2024 YEAR 12 ASSESSMENT TASK 2

Mathematics Extension 2

General Instructions	
	• Reading time – 10 minutes • Working time – 2 hours • Write using black pen. • Calculators approved by NESA may be used. • Answer all questions in the booklet provided. • In Questions 11 – 14, show relevant mathematical reasoning and/or calculations. • A **REFERENCE SHEET** is provided at the back of this paper

Total marks: 69	
	Section I – 10 marks (pages 2 – 5) • Attempt Questions 1 – 10 • Allow about 15 minutes for this section. **Section II – 59 marks** (pages 6 – 9) • Attempt Questions 11 – 14 • Allow about 1 hour 45 minutes for this section.

Section I
10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer page in your booklet to answer Questions 1 – 10.

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1) A student wants to prove that there is an infinite number of prime numbers. To prove this statement by contradiction, what assumption would the student start their proof with?

A) There is only one prime number that is even.

B) There is a finite number of prime numbers.

C) There are an infinite number of prime numbers.

D) All prime numbers are less than 100.

2) Which of the following is the antiderivative of xe^{2x}?

A) $\frac{1}{2}xe^{2x}-\frac{1}{2}e^{2x}+C$

B) $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$

C) $\frac{1}{2}x^2e^{2x}-\frac{1}{4}e^{2x}+C$

D) $\frac{1}{4}x^2e^{2x}+C$

3) Which of the following is equivalent to $-2e^{i\theta}$?

A) $2e^{i(\pi-\theta)}$

B) $2e^{-i\theta}$

C) $2e^{i(\theta-\pi)}$

D) $2e^{i\left(\frac{\pi}{2}-\theta\right)}$

4) A quadratic equation with real coefficients has complex roots α and β. What is the possible value of $\operatorname{Arg}(\alpha)+\operatorname{Arg}(\beta)$?

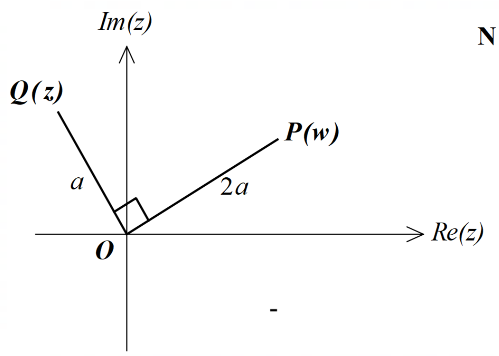
A) $\frac{\pi}{2}$

B) 0

C) π

D) $\frac{3\pi}{2}$

5) In the Argand diagram below the points P and Q represent the complex numbers w and z, respectively where $\angle POQ = 90°$. The distance OP is $2a$ units, and distance OQ is a units. Which of the following is correct?



NOT TO SCALE

A) $w = 2iz$

B) $w = -2iz$

C) $w = -\frac{iz}{2}$

D) $w = -\frac{z}{2i}$

6) Which of the following is equivalent to $\int_0^{\frac{\pi}{6}} \sec(2x)\tan(2x)dx$?

A) $\frac{1}{2}$

B) $-\frac{1}{2}$

C) $\frac{1}{4}$

D) $-\frac{1}{4}$

7) The complex number z is defined such that $|z| = |z-1-i|$. Which of the following describes the set of values Arg(z)?

A) $-\pi < \text{Arg}(z) < \pi$

B) $\text{Arg}(z) = \frac{-3\pi}{4}$ or $\text{Arg}(z) = \frac{\pi}{4}$

C) $-\frac{\pi}{2} < \text{Arg}(z) < \frac{\pi}{2}$

D) $-\frac{\pi}{4} < \text{Arg}(z) < \frac{3\pi}{4}$

8) What is the negation of the converse of the following statement?

"If Nathan practises mathematics daily, then he will be successful in the assessment."

A) Nathan is successful in the assessment, and he did not practise mathematics daily.

B) Nathan is unsuccessful in the assessment, and he did not practise mathematics daily.

C) If Nathan is successful in the assessment, then he did not practise mathematics daily.

D) If Nathan is unsuccessful in the assessment, then he did not practise mathematics daily.

9) Given $a^2 > b^2$, which of the following is always true?

A) $a+b>0$

B) $a>b$

C) $e^{|a|} > e^{|b|}$

D) $|a|+|b|>|a+b|$

10) What value of k will minimise the integral $I(k)=\int_0^1 \left(2x^2-k\right)^2 dx$?

A) $k=0$

B) $k=\frac{2}{3}$

C) $k=\frac{4}{3}$

D) $k=\frac{1}{3}$

End of Section I

Section II (60 marks)
Attempt Questions 11 - 14
Allow about 1 hour 45 minutes for this section

Answer each question on the appropriate pages of your answer booklet. Extra paper is available.
Your responses should include relevant calculations and/or mathematical reasoning.

Question 11 (15 marks)

a) Express $2e^{\frac{i\pi}{3}}$ in:

i) polar form. **1**

ii) Cartesian form. **1**

b) Prove the following statement is false: **2**

" $\exists$ a real number x such that $-x^2 + 2x - 5 \geq 0$."

c) Evaluate $\int_0^1 \frac{x^2}{1+x^6}\, dx$. **3**

d) Solve the equation $z^2 = |z|^2 - 4$. **2**

e) If m is a positive integer, prove by contradiction that $\sqrt{4m+6}$ is always irrational. **3**

f) Find $\int \frac{x^2 - 4x + 2}{(x-2)^3}\, dx$. **3**

End of Question 11

Question 12 (14 marks)

a) Prove that for positive integers a, b , if ab is not divisible by 5 then a is not divisible by 5 and b is not divisible by 5. **2**

b) If $|z| = 2$ and $w = \dfrac{z+3}{z}$, find the locus of w. **3**

c) Consider three positive real numbers x, y, z.

i) Prove that $x^2 + y^2 \geq 2xy$. **1**

ii) Hence, show that $x^2 + y^2 + z^2 \geq xy + yz + zx$. **1**

iii) If $x + y + z = 2$, deduce that $xy + yz + zx \leq \dfrac{4}{3}$. **2**

d) Find $\displaystyle\int \frac{dx}{\sqrt{3+4x-x^2}}$. **2**

e) Use the substitution $t = \tan\dfrac{x}{2}$ to find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$. **3**

End of Question 12

Question 13 (17 marks)

a) Consider the Statement P:

" If a number is divisible by 8, then the last three digits of a number is divisible by 8."

i) Write down the converse of statement P. **1**

ii) Show that the converse of statement P is true. **2**

b) Consider the complex number $z = \cos\theta + i\sin\theta$

i) Show that $\sin k\theta = \dfrac{z^k - z^{-k}}{2i}$ where k is an integer. **2**

ii) Show that $\cos^2\theta\sin^2\theta = -\dfrac{1}{16}\left(z^2 - \dfrac{1}{z^2}\right)^2$. **2**

iii) Use part ii) to express $\cos^2\theta\sin^2\theta$ in the form of $a + b\cos 4\theta$, where a and b are some constants. **2**

c) Show that $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{x(\pi - 2x)} = \frac{2}{\pi}\log_e 2$. **4**

d) Find $\displaystyle\int \sin^{-1}\sqrt{x}\,dx$. **4**

End of Question 13

Question 14 (13 marks)

a) i) Show that $\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}$, where n is positive integer and $n > 1$, **2**

ii) Hence show that for a positive integer x: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{x^2} < 2$ **2**

b) The terms of a sequence u_n is defined recursively by the formula $u_{n+1} = (1-x)u_n + nx$ for all integers $n \geq 1$, and $u_1 = 0$. Use the principle of mathematical induction to show that for all positive integers n, $u_n = \frac{1}{x}\left(nx - 1 + (1-x)^n\right)$. **3**

c) i) Show that $\frac{(1-t)^n}{t} = t^{n-1}\left(\frac{1}{t} - 1\right)^n$ **1**

ii) Given that $I_n = \int_1^x \frac{(1-t)^n}{t}\,dt$, where n = 1, 2, 3, …, show that $I_n = \frac{(1-x)^n}{n} + I_{n-1}$ **3**

iii) Hence, evaluate I_3 when $x = 2$ **2**

End of Exam

Year-12 Assessment Task 2, 2024 Mathematics Extension-2

Marking Guidelines

Section - I

Multiple Choice (1 – 10)

1	B	6	A
2	B	7	D
3	C	8	A
4	B	9	C
5	B	10	B

Sample Solutions

1) Contradiction : There is finite number of primes. (B)

2) $\int x e^{2x} dx = \frac{x e^{2x}}{2} - \int \frac{1}{2} e^{2x} dx$

$= \frac{x e^{2x}}{2} - \frac{1}{4} e^{2x} + C$ (B)

3) $-2e^{i\theta} = 2(-e^{i\theta})$

$= 2(-\cos\theta - i\sin\theta)$

$= 2(\cos(\theta - \pi) + i\sin(\theta - \pi))$

$= 2e^{i(\theta - \pi)}$ (C)

4) α and β are conjugate of each other

If $\alpha = re^{i\theta}$ then $\beta = re^{-i\theta}$

$\operatorname{Arg}\alpha + \operatorname{Arg}\beta = \theta + (-\theta) = 0$ (B)

5) $\vec{OQ} = \frac{1}{2} i \vec{OP}$

$z = \frac{1}{2} i w$

$w = \frac{2z}{i} \cdot \frac{i}{i} = -2iz$ (B)

6) $\int_0^{\pi/6} \sec 2x \tan 2x \, dx$

Let $2x = u$ $\quad x = 0 \Rightarrow u = 0$

$2dx = du$ $\quad x = \frac{\pi}{6} \Rightarrow u = \frac{\pi}{3}$

$dx = \frac{du}{2}$

$\therefore \frac{1}{2}\int_0^{\pi/3} \sec u \tan u \, du = \frac{1}{2}\left[\sec u\right]_0^{\pi/3}$

$= \frac{1}{2}\left[\sec\frac{\pi}{3} - \sec 0\right]$

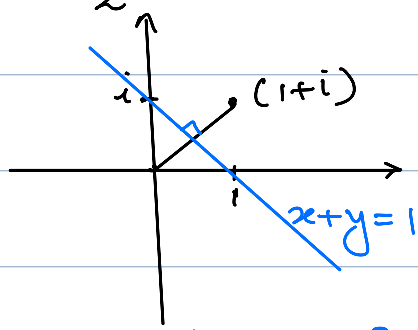
$= \frac{1}{2}\left[2 - 1\right] = \frac{1}{2}$ (A)

7) $|z| = |z - 1 - i|$

$x^2 + y^2 = (x-1)^2 + (y-1)^2$

$x^2 + y^2 = x^2 - 2x + 1 + y^2 - 2y + 2$

$x + y = 1$



$\therefore -\frac{\pi}{4} < \operatorname{Arg}(z) < \frac{3\pi}{4}$ (D)

8) Let P: Nathan practises mathematics daily.

Q: He will be successful in assessment.

Converse of $P \Rightarrow Q$ is $Q \Rightarrow P$

Negation of $Q \Rightarrow P$ or $\neg(Q \Rightarrow P) \Leftrightarrow Q \wedge \neg P$

$\therefore$ Nathan is successful in assessment, and he did not practise mathematics daily. (A)

9) $a^2 > b^2 \Rightarrow a^2 - b^2 > 0$

$(a-b)(a+b) > 0$

$\therefore$ $a+b>0$ and $a>b$ OR $a+b<0$ and $a<b$.

$\therefore$ Can't be 'A' or 'B'.

Can't be 'D' as $a, b \not> 0$.

However $a^2 > b^2 \Rightarrow |a| > |b| \Rightarrow e^{|a|} > e^{|b|}$ C

10) $I(K) = \int_0^1 (2x^2 - K)^2 \, dx$

Differentiate both sides w.r.t K.

$$\frac{dI}{dK} = \frac{d}{dK} \int_0^1 (2x^2 - K)^2 \, dx$$

Using the fundamental theorem of Calculus:

$$\frac{dI}{dK} = \int_0^1 \frac{d}{dK} (2x^2 - K)^2 \, dx$$

$$= \int_0^1 -2(2x^2 - K) \, dx$$

$$= -2 \int_0^1 2x^2 - K \, dx$$

$$= -2 \left[\frac{2x^3}{3} - Kx \right]_0^1$$

$$= -2\left(\frac{2}{3} - K\right)$$

For maximum/minimum value equate $\frac{dI}{dK} = 0$

$-2\left(\frac{2}{3} - K\right) = 0 \Rightarrow K = \frac{2}{3}$

To verify whether $K = \frac{2}{3}$ is minimum or maximum value, test second derivative of $\frac{dI}{dK}$

$$\frac{d^2 I}{dK^2} = 2 > 0 \quad \therefore \text{ Concave up.}$$

Hence $K = \frac{2}{3}$ is minimum value (B)

(OR)

$$I(K) = \int_0^1 (2x^2 - K)^2 \, dx \quad \text{——— (1)}$$

$$= \int_0^1 (4x^4 - 4Kx^2 + K^2) \, dx$$

$$= \left[\frac{4x^5}{5} - \frac{4Kx^3}{3} + K^2 x \right]_0^1$$

$$= \frac{4}{5} - \frac{4K}{3} + K^2$$

Different $I(K)$ w.r.t 'K'

$$\frac{dI}{dK} = -\frac{4}{3} + 2K$$

To minimise ; $\frac{dI}{dK} = 0$

$$-\frac{4}{3} + 2K = 0$$

$$2K = \frac{4}{3}$$

$$K = \frac{2}{3}$$

To verify $K = \frac{2}{3}$ is minimum value, either different $\frac{dI}{dK}$ again w.r.t or substitute $K = \frac{2}{3}$ in $I(K)$ to check whether it is minimum or not.

$\frac{d^2 I}{dK^2} = 2 > 0 \quad \therefore$ Concave up, hence $K = \frac{2}{3}$ is minimum value of given integral.

END OF SECTION I

Question 11

a) i) $2e^{i\pi/3} = 2e^{i\pi/3} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

1 – Correct solution

ii) $2(\frac{1}{2} + i\frac{\sqrt{3}}{2})$ or $1 + i\sqrt{3}$

1 – Correct solution.

b) $\exists x \in \mathbb{R}$ s.t $-x^2+2x-5 \geq 0$

Let $\exists x \in \mathbb{R}$ s.t $-x^2+2x-5 \geq 0$

2 – correct proof

$$-(x^2-2x+1)-4 \geq 0$$

1 – applies an appropriate method to prove that the given statement is false.

$$-(x-1)^2-4 \geq 0$$

$$-[(x-1)^2+4] \geq 0$$

which is false as $(x-1)^2 \geq 0 \ \forall x \in \mathbb{R}$

c) $\displaystyle\int_0^1 \frac{x^2}{1+x^6}dx = \int_0^1 \frac{x^2}{1+(x^3)^2}dx$

Let $x^3 = u$; $x=0 \Rightarrow u=0$ & $x=1 \Rightarrow u=1$

$$3x^2dx = du$$

$$x^2dx = \frac{du}{3}$$

$$\therefore \frac{1}{3}\int_0^1 \frac{du}{1+u^2} = \frac{1}{3}\left[\tan^{-1}u\right]_0^1 = \frac{1}{3}(\tan^{-1}1 - \tan^{-1}0) = \frac{\pi}{12}$$

3 – correct solution

2 – correctly integrate the integral in term of the substituted variable.

1 – rewrite the integral into an appropriate form by using substitution.

d) Let $z = x + iy$

$$z^2 = |z|^2 - 4$$

2 - Correct solution.

$$(x+iy)^2 = x^2 + y^2 - 4$$

1 - Create an equation by expressing $z = x+iy$

$$x^2 - y^2 + 2ixy = x^2 + y^2 - 4$$

$$-y^2 + 2ixy = y^2 - 4$$

Equating real and imaginary parts:

$$-y^2 = y^2 - 4 \quad , \quad 2xy = 0$$

$$2y^2 = 4 \qquad x = 0 \quad (\because y \neq 0)$$

$$y^2 = 2$$

$$y = \pm\sqrt{2}$$

$$\therefore z = \pm\sqrt{2}\, i$$

e) Let $\sqrt{4m+6}$ is rational

$\therefore \sqrt{4m+6} = \frac{p}{q}$, $p, q \in \mathbb{Z}^+$ and p, q are coprime.

$$4m + 6 = \frac{p^2}{q^2}$$

3 - Correct proof

$$2(2m+3) = \frac{p^2}{q^2}$$

2 - Correctly express either 'p' or 'q' as even number by using definition (or) progress towards the proof by showing 'p' and 'q' have common factor.

$$p^2 = 2(2m+3)q^2 \quad \text{—— (1)}$$

$\therefore p^2$ is even $\Rightarrow p$ is even

1 - correctly sets up the proof by contradiction with correct definition.

Let $p = 2k$, Sub in (1)

$$(2k)^2 = 2(2m+3)q^2$$

$$2k^2 = (2m+3)q^2$$

Since $(2m+3)$ is odd, $\therefore q^2 =$ even $\Rightarrow q$ is even

which contradict the assumption of p, q have no common factor.

$\therefore \sqrt{4m+6}$ is irrational.

f)
$$\int \frac{x^2-4x+2}{(x-2)^3}\,dx$$

$$\int \frac{x^2-4x+4-2}{(x-2)^3}\,dx$$

$$\int \frac{(x-2)^2-2}{(x-2)^3}\,dx$$

$$\int \frac{1}{x-2} - \frac{2}{(x-2)^3}\,dx$$

$$\int \frac{1}{x-2}\,dx - 2\int (x-2)^{-3}\,dx$$

$$\ln|x-2| - \frac{2(x-2)^{-2}}{(-2)} + C$$

$$\ln|x-2| + \frac{1}{(x-2)^2} + C$$

3 – correct solution (must have absolute value)

2 – correctly integrate one of the fraction which was decomposed from original fraction.

1 – Correctly express a single fraction into two fractions.

Question - 12

a) P: ab is not divisible by 5.

Q: a is not divisible by 5 and b is not divisible by 5.

$$P \Rightarrow Q \iff \neg Q \Rightarrow \neg P.$$

Given statement can be proved by contraposition.

Contraposition statement:

If a or b is divisible by 5 then ab is divisible by 5.

Proof: Let a is divisible by 5

$a = 5j$, where $j \in \mathbb{Z}^+$

$ab = 5jb$

$= 5(jb)$

$= 5q$, where $q \in \mathbb{Z}^+$ as j, b are integers.

$\therefore$ ab is divisible by 5.

Hence the given statement is proved by contraposition.

2 - correct proof.

1 - Correct contraposition statement

b) $w = \frac{z+3}{z}$

$wz = z + 3$

$wz - z = 3$

$z(w-1) = 3$

$z = \frac{3}{w-1}$

Since $|z| = 2$

$\therefore \left|\frac{3}{w-1}\right| = 2$

$\frac{3}{|w-1|} = 2$

$|w-1| = \frac{3}{2}$

$\therefore$ Locus of w is a circle of centre $(1,0)$ and radius $\frac{3}{2}$.

3 – Correct solution

2 – Expressing $|w-1| = \frac{3}{2}$ or correct approach to find the locus by using $|z| = 2$

1 – correctly manipulating the given expression of w and making z as subject. 'OR' equivalent approach to find the expression for z.

c) i) $(x-y)^2 \geqslant 0$

$x^2 - 2xy + y^2 \geqslant 0$

$x^2 + y^2 \geqslant 2xy$.

1 – correct proof.

ii) $x^2 + y^2 \geqslant 2xy$

$y^2 + z^2 \geqslant 2yz$

$z^2 + x^2 \geqslant 2zx$

1 – correct proof.

Add them; $2(x^2+y^2+z^2) \geqslant 2(xy+yz+zx)$

$x^2+y^2+z^2 \geqslant xy+yz+zx$

iii) $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$

$\geq (xy+yz+zx)+2(xy+yz+zx)$ (from ii)

$= 3(xy+yz+zx)$

$2^2 \geq 3(xy+yz+zx) \qquad (\because x+y+z=2)$

$\therefore \quad (xy+yz+zx) \leq \frac{4}{3}$

2 – Correct solution.

1 – correctly using the inequality in part (ii) (or) finding expression for $(x+y+z)^2$

d) $\int \frac{dx}{\sqrt{3+4x-x^2}}$

2 – Correct expression.

$\int \frac{dx}{\sqrt{7-4+4x-x^2}}$

1 – correctly completing the square in the denominator.

$\int \frac{dx}{\sqrt{7-(x-2)^2}}$

$\sin^{-1}\frac{(x-2)}{\sqrt{7}} + C$

e) $\displaystyle\int_0^{\pi/2} \frac{dx}{2+\cos x}$

$t = \tan\frac{x}{2}$

$dx = \frac{2dt}{1+t^2}$

$x=0 \Rightarrow t=0 \;;\; x=\frac{\pi}{2} \Rightarrow t=1$

$\displaystyle\therefore \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$

$\displaystyle\int_0^1 \frac{2dt}{2+2t^2+1-t^2}$

$\displaystyle\int_0^1 \frac{2\,dt}{3+t^2}$

$\displaystyle 2\int_0^1 \frac{dt}{(\sqrt{3})^2+t^2}$

$\displaystyle \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$\displaystyle \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right)$

$\displaystyle \frac{2}{\sqrt{3}} \times \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$

3 – correct solution

2 – Correctly find the integral of $\frac{2dt}{3+t^2}$.

1 – Correctly rewrites the integral in terms of t.

Question-13

a) i) Converse of Statement P is: 1- Correct statement

"If last three digits of a number is divisible by 8 then a number is divisible by 8."

ii) Let the number be N where the last three digits are abc

i.e. $N = \ldots abc$

$\therefore N = 1000 \times Q + abc \quad , \quad Q \in \mathbb{Z}.$

Since abc is divisible by 8, therefore $abc = 8T, \; T \in \mathbb{Z}$

$\therefore N = 1000Q + 8T$

$= 8(125Q + T)$

$= 8P$ where $P = 125Q + T$ is an integer.

Hence number N is divisible by 8.

Therefore the Converse of statement P is true.

2 - Correct proof

1 - correct progress towards the proof by considering the last three digits of a number is multiple of 8 or equivalent progress towards the proof.

b) i) $z = \cos\theta + i\sin\theta$

$$z^k = (\cos\theta + i\sin\theta)^k$$

$$= \cos k\theta + i\sin k\theta \quad \text{—— (1)}$$

$$\frac{1}{z^k} = z^{-k} = (\cos\theta + i\sin\theta)^{-k}$$

$$= \cos(-k\theta) + i\sin(-k\theta)$$

$$= \cos k\theta - i\sin k\theta \quad \text{—— (2)}$$

Add (1) & (2)

$$z^k + \frac{1}{z^k} = 2\cos k\theta$$

$$\therefore \cos k\theta = \frac{1}{2}\left(z^k + \frac{1}{z^k}\right)$$

Subtract (1) & (2)

$$z^k - \frac{1}{z^k} = 2i\sin k\theta$$

$$\sin k\theta = \frac{1}{2i}\left(z^k - \frac{1}{z^k}\right)$$

2 – Correct expression for $\sin k\theta$.

1 – Correct expression for z^k or z^{-k} in term of Cisθ form or equivalent progress

ii) $\cos^2\theta\sin^2\theta$

$$= (\cos\theta\sin\theta)^2$$

$$= \left[\frac{1}{2}\left(z + \frac{1}{z}\right) \cdot \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^2$$

$$= \left[\frac{1}{4i}\left(z^2 - \frac{1}{z^2}\right)\right]^2$$

$$= -\frac{1}{16}\left(z^2 - \frac{1}{z^2}\right)^2$$

2 - Correct proof

1 – Correct progress towards the proof.

iii) $\cos^2\theta\sin^2\theta = -\frac{1}{16}(2i\sin 2\theta)^2$

$$= \frac{1}{16}(4\sin^2 2\theta)$$

$$= \frac{1}{8}(1 - \cos 4\theta)$$

$$= \frac{1}{8} - \frac{1}{8}\cos 4\theta$$

$$\therefore a = \frac{1}{8} \text{ and } b = -\frac{1}{8}$$

2 – Correct solution

1 – Using part (ii) to express $z^2 - \frac{1}{z^2} = 2i\sin 2\theta$ or equivalent progress

c) Let $\dfrac{1}{x(\pi-2x)} = \dfrac{A}{x} + \dfrac{B}{\pi-2x}$

$$1 = (\pi-2x)A + Bx$$

Sub $x=0$; $1 = \pi A \Rightarrow A = \frac{1}{\pi}$

Sub $x=1$; $1 = (\pi-2)A + B$

$$1 = 1 - \frac{2}{\pi} + B \quad \left(\because A = \frac{1}{\pi}\right)$$

$$B = \frac{2}{\pi}$$

$$\int_{\pi/6}^{\pi/3} \frac{dx}{x(\pi-2x)} = \int_{\pi/6}^{\pi/3} \frac{1}{\pi x} + \frac{2}{\pi(\pi-2x)}\, dx$$

$$= \left[\frac{1}{\pi}\ln|x| + \frac{2}{\pi}\frac{\ln|\pi-2x|}{(-2)}\right]_{\pi/6}^{\pi/3}$$

$$= \left[\frac{1}{\pi}\ln|x| - \frac{1}{\pi}\ln|\pi-2x|\right]_{\pi/6}^{\pi/3}$$

$$= \left[\frac{1}{\pi}\ln\left|\frac{x}{\pi-2x}\right|\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{\pi}\left[\ln\frac{\pi/3}{\pi-2\pi/3} - \ln\frac{\pi/6}{\pi-\pi/3}\right]$$

$$= \frac{1}{\pi}\left[\ln\frac{\pi/3}{\pi/3} - \ln\frac{\pi/6}{2\pi/3}\right]$$

$$= \frac{1}{\pi}\left[\ln 1 - \ln\frac{1}{4}\right]$$

$$= \frac{1}{\pi}\left(0 - \ln 4^{-1}\right)$$

$$= \frac{1}{\pi}\ln 4$$

$$= \frac{1}{\pi}\ln 2^2 = \frac{2}{\pi}\ln 2$$

4 - Correct proof

3 - Correct integral of both expression and progress towards the substitution.

2 - Correct integral of one expression or equivalent progress.

1 - Decompose the given fraction into two fractions.

ii) $\int \sin^{-1}\sqrt{x}\, dx$

4 - Correct expression

Let $\sqrt{x} = \sin t$

3 - Correctly integrates and leave the answer in substitution form (in terms of t or u)

$x = \sin^2 t$

$dx = 2\sin t \cos t \, dt$

2 - Using integration by parts

$= \sin 2t \, dt$

1 - appropriate substitution or similar progress.

$\therefore \int \sin^{-1}\sqrt{x}\, dx = \int t \sin 2t \, dt$

$u = t \qquad dv = \sin 2t$

$du = dt \qquad v = -\dfrac{\cos 2t}{2}$

$= -\dfrac{t\cos 2t}{2} - \int 1\left(-\dfrac{\cos 2t}{2}\right) dt$

$= -\frac{1}{2} t \cos 2t + \frac{1}{2}\int \cos 2t \, dt$

$= -\frac{1}{2} t \cos 2t + \frac{1}{4}\sin 2t + C$

$= -\frac{1}{2} t (1 - 2\sin^2 t) + \frac{1}{4} \cdot 2\sin t \cos t + C$

$= -\frac{1}{2}\sin^{-1}\sqrt{x}\,(1 - 2x) + \frac{1}{2}\sin t\sqrt{1-\sin^2 t} + C$

$= -\frac{1}{2}\sin^{-1}\sqrt{x} + x\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x}\sqrt{1-x} + C$

(OR) $\int \sin^{-1}\sqrt{x}\, dx$

$u = \sin^{-1}\sqrt{x}$ $\quad v' = 1$ $\quad$ (Integration by parts)

$u' = \dfrac{1}{\sqrt{1-x}} \cdot \dfrac{1}{2\sqrt{x}}$ $\quad v = x$

$$\int \sin^{-1}\sqrt{x}\, dx = x\sin^{-1}\sqrt{x} - \int \frac{x}{2\sqrt{x}\sqrt{1-x}}\, dx$$

$$= x\sin^{-1}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{\sqrt{1-x}}\, dx \quad \text{——— (1)}$$

Let $I = \displaystyle\int \frac{\sqrt{x}}{\sqrt{1-x}}\, dx$

Sub $x = \sin^2\theta \implies \sqrt{x} = \sin\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$$I = \int \frac{\sqrt{\sin^2\theta}}{\sqrt{1-\sin^2\theta}}(2\sin\theta\cos\theta)\, d\theta$$

$$= \int \frac{2\sin^2\theta\cos\theta\, d\theta}{\cos\theta}$$

$$= \int 2\sin^2\theta\, d\theta$$

$$= \int (1-\cos 2\theta)\, d\theta$$

$$= \theta - \frac{\sin 2\theta}{2} + C$$

$$= \sin^{-1}\sqrt{x} - \frac{\cancel{2}\sqrt{x}\sqrt{1-x}}{\cancel{2}} + C$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta$
$= 2\sqrt{x}\sqrt{1-x}\,]$

Sub it back in (1)

$$\int \sin^{-1}\sqrt{x}\, dx = x\sin^{-1}\sqrt{x} - \frac{1}{2}\left(\sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x}\right) + C$$

4 - Correct expression

3 - Correctly integrates and leave the answer in terms of substitution form.

2 - Progress to integrate $\int \frac{\sqrt{x}}{\sqrt{1-x}}$ by using substitution

1 - Correctly using integration by parts

Question - 14

a) i) Since $n > 1$

$$n > n-1$$

$$n^2 > n(n-1)$$

$$\therefore \frac{1}{n^2} < \frac{1}{n(n-1)} \quad (\because n > 1)$$

$$= \frac{1}{n-1} - \frac{1}{n}$$

Hence $\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}$

2 - Correct proof

1 - Correct progress towards the required proof.

ii) Since $\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}$

Sub $n=2$; $\frac{1}{2^2} < 1 - \frac{1}{2}$

$n=3$; $\frac{1}{3^2} < \frac{1}{2} - \frac{1}{3}$

$\vdots$

$n=x$ $\frac{1}{x^2} < \frac{1}{x-1} - \frac{1}{x}$

2 - Correct proof

1 - Correct approach towards the proof by using the inequality given in previous part.

Add them:

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{x^2} < 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{x-1}} - \frac{1}{x}$$

Add 1 on both sides

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{x^2} < 1 + 1 - \frac{1}{x}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{x^2} < 2 - \frac{1}{x} < 2 \qquad \left(\because x > 1 \;\; \therefore \frac{1}{x} < 1\right)$$

Hence proved.

b) Prove true for $n=1$.

$L.H.S = u_1 = 0$ $\quad$ $R.H.S = \frac{1}{x}\left[x - 1 + (1-x)^1\right]$

$= \frac{1}{x}\left[\cancel{x} - \cancel{1} + \cancel{1} - \cancel{x}\right] = 0$

$\therefore$ True for $n=1$.

Assume true for $n=k$.

i.e. $u_k = \frac{1}{x}\left[kx - 1 + (1-x)^k\right], \quad k \geq 1$

Prove true for $n=k+1$.

i.e. $u_{k+1} = \frac{1}{x}\left[(k+1)x - 1 + (1-x)^{k+1}\right]$

Proof: $L.H.S = u_{k+1}$

$= (1-x)u_k + kx$ (using recursive formula)

$= (1-x)\cdot\frac{1}{x}\left[kx - 1 + (1-x)^k\right] + kx$ (using assumption)

$= \frac{1}{x}\left[(1-x)kx - (1-x) + (1-x)^{k+1} + kx^2\right]$

$= \frac{1}{x}\left[kx - \cancel{kx^2} - 1 + x + (1-x)^{k+1} + \cancel{kx^2}\right]$

$= \frac{1}{x}\left[(k+1)x - 1 + (1-x)^{k+1}\right]$

$= R.H.S$

$\therefore$ True for $n=k+1$ if it is assumed to be true for $n=k$.

Since $n=1$ is shown true, by the principle of induction, it is true for all positive integers n.

3 - Correct Solution

2 - attempts to use the assumption to prove the required result.

1 - Correctly proves the base case.

c) i) $\frac{(1-t)^n}{t} = \frac{1}{t}\left[t\left(\frac{1}{t}-1\right)\right]^n$ 1 – correct proof

$$= \frac{1}{t}\cdot t^n\left(\frac{1}{t}-1\right)^n$$

$$= t^{n-1}\left(\frac{1}{t}-1\right)^n$$

ii) $I_n = \int_1^x \frac{(1-t)^n}{t}\,dt$

3 – Correct solution

2 – Correct substitution and establishes reduction formula

$$= \int_1^x t^{n-1}\left(\frac{1}{t}-1\right)^n dt$$

Integrate by parts:

1 – Correct integration by parts

$u = \left(\frac{1}{t}-1\right)^n$ $\quad dv = t^{n-1}$

$du = n\left(\frac{1}{t}-1\right)^{n-1}\left(-\frac{1}{t^2}\right)$ $\quad v = \frac{t^n}{n}$

$$\therefore I_n = \left[\frac{t^n}{n}\left(\frac{1}{t}-1\right)^n\right]_1^x - \int_1^x n\left(\frac{1}{t}-1\right)^{n-1}\left(-\frac{1}{t^2}\right)\frac{t^n}{n}\,dt$$

$$= \left[\frac{x^n}{n}\left(\frac{1}{x}-1\right)^n - \frac{1}{n}(1-1)^n\right] + \int_1^x t^{n-2}\left(\frac{1}{t}-1\right)^{n-1} dt$$

$$= \frac{x^n}{n}\left(\frac{1}{x}-1\right)^n + I_{n-1}$$

$$= \frac{x^n(1-x)^n}{x^n\cdot n} + I_{n-1}$$

$$= \frac{(1-x)^n}{n} + I_{n-1}$$

iii) When $x = 2$;

$$I_3 = \frac{(1-2)^3}{3} + I_2 = -\frac{1}{3} + I_2$$

2 – Correct value of I_3.

$$I_2 = \frac{(1-2)^2}{2} + I_1 = \frac{1}{2} + I_1$$

1 – Correctly applies the reduction formula once. (or) Correctly find I_0.

$$I_1 = \frac{(1-2)^1}{1} + I_0 = -1 + I_0$$

$$\therefore I_3 = -\frac{1}{3} + \frac{1}{2} - 1 + I_0$$

$$= -\frac{5}{6} + I_0$$

$$I_0 = \int_1^2 \frac{1}{t}\,dt = \left[\ln t\right]_1^2 = \ln 2 - \ln 1 = \ln 2$$

$$\therefore I_3 = -\frac{5}{6} + \ln 2.$$

END OF EXAM