statement and its converse are both false.

6 Given that a, b, c, d, e and f are real numbers with $a < b < c$ and $d < e < f$, which of the following is **ALWAYS** correct?

A. $ad < be < cf$

B. $\frac{a}{d} < \frac{b}{e} < \frac{c}{f}$

C. $a+d < b+e < c+f$

D. $a-d < b-e < c-f$

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7 If $(1+i)(1+2i)(1+3i)(1+4i)\times\cdots\times(1+ni) \equiv a+bi$ for some positive integer n and real values of a and b, which of the following expressions is equivalent to $2\times5\times10\times17\times\cdots\times(1+n^2)$?

A. a^2+b^2

B. a^2-b^2

C. $\sqrt{a^2+b^2}$

D. $\sqrt{a^2-b^2}$

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8 What is the value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{\sin x \cos x}$?

A. $\log_e \sqrt{2}$

B. $\log_e \sqrt{3}$

C. $-\log_e \sqrt{2}$

D. $-\log_e \sqrt{3}$

9 Which of the following integrals has the smallest value?

A. $\int_0^{\frac{\pi}{6}} \sin^2 x \, dx$

B. $\int_0^{\frac{\pi}{6}} (1 - \sin x) \, dx$

C. $\int_0^{\frac{\pi}{6}} (1 - \sin^2 x) \, dx$

D. $\int_0^{\frac{\pi}{6}} (1 - \sin x)^2 \, dx$

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10 Consider the statement $\int_1^4 f(x) \, dx = \int_1^4 f(a - x) \, dx$.

For which value of a is this statement true for all continuous functions $f(x)$?

A. 0

B. 1

C. 4

D. 5

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate section of the writing booklet. Extra writing paper is available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 section of the writing booklet.

(a) Prove that the difference between the squares of a pair of successive odd integers is always divisible by 8. **2**

(b) (i) Find real values A, B and C such that $\dfrac{4-2x}{(x-1)(x^2+1)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$. **2**

(ii) Hence find $\displaystyle\int \frac{4-2x}{(x-1)(x^2+1)}\,dx$. **2**

(c) Use integration by parts to find $\int \sin^{-1} x\,dx$. **2**

(d) (i) Express $1+\sqrt{3}i$ in exponential form. **2**

(ii) Hence state the condition on the integer n for $\left(1+\sqrt{3}i\right)^{\frac{n}{2}}$ to be purely imaginary. **2**

(e) Prove using mathematical induction that, for all integers $n \geq 1$, **3**

$$1^2 + 2^2 + 3^2 + \cdots + n^2 > \frac{n^3}{3}.$$

End of Question 11

Question 12 (15 marks) Use the Question 12 section of the writing booklet.

(a) (i) Find all pairs of integers a and b such that $(a+ib)^2 = 8+6i$. **2**

(ii) Hence solve $z^2 + 2z(1+2i) - (11+2i) = 0$. **2**

(b) On an Argand diagram, sketch the subset of the complex plane represented by **2**

$$z\overline{z} - 3(z+\overline{z}) \leq 0.$$

(c) Find $\displaystyle\int \frac{1}{2x^2 - 2x + 1}\,dx$. **3**

(d) Let $I_n = \displaystyle\int_0^1 \frac{x^n}{x^2+1}\,dx$, where n is a non-negative integer.

(i) Find the value of I_0. **2**

(ii) Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$. **2**

(iii) Hence find the value of $\displaystyle\int_0^1 \frac{x^4}{x^2+1}\,dx$. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 section of the writing booklet.

(a) Evaluate $\displaystyle\int_0^1 \sqrt{\frac{1-x}{3+x}}\,dx$. **3**

(b) Use the substitution $t=\tan\dfrac{x}{2}$ to find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2-\cos x}\,dx$. **3**

(c) Consider the statement P: *"If n^3 is a multiple of 3, then n is a multiple of 3."*

(i) Write down the contrapositive of P. **1**

(ii) Prove statement P by contraposition. **3**

(d) (i) Use de Moivre's theorem to prove $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **2**

(ii) Solve the equation $\cos 5\theta = 0$ and hence show that $\cos^2\left(\dfrac{3\pi}{10}\right)=\dfrac{5-\sqrt{5}}{8}$. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 section of the writing booklet.

(a) A sequence is defined recursively by $t_{n+1} = \dfrac{t_n}{t_n + 1}$ for integers $n \geq 1$, where $t_1 = 2$. **3**

Prove by mathematical induction that $t_n = \dfrac{2}{2n-1}$ for all positive integers n.

(b) (i) Show that $a^2 + b^2 \geq 2ab$ for all real values of a and b. **1**

(ii) Hence show for $a, b, c \in \mathbb{R}$ that $\dfrac{a^2 + b^2 + c^2}{3} \geq \left(\dfrac{a+b+c}{3}\right)^2$. **2**

(c) Consider two distinct rational numbers g and h such that $g < h$.

(i) Show that $g < \dfrac{g+h}{2} < h$. **2**

(ii) Hence show there are infinitely many rational numbers by showing there always exists a rational number x such that $g < x < h$. **2**

(d) (i) Explain why, for $0 \leq x \leq 1$, the following inequality holds: **2**

$$\frac{1}{2}\int_0^1 x^2 (1-x)^2 \, dx \; < \; \int_0^1 \frac{x^2 (1-x)^2}{1+x} \, dx \; < \; \int_0^1 x^2 (1-x)^2 \, dx$$

(ii) Hence show that $\dfrac{166}{240} < \log_e 2 < \dfrac{167}{240}$. **3**

End of Paper

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YEAR 12 ASSESSMENT TASK 2 2023
MATHEMATICS EXTENSION 2
MARKING GUIDELINES

Section I

Multiple-choice Answer Key

Question	Answer
1	D
2	A
3	B
4	C
5	B

Question	Answer
6	C
7	A
8	B
9	A
10	D

Questions 1 – 10

Sample solution	
1.	The negation of "for all… statement *P*" is "there exists… not statement *P*".
2.	$z^2 = 4e^{i\frac{4\pi}{3}}$ $z = \pm\left(4e^{i\frac{4\pi}{3}}\right)^{\frac{1}{2}}$ $= \pm 2e^{i\frac{2\pi}{3}}$ $= -1+\sqrt{3}i\ ,\ 1-\sqrt{3}i$
3.	$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}\,dx$ $= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$ $= -\frac{1}{2}\times\left(e^{-1}-e^0\right)$ $= \frac{\frac{1}{e}-1}{-2}$ $= \frac{e-1}{2e}$
4.	The locus represented by $\arg\left(\frac{z+1}{z+i}\right) = \arg\left(\frac{z-(-1)}{z-(-i)}\right) = \frac{\pi}{4}$ is an anti-clockwise major circular arc from $z=-1$ to $z=-i$. (More specifically, one with an angle at the circumference of $\frac{\pi}{4}$ radians / angle at the centre of $\frac{\pi}{2}$ radians.)
5.	The converse of *"If m^2 is odd and n^2 is even, then mn is even."* is *"If mn is even, then m^2 is odd and n^2 is even."* The given statement is true, as n^2 is even, then *n* is even, which means *mn* is even. The converse is false as *m* could be even, then m^2 would be even.

Questions 1 – 10 (continued)

Sample solution	
6.	$a<b$ and $d<e$ yields $a+d<b+e$. Similarly, $b+e<c+f$, therefore $a+d<b+e<c+f$ is true. $0<1<2$ and $-2<0<2$ is a counterexample for option D as $0-(-2)\not<1-0\not<2-2$. $-10<2<4$ and $-1<1<2$ is a counterexample for options A and B as $-10\times(-1)\not<2\times1<4\times2$ and $\frac{-10}{-1}\not<\frac{2}{1}\not<\frac{4}{2}$.
7.	$2\times5\times10\times17\times\cdots\times(1+n^2)$ $=(1+1^2)\times(1+2^2)\times(1+3^2)\times\cdots\times(1+n^2)$ $=\lvert 1+i\rvert^2\times\lvert 1+2i\rvert^2\times\lvert 1+3i\rvert^2\times\cdots\times\lvert 1+ni\rvert^2$ $=(\lvert 1+i\rvert\times\lvert 1+2i\rvert\times\lvert 1+3i\rvert\times\cdots\times\lvert 1+ni\rvert)^2$ $=\lvert(1+i)(1+2i)(1+3i)\times\cdots\times(1+ni)\rvert^2$ $=\lvert a+bi\rvert^2$ $=a^2+b^2$
8.	$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{dx}{\sin x\cos x}=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2 x}{\sin x\cos x\sec^2 x}dx$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2 x}{\tan x}dx$ $=\left[\log_e\lvert\tan x\rvert\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $=\log_e\left\lvert\tan\frac{\pi}{3}\right\rvert-\log_e\left\lvert\tan\frac{\pi}{4}\right\rvert$ $=\log_e\lvert\sqrt{3}\rvert-\log_e\lvert 1\rvert$ $=\log_e\sqrt{3}$
9.	For $0\le x\le\frac{\pi}{6}$, $1-\sin^2 x=\cos^2 x>\sin^2 x$, so $\int_0^{\frac{\pi}{6}}(1-\sin^2 x)dx>\int_0^{\frac{\pi}{6}}\sin^2 x\,dx$ Since $0<\sin x<1$, therefore $0<1-\sin x<1$, making $(1-\sin x)^2<1-\sin x$. As such, $\int_0^{\frac{\pi}{6}}(1-\sin x)dx>\int_0^{\frac{\pi}{6}}(1-\sin x)^2\,dx$. For $0\le x\le\frac{\pi}{6}$, $0\le\sin x\le\frac{1}{2}$, making $0\le 2\sin x\le 1$, therefore $1-2\sin x\ge 0$. $(1-\sin x)^2=1-2\sin x+\sin^2 x$ $\ge\sin^2 x$ Therefore, $\int_0^{\frac{\pi}{6}}(1-\sin x)^2\,dx>\int_0^{\frac{\pi}{6}}\sin^2 x\,dx$, making $\int_0^{\frac{\pi}{6}}\sin^2 x\,dx$ the integral with the smallest value.
10.	For the given identity to be true for **all** continuous function $f(x)$, this function must have an axis of symmetry at $x=2.5$, meaning that $\int_0^5 f(x)dx=\int_0^5 f(5-x)dx$.

Section II

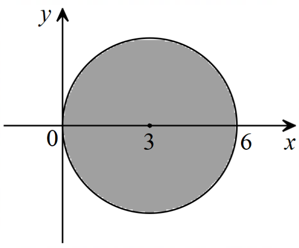
Question 11

Sample solution		Suggested marking criteria
(a)	Let the two consecutive pair of successive odd integers be $2n-1$ and $2n+1$ for some positive integer n. $\begin{aligned}(2n+1)^2-(2n-1)^2 &= \left[(2n+1)+(2n-1)\right]\left[(2n+1)-(2n-1)\right]\\ &= 4n\times 2\\ &= 8n,\ \text{which is divisible by 8 since } n \text{ is a positive integer.}\end{aligned}$	• 2 – correct proof • 1 – appropriately sets up the proof by algebraically defining two successive odd integers
(b)	(i) $\dfrac{4-2x}{(x-1)(x^2+1)} \equiv \dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$ $4-2x \equiv A(x^2+1)+(Bx+C)(x-1)$ Let $x=1$: $2=2A$ $1=A$ Let $x=0$: $4=A-C$ $4=1-C$ $C=-3$ Let $x=2$: $0=5A+2B+C$ $0=5+2B-3$ $2B=-2$ $B=-1$ $\dfrac{4-2x}{(x-1)(x^2+1)} \equiv \dfrac{1}{x-1}+\dfrac{-x-3}{x^2+1}$	• 2 – correct solution • 1 – evaluates one of the three coefficients
	(ii) $\begin{aligned}\int \frac{4-2x}{(x-1)(x^2+1)}\,dx &= \int \frac{1}{x-1}+\frac{-x-3}{x^2+1}\,dx\\ &= \int \frac{1}{x-1}\,dx-\int \frac{x}{x^2+1}\,dx-\int \frac{3}{x^2+1}\,dx\\ &= \ln\lvert x-1\rvert-\frac{1}{2}\ln\lvert x^2+1\rvert-3\tan^{-1}x+c\end{aligned}$	• 2 – correct answer • 1 – applies the partial fraction to the integrand and correctly integrates one of the fractions
(c)	$\begin{aligned}\int \sin^{-1}x\,dx &= x\sin^{-1}x-\int \frac{x}{\sqrt{1-x^2}}\,dx \quad \begin{pmatrix} u=\sin^{-1}x & v=x\\ du=\frac{dx}{\sqrt{1-x^2}} & dv=dx\end{pmatrix}\\ &= x\sin^{-1}x+\int \frac{-2x}{2\sqrt{1-x^2}}\,dx\\ &= x\sin^{-1}x+\sqrt{1-x^2}+c\end{aligned}$	• 2 – correct solution • 1 – uses integration by parts

Question 11 (continued)

<table>
<tr><th colspan="2">Sample solution</th><th>Suggested marking criteria</th></tr>
<tr><td>(d)</td><td>(i)
$\left|1+\sqrt{3}i\right| = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2$
$\arg\left(1+\sqrt{3}i\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$
$1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}$</td><td>• 2 – correct exponential form
• 1 – correct modulus or argument</td></tr>
<tr><td></td><td>(ii)
$\left(1+\sqrt{3}i\right)^{\frac{n}{2}} = \left(2e^{i\frac{\pi}{3}}\right)^{\frac{n}{2}} = 2^{\frac{n}{2}}e^{i\frac{n\pi}{6}}$
For this number to be purely imaginary,
$\frac{n\pi}{6} = \frac{\pi}{2}+k\pi$, where $k \in \mathbb{Z}$
$\frac{n}{6} = \frac{1}{2}+k$
$n = 3+6k = 3(2k+1)$
i.e. n must be an odd multiple of 3.</td><td>• 2 – correct condition on n
• 1 – find one correct value of n that meets the criteria</td></tr>
<tr><td>(e)</td><td>Let $S(n)$ be the statement that $1^2+2^2+3^2+\cdots+n^2 > \frac{n^3}{3}$.
<u>Show $S(1)$ is true:</u>
$\text{LHS} = 1^2 = 1 \quad \text{RHS} = \frac{1^3}{3} = \frac{1}{3}$
Since LHS > RHS, $S(1)$ is true.
<u>Assume $S(k)$ is true, i.e.:</u> $1^2+2^2+3^2+\cdots+k^2 > \frac{k^3}{3}$, for some integer $k \ge 1$.
<u>Show $S(k+1)$ is true, i.e.:</u> $1^2+2^2+3^2+\cdots+k^2+(k+1)^2 > \frac{(k+1)^3}{3}$
$\text{LHS} = 1^2+2^2+3^2+\cdots+k^2+(k+1)^2$
$> \frac{k^3}{3}+(k+1)^2$
$= \frac{k^3}{3}+\frac{3k^2+6k+3}{3}$
$= \frac{k^3+3k^2+3k+1}{3}+\frac{3k+2}{3}$
$= \frac{(k+1)^3}{3}+k+\frac{2}{3}$
$> \frac{(k+1)^3}{3}$, since $k \ge 1$
$= \text{RHS}$
$\therefore S(k+1)$ is true if $S(k)$ is assumed true.
Since $S(1)$ was shown true, by mathematical induction, $S(n)$ is true for all integers n, where $n \ge 1$.</td><td>• 3 – correct proof
• 2 – attempts to proof the result using the inductive assumption
• 1 – correctly shows the base case</td></tr>
</table>

Question 12

	Sample solution	Suggested marking criteria
(a)	(i) $(a+ib)^2 = 8+6i$ $(a^2-b^2)+2abi = 8+6i$ Equating real and imaginary parts: $a^2-b^2=8$ $2ab=6$ By inspection, $a=3, b=1$ and $a=-3, b=-1$.	• 2 – correct values for a and b • 1 – correctly finds one of the values of a and b
	(ii) $\Delta = [2(1+2i)]^2 - 4\times1\times[-(11+2i)]$ $= 4(1+4i-4)+4(11+2i)$ $= 4+16i-16+44+8i$ $= 32+24i$ $z = \dfrac{-2(1+2i)\pm\sqrt{32+24i}}{2\times1}$ $= \dfrac{-2-4i\pm2\sqrt{8+6i}}{2}$ $= -1-2i\pm(3+i)$ $= 2-i\ ,\ -4-3i$	• 2 – correct solution • 1 – applies an appropriate method to solve the complex quadratic equation
(b)	Let $z=x+iy$: $z\bar{z}-3(z+\bar{z})\le 0$ $(x+iy)(x-iy)-3[(x+iy)+(x-iy)]\le 0$ $x^2+y^2-3\times2x\le 0$ $x^2-6x+y^2\le 0$ $(x-3)^2+y^2\le 9$ 	• 2 – correct region • 1 – finds the Cartesian equation of the region and sketches a circular boundary
(c)	$\displaystyle\int_0^1 \frac{1}{2x^2-2x+1}\,dx = \int_0^1 \frac{2}{4x^2-4x+2}\,dx$ $\displaystyle= \int_0^1 \frac{2}{(2x-1)^2+1}\,dx$ $= \left[\tan^{-1}(2x-1)\right]_0^1$ $= \tan^{-1}1-\tan^{-1}(-1)$ $= \dfrac{\pi}{4}-\left(-\dfrac{\pi}{4}\right)$ $= \dfrac{\pi}{2}$	• 3 – correct solution • 2 – correctly integrates the integrand • 1 – correctly manipulates the integrand into a standard form

Question 12 (continued)

Sample solution		Suggested marking criteria
(d)	(i) $$\begin{aligned} I_0 &= \int_0^1 \frac{1}{x^2+1}\,dx \\ &= \left[\tan^{-1} x\right]_0^1 \\ &= \tan^{-1}1 - \tan^{-1}0 \\ &= \frac{\pi}{4} \end{aligned}$$	• 2 – correct solution • 1 – correctly integrates the integrand
	(ii) $$\begin{aligned} I_n &= \int_0^1 \frac{x^n}{x^2+1}\,dx \\ &= \int_0^1 \frac{x^{n-2}\left(x^2\right)}{x^2+1}\,dx \\ &= \int_0^1 \frac{x^{n-2}\left(x^2+1-1\right)}{x^2+1}\,dx \\ &= \int_0^1 \frac{x^{n-2}\left(x^2+1\right)}{x^2+1}\,dx - \int_0^1 \frac{x^{n-2}}{x^2+1}\,dx \\ &= \int_0^1 x^{n-2}\,dx - I_{n-2} \\ &= \left[\frac{x^{n-1}}{n-1}\right]_0^1 - I_{n-2} \\ &= \left(\frac{1^{n-1}}{n-1} - \frac{0^{n-1}}{n-1}\right) - I_{n-2} \\ &= \frac{1}{n-1} - I_{n-2} \end{aligned}$$	• 2 – correct solution • 1 – uses correct integration methods to attempt to reduce the integral
	(iii) $$\begin{aligned} \int_0^1 \frac{x^4}{x^2+1}\,dx &= I_4 \\ &= \frac{1}{3} - I_2 \\ &= \frac{1}{3} - \left(1 - I_0\right) \\ &= -\frac{2}{3} + \frac{\pi}{4} \end{aligned}$$	• 2 – correct answer • 1 – correctly applies the reduction formula once

Question 13

	Sample solution	Suggested marking criteria
(a)	$\displaystyle\int_0^1 \sqrt{\frac{1-x}{3+x}}\,dx = \int_0^1 \frac{1-x}{\sqrt{(3+x)(1-x)}}\,dx$ $\displaystyle= \frac{1}{2}\int_0^1 \frac{2-2x}{\sqrt{3-2x-x^2}}\,dx$ $\displaystyle= \frac{1}{2}\int_0^1 \frac{-2-2x}{\sqrt{3-2x-x^2}}\,dx + \frac{1}{2}\int_0^1 \frac{4}{\sqrt{3-2x-x^2}}\,dx$ $\displaystyle= \frac{1}{2}\int_0^1 \frac{-2-2x}{\sqrt{3-2x-x^2}}\,dx + 2\int_0^1 \frac{1}{\sqrt{4-(x+1)^2}}\,dx$ $\displaystyle= \frac{1}{2}\left[2\sqrt{3-2x-x^2}\right]_0^1 + 2\left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_0^1$ $\displaystyle= \frac{1}{2}\left(0-2\sqrt{3}\right) + 2\left[\sin^{-1}1 - \sin^{-1}\left(\frac{1}{2}\right)\right]$ $\displaystyle= -\sqrt{3} + 2\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$ $\displaystyle= \frac{2\pi}{3} - \sqrt{3}$	• 3 – correct solution • 2 – correctly integrates part of the integrand using a suitable technique • 1 – manipulates the integral into a combination of standard forms
(b)	$\displaystyle\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2-\cos x}\,dx = \int_0^1 \frac{3\sqrt{3}}{2-\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$ $\quad \left(\begin{array}{l} t=\tan\frac{x}{2} \Rightarrow dx = \frac{2dt}{1+t^2} \\ x=0 \Rightarrow t=0\ ,\ x=\frac{\pi}{2} \Rightarrow t=1 \end{array}\right)$ $\displaystyle= \int_0^1 \frac{6\sqrt{3}}{2(1+t^2)-(1-t^2)}\,dt$ $\displaystyle= \int_0^1 \frac{6\sqrt{3}}{1+3t^2}\,dt$ $\displaystyle= 6\int_0^1 \frac{\sqrt{3}}{1+\left(\sqrt{3}t\right)^2}\,dt$ $\displaystyle= 6 \times \left[\tan^{-1}\left(\sqrt{3}t\right)\right]_0^1$ $\displaystyle= 6\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right)$ $\displaystyle= 6 \times \frac{\pi}{3}$ $= 2\pi$	• 3 – correct solution • 2 – correctly manipulates the integrand into a standard form • 1 – correctly rewrites the integral in terms of *t*
(c) (i)	The contrapositive statement is *"If n is not a multiple of 3, then n^3 is not a multiple of 3."*	• 1 – correct contrapositive
(ii)	If *n* is not a multiple of 3, then $n = 3k \pm 1$ for some integer *k*. $n^3 = (3k \pm 1)^3$ $= (3k)^3 \pm 3(3k)^2 + 3(3k) \pm 1^3$ $= 3\left(9k^3 \pm 9k^2 + 3k\right) \pm 1$ $= 3\ell \pm 1$, where $\ell = 9k^3 \pm 9k^2 + 3k$ is an integer. Since n^3 is 1 more or 1 less than a multiple of 3, it cannot be a multiple of 3, therefore if *n* is not a multiple of 3, then n^3 is not a multiple of 3. By contraposition, statement *P* is true.	• 3 – correct proof • 2 – uses appropriate algebraic techniques towards the required result • 1 – correctly sets up proof by appropriately defining *n*

Question 13 (continued)

Sample solution		Suggested marking criteria
(d)	(i) $\left(\cos\theta + i\sin\theta\right)^5 = \cos^5\theta + 5\cos^4\theta\sin\theta i + 10\cos^3\theta\sin^2\theta i^2$ $= +10\cos^2\theta\sin^3\theta i^3 + 5\cos\theta\sin^4\theta i^4 + \sin^5\theta i^5$ $\cos 5\theta + i\sin 5\theta = \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right)$ $= +i\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)$ Equating real parts: $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)^2$ $= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$ $= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	• 2 – correct proof • 1 – applies de Moivre's theorem to express $\cos 5\theta + i\sin 5\theta$ as powers of $\cos\theta$ and $\sin\theta$
	(ii) $\cos 5\theta = 0$ $5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$ $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$ $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$ $\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right) = 0$ From $\cos 5\theta = 0$, $16\cos^4\theta - 20\cos^2\theta + 5 = 0$ has solutions $\cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$. By the quadratic formula, $\cos^2\theta = \frac{-(-20) \pm \sqrt{(-20)^2 - 4\times 16\times 5}}{2\times 16}$ $= \frac{20 \pm \sqrt{80}}{32}$ $= \frac{5 \pm \sqrt{5}}{8}$ Since $\cos^2\frac{\pi}{10} = \cos^2\frac{9\pi}{10}$ and $\cos^2\frac{3\pi}{10} = \cos^2\frac{7\pi}{10}$, with $\cos^2\frac{3\pi}{10} < \cos^2\frac{\pi}{10}$, therefore $\cos^2\frac{3\pi}{10} = \frac{5-\sqrt{5}}{8}$.	• 3 – correct proof • 2 – correctly deduces the solutions to the reducible quadratic equation from the solutions of $\cos 5\theta = 0$ and by applying the quadratic formula • 1 – correctly solves $\cos 5\theta = 0$

Question 14

<table>
<tr><th colspan="2">Sample solution</th><th>Suggested marking criteria</th></tr>
<tr><td>(a)</td><td>

Let $S(n)$ be the statement $t_n = \dfrac{2}{2n-1}$.

<u>Show $S(1)$ is true:</u>

$$\begin{aligned} \text{LHS} &= t_1 \\ &= 2 \end{aligned} \qquad \begin{aligned} \text{RHS} &= \frac{2}{2\times 1-1} \\ &= \frac{2}{1} \\ &= 2 \end{aligned}$$

Since LHS = RHS, $S(1)$ is true.

<u>Assume $S(k)$ is true, i.e.:</u> $t_k = \dfrac{2}{2k-1}$

<u>Show $S(k+1)$ is true, i.e.:</u> $t_{k+1} = \dfrac{2}{2(k+1)-1} = \dfrac{2}{2k+1}$

$$\begin{aligned} \text{LHS} &= t_{k+1} \\ &= \frac{t_k}{t_k+1} \\ &= \frac{\left(\frac{2}{2k-1}\right)}{\frac{2}{2k-1}+1} \\ &= \frac{\left(\frac{2}{2k-1}\right)}{\left(\frac{2+(2k-1)}{2k-1}\right)} \\ &= \frac{\left(\frac{2}{2k-1}\right)}{\left(\frac{2k+1}{2k-1}\right)} \\ &= \frac{2}{2k+1} \\ &= \text{RHS} \end{aligned}$$

$\therefore S(k+1)$ is true if $S(k)$ is assumed true.

Since $S(1)$ was shown true, by mathematical induction, $S(n)$ is true for all integers n, where $n \geq 1$.

</td><td>

- 3 – correct proof
- 2 – attempts to proof the result using the inductive assumption

1 – correctly shows the base case

</td></tr>
<tr><td>(b)</td><td>

(i)

$$\begin{aligned} (a-b)^2 &\geq 0 \\ a^2-2ab+b^2 &\geq 0 \\ a^2+b^2 &\geq 2ab \end{aligned}$$

</td><td>

- 1 – correct proof

</td></tr>
<tr><td></td><td>

(ii)

$$\begin{aligned} a^2+b^2 &\geq 2ab \\ a^2+c^2 &\geq 2ac \\ b^2+c^2 &\geq 2bc \quad + \\ \hline 2a^2+2b^2+2c^2 &\geq 2ab+2ac+2bc \\ 2a^2+2b^2+2c^2 &\geq (a+b+c)^2-a^2-b^2-c^2 \\ 3a^2+3b^2+3c^2 &\geq (a+b+c)^2 \\ \frac{3a^2+3b^2+3c^2}{9} &\geq \frac{(a+b+c)^2}{9} \\ \frac{a^2+b^2+c^2}{3} &\geq \left(\frac{a+b+c}{3}\right)^2 \end{aligned}$$

</td><td>

- 2 – correct proof
- 1 – uses the identity from (i) and appropriate algebraic techniques towards the required inequality

</td></tr>
</table>

Question 14 (continued)

Sample solution			Suggested marking criteria
(c)	(i)	$g < h$ $g + g < g + h$ $2g < g + h$ $g < \frac{g+h}{2}$ $g < h$ $g + h < h + h$ $g + h < 2h$ $\frac{g+h}{2} < h$ $g < \frac{g+h}{2} < h$	• 2 – correct proof • 1 – reconciles part of the compound inequality
	(ii)	Suppose g and h are both rational, i.e $g = \frac{a}{b}$ and $h = \frac{c}{d}$, where a, b, c and d are integers, with a and b coprime, c and d coprime and $b, d \neq 0$. $\frac{g+h}{2} = \frac{\frac{a}{b}+\frac{c}{d}}{2}$ $= \frac{ad+bc}{2bd}$ $ad + bc$ and $2bd$ are integers and $2bd \neq 0$ (since $b, d \neq 0$), therefore $\frac{g+h}{2}$ is also rational and from (i), lies between g and h, which are by definition rational. Iterating this process will continue to generate rational numbers between two rational numbers, thus there are infinitely many rational numbers.	• 2 – correct proof • 1 – justifies that $\frac{g+h}{2}$ is rational (algebraically or in words)
(d)	(i)	$0 \le x \le 1$ $1 \le x+1 \le 2$ $\frac{1}{2} \le \frac{1}{x+1} \le 1$ $\frac{1}{2}x^2(1-x)^2 \le \frac{x^2(1-x)^2}{x+1} \le x^2(1-x)^2 \quad \left(\text{since } x^2(1-x)^2 \ge 0\right)$ $\frac{1}{2}\int_0^1 x^2(1-x)^2\,dx < \int_0^1 \frac{x^2(1-x)^2}{1+x}\,dx < \int_0^1 x^2(1-x)^2\,dx$	• 2 – correct proof • 1 – establishes $\frac{1}{2} \le \frac{1}{x+1} \le 1$
	(ii)	$\int_0^1 \frac{x^2(1-x)^2}{1+x}\,dx$ $= \int_0^1 \frac{x^2\left(1-2x+x^2\right)}{1+x}\,dx$ $= \int_0^1 \frac{x^2-2x^3+x^4}{1+x}\,dx$ $= \int_0^1 \frac{(x+1)\left(x^3-3x^2+4x-4\right)+4}{1+x}\,dx$ $= \int_0^1 x^3-3x^2+4x-4+\frac{4}{1+x}\,dx$ $= \left[\frac{x^4}{4}-x^3+2x^2-4x+4\ln\lvert 1+x\rvert\right]_0^1$ $= \frac{1}{4}-1+2-4+4\log_e\lvert 1+1\rvert$ $= -\frac{11}{4}+4\log_e 2$ $\int_0^1 x^2(1-x)^2\,dx$ $= \int_0^1 x^2-2x^3+x^4\,dx$ $= \left[\frac{x^3}{3}-\frac{x^4}{2}+\frac{x^5}{5}\right]_0^1$ $= \frac{1}{3}-\frac{1}{2}+\frac{1}{5}$ $= \frac{1}{30}$ $\frac{1}{60} < -\frac{11}{4}+4\log_e 2 < \frac{1}{30}$ $\frac{1}{60}+\frac{11}{4} < 4\log_e 2 < \frac{1}{30}+\frac{11}{4}$ $\frac{83}{30} < 4\log_e 2 < \frac{167}{60}$ $\frac{83}{120} < \log_e 2 < \frac{167}{240}$ $\frac{166}{240} < \log_e 2 < \frac{167}{240}$	• 3 – correct proof • 2 – correctly evaluates $\int_0^1 \frac{x^2(1-x)^2}{1+x}\,dx$ • 1 – correctly evaluates $\int_0^1 x^2(1-x)^2\,dx$