2021 Caringbah EXT2 AT3 Exam

Maths extension 2 (Sydney Technical High School)

Caringbah High School

Year 12 2021
Mathematics Extension 2
HSC Course
Assessment Task 3

General Instructions

- Reading time – 5 minutes
- Working time – 70 minutes
- Write using a black pen
- Board-approved calculators may be used
- A reference sheet is provided.
- In Questions 6-9, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 45

Section I **5 marks**
Attempt Questions 1-5
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II 40 **marks**
Attempt Questions 6-9
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.[SL1]

Name: ____ **Class:** ____________

Marker's Use Only						
Section I	**Section II**				**Total**	
Q 1-5	**Q6**	**Q7**	**Q8**	**Q9**		
/5	/10	/10	/10	/10	/45	%

Section I

5 marks
Attempt Questions 1-5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5

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1. Which expression is equal to $\int \tan x \, dx$?

(A) $\sec^2 x + C$

(B) $-\ln|\cos x| + C$

(C) $\dfrac{\tan^2 x}{2} + C$

(D) $\ln|\sec x + \tan x| + C$

2. Which of the following is an expression of $\int x^3 e^{x^4} \, dx$?

(A) $x^3 e^{x^4} + C$

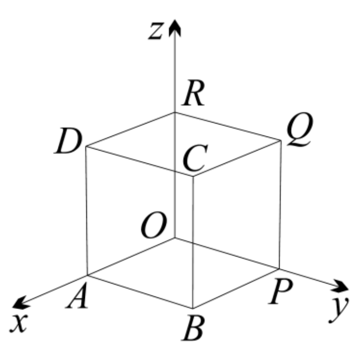
(B) $e^{x^4} + C$

(C) $4e^{x^4} + C$

(D) $\dfrac{1}{4} e^{x^4} + C$

3. The diagram shows a cube of side 2 units.

What is the length of the diagonal OC ?



(A) $\sqrt{6}$

(B) $2\sqrt{2}$

(C) $2\sqrt{3}$

(D) $3\sqrt{2}$

4. Which of the following is the first line of working for the expression:

$$\int x \cos x \, dx$$

(A) $-x \sin x - \int \cos x \, dx$

(B) $-x \cos x - \int \sin x \, dx$

(C) $x \sin x - \int \cos x \, dx$

(D) $x \cos x - \int \sin x \, dx$

5. Find any values of λ for which $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular if

$$\underset{\sim}{a} = \begin{bmatrix} 1+\lambda \\ 2\lambda \\ \lambda+2 \end{bmatrix} \text{ and } \underset{\sim}{b} = \begin{bmatrix} 2 \\ 5-4\lambda \\ 2\lambda-4 \end{bmatrix}$$

(A) $\lambda = 0$

(B) $\lambda = -1$

(C) $\lambda = 1$

(D) λ has no solution

End of Section I

Section II

60 marks
Attempt Questions 6–9
Allow about 1 hour for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (10 marks) Use a SEPARATE writing booklet.

(a) The points P and Q have position vectors $1\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$ and $4\hat{\imath} - 5\hat{\jmath} - 6\hat{k}$ respectively. Find

(i) $\overrightarrow{PQ}$ **1**

(ii) $\overrightarrow{QP} - 2\overrightarrow{PQ}$ **1**

(iii) $\left|\overrightarrow{PQ}\right|$ **1**

(b) Evaluate **2**

$$\int_0^{\frac{\pi}{3}} \cos^3\theta \, d\theta$$

(c) Find the Cartesian equation of the line **2**

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

(d) Use the substitution $u = \sqrt{x}$, or otherwise, evaluate **3**

$$\int_0^1 \frac{1}{(1+x)\sqrt{x}} \, dx$$

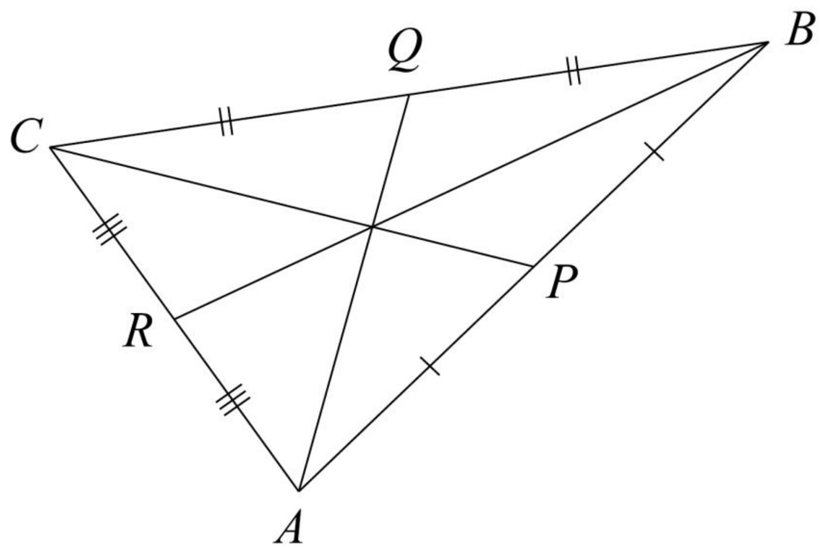
End of Question 6

Question 7 (10 marks) Use a SEPARATE writing booklet.

(a) Find **3** [SL2]

$$\int x \tan^{-1} x \, dx$$

(b) The diagram above shows ΔABC, where A, B and C have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. The points P, Q and R bisect the intervals AB, BC and CA respectively. **3** [SL3]



(i) Show that $\overrightarrow{AQ} = \frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{b}) - \underset{\sim}{a}$ **1**

(ii) Hence show that $\overrightarrow{AQ} + \overrightarrow{BR} + \overrightarrow{CP} = 0$ **2**

(c) Let $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \; dx,$ for intergeters $n \geq 0$

(i) Show that $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$ for $n \geq 2$ **2**

(ii) Hence evaluate $I_2 = \int_0^{\frac{\pi}{2}} x^2 \sin x \; dx$ **2**

End of Question 7

Question 8 (10 marks) Use a SEPARATE writing booklet.

(a) Determine $\int \sqrt{\frac{2-x}{2+x}}\, dx$ **3**

(b) Given $\underset{\sim}{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} -7 \\ 6 \\ -5 \end{bmatrix}$

(i) What is the scalar product of the two vectors **1**

(ii) Hence find the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ **2**

(c) Using the substitution $t = \tan\frac{\theta}{2}$ find **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{4\sin\theta - 2\cos\theta + 6}\, d\theta$$

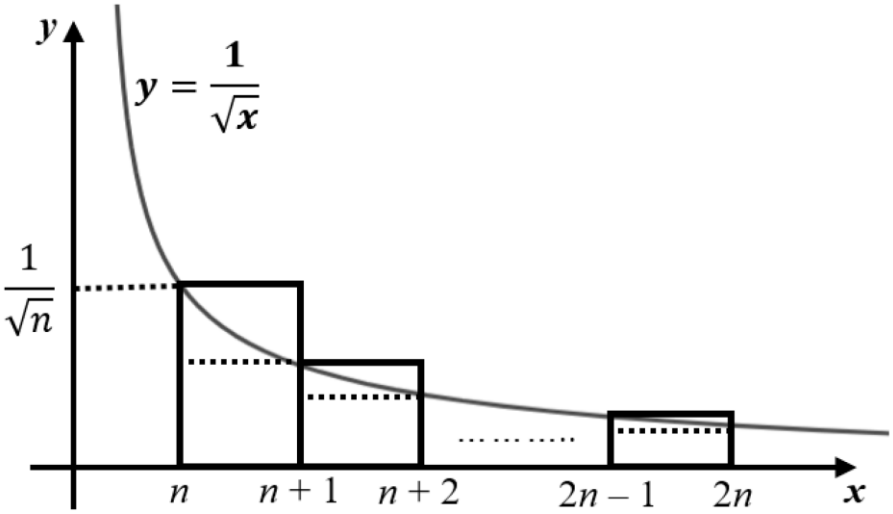
End of Question 8

Question 9 (10 marks) Use a SEPARATE writing booklet.

(a) Find the exact values of x such that the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 60° if: **3**

$$\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} \text{ and } \underset{\sim}{b} = \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix}$$

(b) In the diagram below, the graph of $y = \frac{1}{\sqrt{x}}$ has been drawn, and n rectangles have been constructed between $x = n$ and $x = 2n$, each of width 1 unit



i) Show that $\displaystyle\int_n^{2n} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{n}\,(\sqrt{2}-1)$ **1**

ii) Let $S_n = \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{2n}}$, show that **4**

$$2\sqrt{n}\,(\sqrt{2}-1) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}\,(\sqrt{2}-1)$$

iii) Hence find, correct to three decimal places **2**

$$\frac{1}{\sqrt{1\,000\,000}} + \frac{1}{\sqrt{1\,000\,001}} + \frac{1}{\sqrt{1\,000\,002}} + \cdots + \frac{1}{\sqrt{2\,000\,000}}$$

End of Paper

Extra Writing Space
Please clearly indicate question number.

Multiple Choice Answer Sheet Name______________________

Colour the oval A, B, C or D that best answers the question.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

1. A ○ B ○ C ○ D ○

2. A ○ B ○ C ○ D ○

3. A ○ B ○ C ○ D ○

4. A ○ B ○ C ○ D ○

5. A ○ B ○ C ○ D ○

2021 EXT 2 AT3 SOLUTIONS

Q1 $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$ **B**

Q2 $\int x^3 e^{x^4} dx = \frac{1}{4}\int 4x^3 e^{x^4} dx = \frac{1}{4} e^{x^4} + C$ **D**

Q3 $|OC| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$ **C**

Q4 $u = x \quad v' = \cos x \quad \therefore u' = 1 \quad v = sinx$

$\therefore \int x \cos x\, dx = x \sin x - \int 1 \cos x\, dx$ **C**

Q5 $\underset{\sim}{a} = \begin{bmatrix} 1+\lambda \\ 2\lambda \\ \lambda+2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 2 \\ 5-4\lambda \\ 2\lambda-4 \end{bmatrix}$ and $\underset{\sim}{a} \perp \underset{\sim}{b}$

$\therefore \underset{\sim}{a} . \underset{\sim}{b} = 2 + 2\lambda + 10\lambda - 8\lambda^2 + 2\lambda^2 - 8 = 0$

$\therefore \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \therefore \lambda = 1$ **C**

Q6a) i) $\overrightarrow{PQ} = \begin{bmatrix} 4-1 \\ -5-(-2) \\ -6-(-3) \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$

ii) $\overrightarrow{QP} - 2\overrightarrow{PQ} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -9 \\ 9 \\ 9 \end{bmatrix}$

iii) $|\overrightarrow{PQ}| = \sqrt{3^2 + (-3)^2 + (-3)^2} = \sqrt{27}$

Q6b) $\int_0^{\frac{\pi}{3}} \cos^3\theta\, d\theta = \int_0^{\frac{\pi}{3}} \cos^2\theta \cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{3}} (1 - \sin^2\theta)\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{3}} \cos\theta - \sin^2\theta\cos\theta\, d\theta$

$= \sin\theta - \frac{1}{3}\sin^3\theta]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{2} - \frac{1}{3} \times \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$

Q6c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ -4 \end{bmatrix} \therefore x = 3 - \lambda \quad y = 5 - 4\lambda$

$\therefore y = 5 - 4(3 - x) \therefore y = 4x - 7$

Q6d) $u = \sqrt{x} \therefore du = \frac{1}{2\sqrt{x}} dx$

$\therefore \int_0^1 \frac{1}{(1+x)\sqrt{x}} dx = 2\int \frac{1}{1+u^2} du = 2\tan^{-1} u$

$= 2\tan^{-1}\sqrt{x}\ ]_0^1 = \frac{\pi}{2}$

Q7a) Let $u = \tan^{-1} x \qquad v' = x$

$\therefore u' = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2}$

$\therefore \int x \tan^{-1} x\, dx = \frac{x^2}{2}\tan^{-1} x - \int \frac{x^2}{2} \times \frac{1}{1+x^2} dx$

$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2} dx$

$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int 1 - \frac{1}{1+x^2} dx$

$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}[x - \tan^{-1} x] + C$

Q7b) i) $\overrightarrow{AQ} = \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB} = (\underset{\sim}{c} - \underset{\sim}{a}) + \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{c})$

$= \underset{\sim}{c} - \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{c} = \frac{1}{2}\underset{\sim}{c} - \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

$= \frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{b}) - \underset{\sim}{a}$

ii) As $\overrightarrow{AQ} = \frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{b}) - \underset{\sim}{a}$

Then $\overrightarrow{BR} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c}) - \underset{\sim}{b}$ and $\overrightarrow{CP} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) - \underset{\sim}{c}$

$\therefore \overrightarrow{AQ} + \overrightarrow{BR} + \overrightarrow{CP} =$

$\frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{b}) - \underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c}) - \underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) - \underset{\sim}{c} = 0$

Q7c) i) $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\ dx$ Let $u = x^n$; $v' = \sin x$

$\therefore u' = nx^{n-1} \qquad v = -\cos x$

$\therefore I_n = [-x^n \cos x\ ]_0^{\frac{\pi}{2}} + n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x\ dx$

Let $u = x^{n-1}$; $v' = \cos x$

$\therefore u' = (n-1)x^{n-2}; v = \sin x$

$\therefore I_n = n\left([x^{n-1}\sin x\ ]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)x^{n-2}\sin x\ dx\right)$

$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\sin x\ dx$

$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$

Q7c) ii) $\int_0^{\frac{\pi}{2}} x^2 \sin x \ dx = 2\left(\frac{\pi}{2}\right) - 2 \times I_0$

$I_0 = \int_0^{\frac{\pi}{2}} \sin x \ dx = [-\cos x \ ]_0^{\frac{\pi}{2}} = 1$

$\therefore \int_0^{\frac{\pi}{2}} x^2 \sin x \ dx = \pi - 2$

Q8a) $\int \sqrt{\frac{2-x}{2+x}} \ dx \ \times \frac{\sqrt{2-x}}{\sqrt{2-x}} = \int \frac{2-x}{\sqrt{4-x^2}} dx$

$= \int \frac{2}{\sqrt{4-x^2}} dx - \int \frac{x}{\sqrt{4-x^2}} dx$

$= 2\int \frac{1}{\sqrt{4-x^2}} dx + \frac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}} dx$

$Let \ \ u = 4 - x^2 \qquad du = -2x \ dx$

$\therefore \frac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}} dx = \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}[2\sqrt{u}]$

$\therefore \int \sqrt{\frac{2-x}{2+x}} \ dx = 2\sin^{-1}\frac{x}{2} + \sqrt{4-x^2} + C$

Q8b) i) $\underset{\sim}{u} \cdot \underset{\sim}{v} = (3 \times -7) + (-2 \times 6) + (1 \times -5)$

$= -38$

Q8b) ii) $\text{proj}_{\underset{\sim}{v}} \underset{\sim}{u} = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\underset{\sim}{v} \cdot \underset{\sim}{v}} \times \underset{\sim}{v}$

$= \frac{-38}{(-7)^2+6^2+(-5)^2} \times \underset{\sim}{v}$

$= \frac{-19}{55} \times \underset{\sim}{v}$

$\therefore \text{proj}_{\underset{\sim}{v}} \underset{\sim}{u} = \frac{-19}{55} \times \begin{bmatrix} -7 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{133}{55} \\ -\frac{114}{55} \\ \frac{19}{11} \end{bmatrix}$

Q8c) $\int_0^{\frac{\pi}{2}} \frac{1}{4\sin\theta - 2\cos\theta + 6} \ d\theta$

$t = \tan\frac{\theta}{2} \quad \theta = 0, t = 0 \ \ ; \ \ \theta = \frac{\pi}{2}, t = 1$

$= \int_0^1 \frac{1}{\frac{8t}{1+t^2} - \frac{2(1-t^2)}{1+t^2} + 6} \times \frac{2}{1+t^2} dt$

$= \int_0^1 \frac{1}{4t-1+t^2+3+3t^2} dt = \int_0^1 \frac{1}{4t^2+4t+2} dt$

$= \int_0^1 \frac{1}{(4t^2+4t+1)+1} dt = \frac{1}{2}\int_0^1 \frac{2}{(2t+1)^2+1} dt$

$= \frac{1}{2}[\tan^{-1}(2t+1)\,]_0^1 = \frac{1}{2}(\tan^{-1}(3) - \frac{\pi}{4})$

Q9a) $\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times x) + (2 \times 2) + (x \times 1) = 2x + 4$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \sqrt{1^2 + 2^2 + x^2}\sqrt{x^2 + 2^2 + 1^2} \times \cos 60°$

$\therefore 2x + 4 = \sqrt{x^2+5} \times \sqrt{x^2+5} \times \frac{1}{2}$

$\therefore 4x + 8 = x^2 + 5 \quad \therefore x^2 - 4x - 3 = 0$

$\therefore x = \frac{4 \pm \sqrt{4^2 - 4\times1\times(-3)}}{2} = \frac{4\pm\sqrt{28}}{2} = 2 \pm \sqrt{7}$

Q9b)i) $\displaystyle\int_n^{2n} \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_n^{2n}$

$= 2\sqrt{2n} - 2\sqrt{n}$

$= 2\sqrt{n}(\sqrt{2} - 1)$

Q9b) ii) Area of the lower rectangles < Area under the curve

$$\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{n}} < \int_n^{2n} \frac{dx}{\sqrt{x}}$$

$$S_n < 2\sqrt{n}(\sqrt{2} - 1)$$

Area under the curve < Area of the upper rectangles

$$\int_n^{2n} \frac{dx}{\sqrt{x}} < \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{2n-1}}$$

$$2\sqrt{n}(\sqrt{2} - 1) < \frac{1}{\sqrt{n}} + \left(\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{2n-1}} + \frac{1}{\sqrt{2n}}\right) - \frac{1}{\sqrt{2n}}$$

$$2\sqrt{n}(\sqrt{2} - 1) - \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2n}} < S_n \qquad 2\sqrt{n}(\sqrt{2} - 1) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n$$

$$\therefore 2\sqrt{n}(\sqrt{2} - 1) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}(\sqrt{2} - 1)$$

Q9b) iii)

Let $n = 1\,000\,000$

$$2\sqrt{1000000}(\sqrt{2} - 1) + \frac{1-\sqrt{2}}{\sqrt{2000000}} < S_{1000000} < 2\sqrt{1000000}(\sqrt{2} - 1)$$

$$828.4268 < S_{1000000} < 828.4271$$

$\therefore S_{1000000} = 828.427$ (to 3 decimal places)

$$\frac{1}{\sqrt{1000000}} + \frac{1}{\sqrt{1000001}} + \ldots + \frac{1}{\sqrt{2000000}} = \frac{1}{\sqrt{1000000}} + S_{1000000}$$

$= 828.428$ (to 3 decimal places)