Caringbah High School

Year 12 2022
Mathematics Extension 2
HSC Course
Assessment Task 1

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- NESA-approved calculators may be used
- A reference sheet is provided
- In Questions 6-9, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 40

Section I **5 marks**
Attempt Questions 1-5
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **35 marks**
Attempt Questions 6-9
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ____ **Class:** ____________

Marker's Use Only						
Section I	**Section II**				**Total**	
Q 1-5	**Q6**	**Q7**	**Q8**	**Q9**		
/5	/9	/8	/9	/9	/40	%

Section I

5 marks
Attempt Questions 1-5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5

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1. What is $-1+i\sqrt{3}$ in modulus-argument form?

(A) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

(B) $\sqrt{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

(C) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

(D) $\sqrt{2}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

2. Consider the statement:
P: For every integer x, there exists an integer y such that $y^2 > x$.
Which of the following statements is $\neg P$?

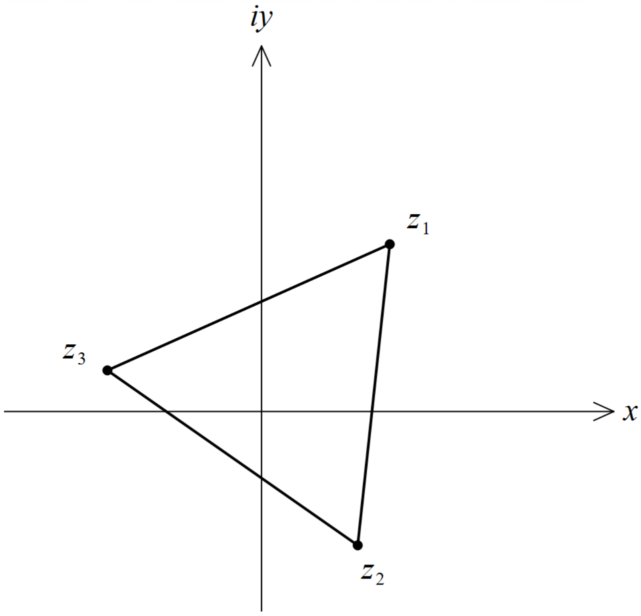
(A) $\exists x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ y^2 > x$

(B) $\exists x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ y^2 \leq x$

(C) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ y^2 > x$

(D) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ y^2 \leq x$

3. Let z_1, z_2 and z_3 form an equilateral triangle as shown below.



Which of the following statements is true?

(A) $\dfrac{z_2 - z_1}{z_1 - z_3} = \text{cis}\dfrac{\pi}{6}$

(B) $\dfrac{z_2 - z_1}{z_3 - z_1} = \text{cis}\dfrac{\pi}{6}$

(C) $\dfrac{z_2 - z_1}{z_1 - z_3} = \text{cis}\dfrac{\pi}{3}$

(D) $\dfrac{z_2 - z_1}{z_3 - z_1} = \text{cis}\dfrac{\pi}{3}$

4. $z = x + iy$ is a complex number, such that it is NOT purely imaginary.

The equation of the locus of z such that $\text{Re}\left(z - \dfrac{1}{z}\right) = 0$ is:

(A) $x = 0$

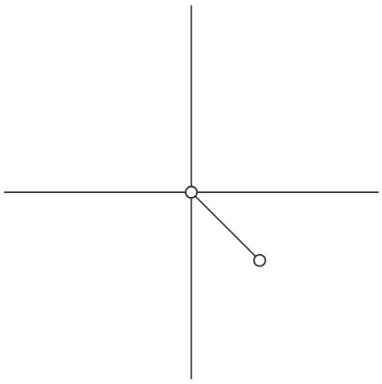
(B) $y = 0$

(C) $x^2 + y^2 = 1$

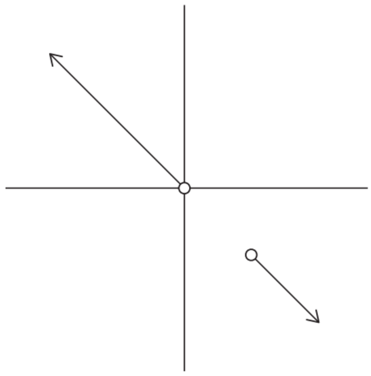
(D) $x + y - \dfrac{1}{x+y} = 0$

5. Which diagram best represents the solutions to the equation $\arg(z) = \arg(z+1-i)$?

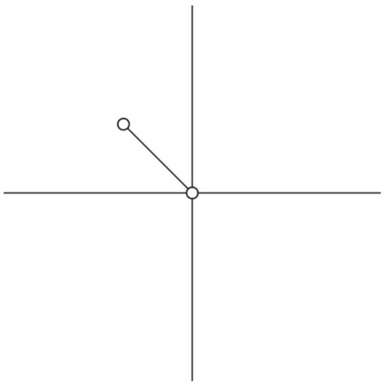
(A)



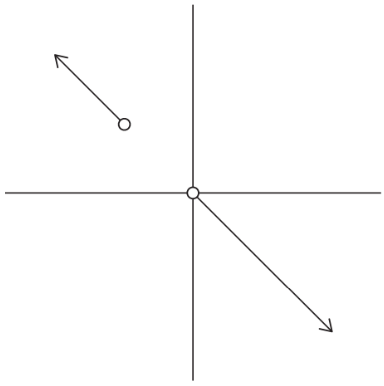
(B)



(C)



(D)



End of Section I

Section II

35 marks
Attempt Questions 6–9
Allow about 52 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (9 marks) Use a SEPARATE writing booklet.

(a) Given two complex numbers, $z = 2 + 5i$ and $w = -1 + 2i$, find:

(i) $z - 2w$ **1**

(ii) zw **1**

(iii) $\overline{z + iw}$ **2**

(b) It is given that $P(z) = z^3 - 4z^2 + 14z - 20$ has a root at $z = 1 + 3i$. **3**
Factorise $P(z)$ fully over $\mathbb{R}$.

(c) Prove by contradiction that $\log_3 5$ is irrational. **2**

Question 7 (8 marks) Use a SEPARATE writing booklet.

(a) Solve the equation $z^2 - (1 - 4i)z - 5 + i = 0$, leaving your answers in the form $a + ib$ where a and b are real. **3**

(b) **(i)** Sketch the following on the same Argand diagram: **3**

$$|z - 1 - 2i| = 2 \quad \text{and} \quad \arg(z - 1 - 2i) = \frac{\pi}{3}$$

(ii) Find the complex number which represents the intersection of these two equations. **2**

Question 8 (9 marks) Use a SEPARATE writing booklet.

(a) Prove by contrapositive that if $3n+7$ is even, then n is odd. **2**

(b) Let $z = 3\text{cis}\dfrac{\pi}{6}$ and $w = 2\text{cis}\dfrac{\pi}{4}$.

(i) Express z and w in Cartesian form. **2**

(ii) Hence, express $\dfrac{w}{z}$ in both modulus-argument and Cartesian form. **3**

(iii) Hence, find the exact values of $\cos\dfrac{\pi}{12}$ and $\sin\dfrac{\pi}{12}$. **2**

Question 9 (9 marks) Use a SEPARATE writing booklet.

(a) On an Argand diagram, the points A, B, C and D represent the complex numbers $\alpha, \beta, \gamma,$ and δ respectively.
If $\alpha+\gamma=\beta+\delta$, deduce that $ABCD$ is a parallelogram. **2**

(b) Consider the following statement:

If n divides $(a+b)$ and n divides $(b+c)$, then n divides $(a+c)$.

Prove or disprove this statement. **2**

(c) **(i)** On an Argand diagram, sketch the curve defined by $\arg\left(\dfrac{z+2}{z-2}\right)=\dfrac{\pi}{3}$ **2**

(ii) Determine the Cartesian equation of this locus. **3**

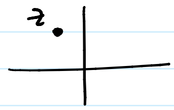
End of Paper

Mathematics Extension 2 AT1 Solutions

Section 1

1. $-1+i\sqrt{3} = z$

$|z| = \sqrt{(-1)^2+(\sqrt{3})^2}$
$= \sqrt{4}$
$= 2$

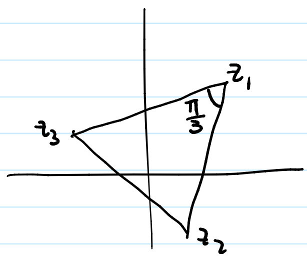


$\arg(z) = \tan^{-1}(\sqrt{3}/-1)$
$= \frac{2\pi}{3}$

$\therefore -1+i\sqrt{3} = 2\,\text{cis}\,2\pi/3$ (C) ✓

2. $P: \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, y^2 > x$
$\neg P: \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, y^2 \leq x$ (B) ✓

3.



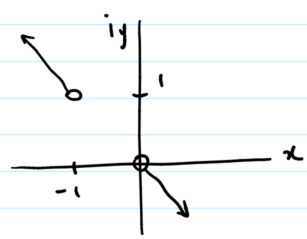
(D) ✓

4. $z = x+iy \quad (x \neq 0)$

$z - \frac{1}{z} = x+iy - \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$
$= x+iy - \frac{x-iy}{x^2+y^2}$
$= x - \frac{x}{x^2+y^2} + i\left(y+\frac{y}{x^2+y^2}\right)$

$\text{Re}(z-\frac{1}{z}) = 0$
$x - \frac{x}{x^2+y^2} = 0$
$x = \frac{x}{x^2+y^2}$
$1 = \frac{1}{x^2+y^2} \quad (x \neq 0)$
$\therefore x^2+y^2 = 1$ (C) ✓

5. $\arg(z) = \arg(z+1-i)$
$= \arg(z-(-1+i))$



(D) ✓

Section 2

Question 6

(a) $z = 2+5i$, $w = -1+2i$

(i) $z - 2w = 2+5i - 2(-1+2i)$
$= 2+5i+2-4i$
$= 4+i$

(ii) $zw = (2+5i)(-1+2i)$
$= -2+4i-5i-10$
$= -12-i$

(iii) $\overline{z+iw} = \overline{2+5i+i(-1+2i)}$
$= \overline{2+5i-i-2}$
$= \overline{4i}$
$= -4i$

(b) $P(z) = z^3 - 4z^2 + 14z - 20$, root $z = 1+3i$

Since $P(z)$ has real coefficients, $\bar{z} = 1-3i$ is also a root
Then, sum of roots $\alpha+\beta+\gamma = \frac{-b}{a}$
$1+3i+1-3i+\gamma = --4/1$
$2+\gamma = 4$
$\gamma = 2$

$\therefore P(z) = (z-2)(z-(1+3i))(z-(1-3i))$
$= (z-2)(z^2-2z+10)$

(c) RTP: $\log_3 5$ is irrational

Suppose $\log_3 5$ is rational
ie. $\log_3 5 = \frac{p}{q}$ where p, q are integers with no common factors except 1.
$q \log_3 5 = p$
$\log_3 5^q = p$
$5^q = 3^p$, which is a contradiction since 3 and 5 are co-prime
$\therefore$ By contradiction, $\log_3 5$ is irrational.

Question 7

(a) $z^2-(-1+4i)z-5+i=0$

$\Delta = B^2-4AC$

$= (-1+4i)^2-4(1)(-5+i)$

$= 1-8i-16+20-4i$

$= 5-12i$

$z = \dfrac{-(-1+4i)\pm\sqrt{5-12i}}{2(1)}$

$= \dfrac{1-4i\pm\sqrt{5-12i}}{2}$ ✓

$= \dfrac{1-4i\pm(3-2i)}{2}$

$= \dfrac{1-4i+3-2i}{2}, \dfrac{1-4i-3+2i}{2}$

$= \dfrac{4-6i}{2}, \dfrac{-2-2i}{2}$

$= 2-3i, -1-i$ ✓

Let $(x+iy)^2 = 5-12i$

$x^2+2ixy-y^2 = 5-12i$

$\begin{cases} x^2-y^2 = 5 & — (1) \\ 2xy = -12 & — (2) \end{cases}$

$(x^2+y^2)^2 = (x^2-y^2)^2+(2xy)^2$

$= 5^2+(-12)^2$

$= 169$

$x^2+y^2 = 13$ — (3)

(1) + (3)

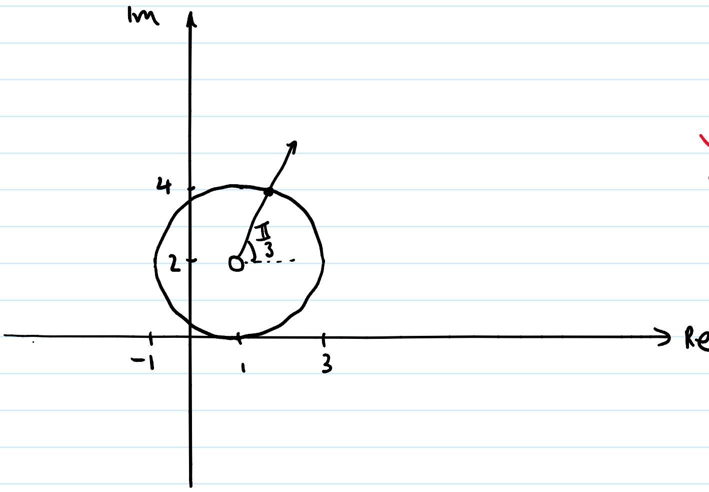
$2x^2 = 18$

$x^2 = 9$

$x = \pm 3 \rightarrow$ Sub in (2) $y = \mp 2$ ✓

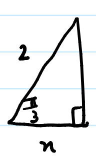
(b) (i) $|z-1-2i| = 2$ and $\arg(z-1-2i) = \pi/3$

$|z-(1+2i)| = 2$ $\quad\arg(z-(1+2i)) = \dfrac{\pi}{3}$



✓ circle centre
✓ circle radius
✓ Ray

(ii)



$\cos\dfrac{\pi}{3} = \dfrac{x}{2}$

$\dfrac{1}{2} = \dfrac{x}{2}$

$x=1, \quad y=\sqrt{3}$ ✓

$\therefore$ POI at $z = 2+i(2+\sqrt{3})$ ✓

Question 8

(a) RTP: If $3n+7$ is even, then n is odd

Suppose n is even, ie. $n=2k$, $k\in\mathbb{Z}$ ✓
Then
$$\begin{aligned}3n+7 &= 3(2k)+7\\ &= 6k+7\\ &= 2(3k+3)+1\\ &= 2A+1, \text{ which is odd } (A\in\mathbb{Z})\end{aligned}$$ ✓

$\therefore$ If n is even, then $3n+7$ is odd
$\therefore$ By contrapositive, if $3n+7$ is even, then n is odd

(b) $z=3\operatorname{cis}\frac{\pi}{6}$, $w=2\operatorname{cis}\frac{\pi}{4}$

(i)
$$\begin{aligned}z &= 3\left(\cos\tfrac{\pi}{6}+i\sin\tfrac{\pi}{6}\right)\\ &= 3\left(\sqrt{3}/2+i\cdot 1/2\right)\\ &= \frac{3\sqrt{3}}{2}+i\,\frac{3}{2}\end{aligned}$$ ✓

$$\begin{aligned}w &= 2\left(\cos\tfrac{\pi}{4}+i\sin\tfrac{\pi}{4}\right)\\ &= 2\left(1/\sqrt{2}+i\cdot 1/\sqrt{2}\right)\\ &= \frac{2}{\sqrt{2}}+i\cdot\frac{2}{\sqrt{2}}\\ &= \sqrt{2}+i\sqrt{2}\end{aligned}$$ ✓

(ii)
$$\begin{aligned}\frac{w}{z} &= \frac{2\operatorname{cis}\frac{\pi}{4}}{3\operatorname{cis}\frac{\pi}{6}}\\ &= \frac{2}{3}\operatorname{cis}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\\ &= \frac{2}{3}\operatorname{cis}\frac{\pi}{12}\end{aligned}$$ ✓

$$\begin{aligned}\frac{w}{z} &= \frac{\sqrt{2}+i\sqrt{2}}{\frac{3\sqrt{3}}{2}+i\frac{3}{2}}\\ &= \frac{2}{3}\cdot\frac{\sqrt{2}+i\sqrt{2}}{\sqrt{3}+i}\times\frac{\sqrt{3}-i}{\sqrt{3}-i}\\ &= \frac{2}{3}\cdot\frac{\sqrt{6}-i\sqrt{2}+i\sqrt{6}+\sqrt{2}}{3+1}\\ &= \frac{2}{3}\cdot\frac{(\sqrt{6}+\sqrt{2})+i(\sqrt{6}-\sqrt{2})}{4}\\ &= \frac{1}{6}\left[\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2})\right]\\ &= \frac{\sqrt{6}+\sqrt{2}}{6}+i\,\frac{\sqrt{6}-\sqrt{2}}{6}\end{aligned}$$ ✓ ✓

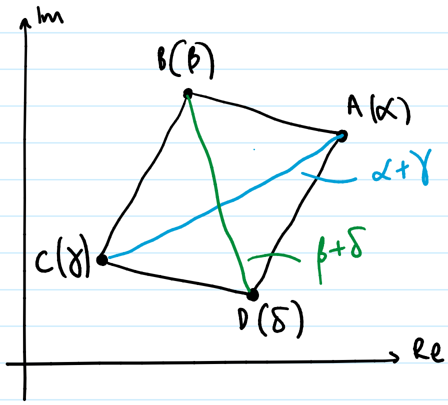
(iii) Equating real & imaginary parts of $\frac{w}{z}$

$$\frac{2}{3}\cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{6}$$
$$\therefore \cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}$$ ✓

$$\frac{2}{3}\sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{6}$$
$$\therefore \sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}$$ ✓

Question 9

(a) A, B, C, D represent $\alpha, \beta, \gamma, \delta$



$\alpha+\gamma = \beta+\delta$

$\frac{1}{2}(\alpha+\gamma) = \frac{1}{2}(\beta+\delta)$ ✓

$\therefore$ Midpoints of AC and BD are the same

$\therefore$ Diagonals AC and BD bisect each other ✓

$\therefore$ ABCD is a parallelogram.

(b) RTP(?): If n divides $(a+b)$ and n divides $(b+c)$ then n divides $(a+c)$.

Let $\underbrace{a+b = nP}_{①}$ and $\underbrace{b+c = nQ}_{②}$, $P, Q \in \mathbb{Z}$

① + ②

$a + 2b + c = nQ + nP$

$a + c = n(P+Q) - 2b$, which isn't divisible by n?

Not necessary

Try counter example to disprove

Let $a=4$, $b=8$, $c=7$, $n=3$

$a+b = 4+8 = 12$, divisible by 3 ✓

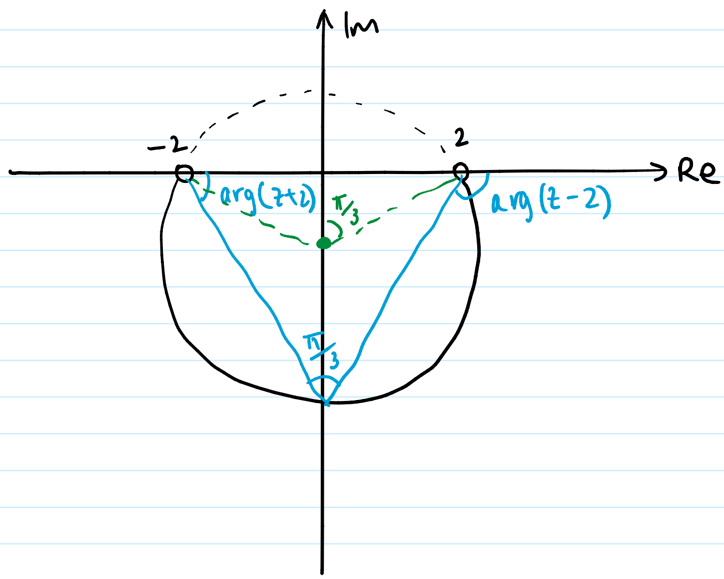
$b+c = 8+7 = 15$, divisible by 3

$a+c = 4+7 = 11$, not divisible by 3 ✓

$\therefore$ By counterexample, the statement is false.

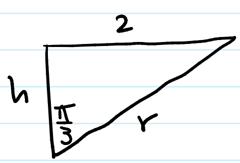
(c) (i) $\arg\left(\frac{z+2}{z-2}\right)=\frac{\pi}{3}$

$\arg(z+2)-\arg(z-2)=\frac{\pi}{3}$



✓ major arc incl. open circles
✓ showing blue angles

(ii)



$\sin\frac{\pi}{3}=\frac{2}{r}$

$\frac{\sqrt{3}}{2}=\frac{2}{r}$

$r=\frac{4}{\sqrt{3}}$ ✓

$\tan\frac{\pi}{3}=\frac{2}{h}$

$\sqrt{3}=\frac{2}{h}$

$h=\frac{2}{\sqrt{3}}$ ✓

Circle with centre $\left(0,-\frac{2}{\sqrt{3}}\right)$ and radius $\frac{4}{\sqrt{3}}$ units

$\therefore\ x^2+\left(y+\frac{2}{\sqrt{3}}\right)^2=\frac{16}{3}$ ✓