Caringbah High School

Year 12 2023
Mathematics Extension 2
HSC Course
Assessment Task 3

General Instructions
- Reading time – 5 minutes
- Working time – 75 minutes
- Write using a black pen
- NESA-approved calculators may be used
- A reference sheet is provided.
- In Questions 6-9, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 45

Section I **5 marks**
Attempt Questions 1-5
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **40 marks**
Attempt Questions 6-9
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ______________________________ **Class:** ____________

Marker's Use Only						
Section I	**Section II**				**Total**	
Q 1-5	**Q6**	**Q7**	**Q8**	**Q9**		
/5	/10	/10	/10	/10	/45	%

Section I

5 marks
Attempt Questions 1-5
Allow about 10 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5

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1. What is the correct first step in integrating the following?

$$\int xe^x \, dx$$

(A) $= xe^x - \int x \, dx$

(B) $= xe^x - \int e^x \, dx$

(C) $= xe^x - \int xe^x \, dx$

(D) $= xe^x + \int xe^x \, dx$

2. The algebraic fraction $\frac{x+1}{2(x+h)^2}$, where h is a non-zero real number can be written in partial fraction form, where A and B are real numbers, as which of the following?

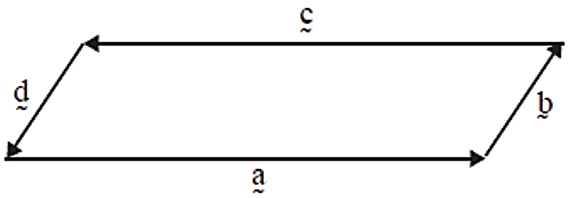
(A) $\dfrac{A}{x+h} + \dfrac{B}{x+h}$

(B) $\dfrac{A}{2x+h} + \dfrac{B}{(x+h)^2}$

(C) $\dfrac{A}{x+h} + \dfrac{B}{(x+h)^2}$

(D) $\dfrac{A}{2(x+h)} + \dfrac{B}{x+h}$

3. In the parallelogram $\left|\underset{\sim}{a}\right| = 2\left|\underset{\sim}{b}\right|$



Which one of the following statements is true ?

(A) $\underset{\sim}{a} = 2\underset{\sim}{b}$

(B) $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$

(C) $\underset{\sim}{a} + \underset{\sim}{c} = 0$

(D) $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

4. Of the following expressions which does NOT contain a logarithmic integral?

(A) $\dfrac{x-1}{x^3+1}$

(B) $\dfrac{x+1}{x^3-1}$

(C) $\dfrac{x-1}{x^3-1}$

(D) $\dfrac{x^2+1}{x^3+1}$

5. A, B and C are three collinear points with position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. B lies between A and C.

If $2\left|\overrightarrow{BC}\right| = \left|\overrightarrow{AB}\right|$, the $\underset{\sim}{c}$ is equal to which of the following?

(A) $\frac{3}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

(B) $\frac{3}{2}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$

(C) $\frac{1}{2}\underset{\sim}{a} - \frac{3}{2}\underset{\sim}{b}$

(D) $\frac{1}{2}\underset{\sim}{b} - \frac{3}{2}\underset{\sim}{a}$

End of Section I

Section II

40 marks
Attempt Questions 6–9
Allow about 65 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (10 marks) Use a SEPARATE writing booklet.

(a) Evaluate **3**

$$\int_0^{\frac{\pi}{3}} \cos^3 3x \sin 3x \, dx$$

(b) The equation of intersecting line L and M are given below with respect to the origin O. **2**

$$L: \quad \underset{\sim}{r} = \begin{bmatrix} 11 \\ 2 \\ 17 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}$$

$$M: \quad \underset{\sim}{r} = \begin{bmatrix} -5 \\ 11 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} p \\ 2 \\ 2 \end{bmatrix}$$

If L and M are perpendicular what is the value of p ?

(c) Evaluate the following **3**

$$\int_0^1 \frac{x+1}{x^2+1} dx$$

(d) Given $\underset{\sim}{u} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ find $\underset{\sim}{w}$ given: **2**

$$\left|\underset{\sim}{w}\right|^2 = \left|\underset{\sim}{u}\right|^2 + \left|\underset{\sim}{v}\right|^2$$

End of Question 6

Question 7 (10 marks) Use a SEPARATE writing booklet.

(a) Evaluate **2**

$$\int \frac{dx}{\sqrt{9-4x^2}}$$

(b) Use the substitution $u = 1 + x^2$ to evaluate **3**

$$\int_0^{\sqrt{3}} \frac{x^3}{\sqrt{(1+x^2)^3}}\,dx$$

(c) **i)** Find real numbers a and b such that

$$\frac{5x^2-3x+1}{(x^2+1)(x-2)} \equiv \frac{ax+1}{x^2+1} + \frac{b}{x-2}$$ **2**

ii) Find: **3**

$$\int \frac{5x^2-3x+1}{(x^2+1)(x-2)}\,dx$$

End of Question 7

Question 8 (10 marks) Use a SEPARATE writing booklet.

(a) Evaluate: **3**

$$\int \frac{1}{2x \ln 2x} \, dx$$

(b) Evaluate: **4**

$$\int \sin^{-1} x \, dx$$

(c) Evaluate: **3**

$$\int_{-2}^{-1} \frac{1}{\sqrt{-x^2 - 4x - 3}} \, dx$$

End of Question 8

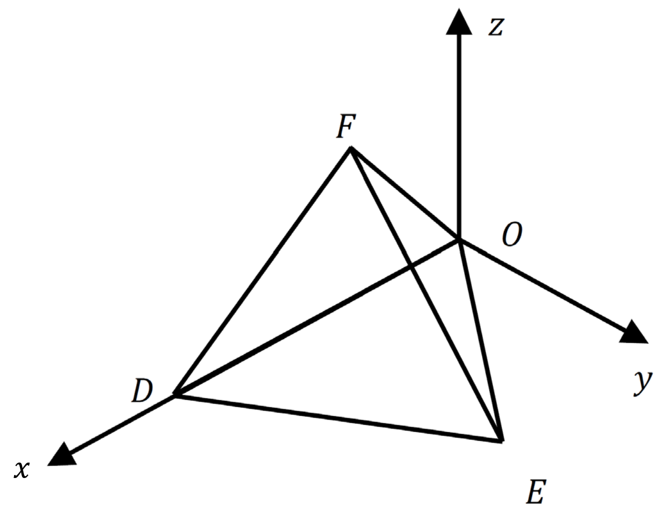
Question 9 (10 marks) Use a SEPARATE writing booklet.

(a) By using $t = \tan\frac{x}{2}$ find the integral **2**

$$\int \frac{dx}{1 + sinx}$$

(b) Let $ODEF$ be a tetrahedron with faces that are equilateral triangles all with side lengths of 2 units. **4**

Its base ODE lies flat on the xy-plane with two vertices at O and D , with F above the xy-plane.



Prove, using vectors, the coordinates of the vertex F are $\left(1, \frac{1}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right)$

(c) Consider the integral **4**

$$I_n = \int_0^1 x^n e^{\frac{x^2}{2}}\, dx.$$

The recurrence relationship linking I_n and I_{n-2} can be written in the form

$$I_n = a - f(n) \times I_{n-2},$$

where n is an integer greater than or equal to 2, a is a constant and $f(n)$ is a function of n.

Find a and $f(n)$.

End of Examination

Multiple Choice Answer Sheet Name____________________

Colour the oval A, B, C or D that best answers the question.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ☒ B ☒ C ○ D ○

1. A ○ B ○ C ○ D ○

2. A ○ B ○ C ○ D ○

3. A ○ B ○ C ○ D ○

4. A ○ B ○ C ○ D ○

5. A ○ B ○ C ○ D ○

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Q1. $xe^x - \int e^x\,dx$ (B)

Q2. $\frac{A}{x+h} + \frac{B}{(x+h)^2}$ (C)

The 2 can be absorbed into the values of A & B so not necessary for the denominator.

Q3. $\underset{\sim}{a} = -\underset{\sim}{c} \;\therefore\; \underset{\sim}{a} + \underset{\sim}{c} = 0$ (C)

Q4. $\frac{x-1}{x^3-1} = \frac{(x-1)}{(x-1)(x^2+x+1)} = \frac{1}{x^2+x+1}$ (C)

Q5. $\frac{3}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$ (A)

Q6. a) Let $u = \cos 3x \;\therefore\; du = -3\sin 3x\,dx$

$\int \cos^3 3x \sin 3x\,dx = -\frac{1}{3}\int u^3\,du = -\frac{u^4}{12} + c$ ❶

$= -\left.\frac{\cos^4 3x}{12}\right]_0^{\frac{\pi}{3}}$ ❷

$= \frac{1}{12}$ ❸

b) $\begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} p \\ 2 \\ 2 \end{bmatrix} = 0$ ❶

$\therefore\; -2p + 2 - 8 = 0 \;\therefore\; p = -3$ ❷

c) $\int_1^2 \frac{x+1}{x^2+1}\,dx = \int_1^2 \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$ ❶

$= \frac{1}{2}\int_1^2 \frac{2x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx$

$= \frac{1}{2}\ln|x^2+1| + \tan^{-1}x\,\Big]_0^1$ ❷

$= \frac{\ln 2}{2} + \frac{\pi}{4}$ ❸

d) As $\left|\underset{\sim}{w}\right|^2 = \left|\underset{\sim}{u}\right|^2 + \left|\underset{\sim}{v}\right|^2$

therefore the vectors represent the sides of a right triangle. ❶

$\therefore\; \underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v} = \left(2\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right) + \left(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

$= 3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ ❷

Q7. a) $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}$ ❶

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$ ❷

b) $u = 1 + x^2 \;\therefore\; du = 2x\,dx \;\therefore\; dx = \frac{1}{2x}\,du$ ❶

$\int_0^{\sqrt{3}} \frac{x^3}{\sqrt{(1+x^2)^3}}\,dx = \frac{1}{2}\int \frac{u-1}{u^{\frac{3}{2}}}\,du$

$= \frac{1}{2}\int u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\,du$

$= \frac{1}{2}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]$ ❷

$= \left[(1+x^2)^{\frac{1}{2}} + (1+x^2)^{-\frac{1}{2}}\right]_0^{\sqrt{3}} = \frac{1}{2}$ ❸

c)i) $\frac{5x^2-3x+1}{(x^2+1)(x-2)} \equiv \frac{ax+1}{x^2+1} + \frac{b}{x-2}$

$\therefore\; 5x^2 - 3x + 1$

$= (x-2)(ax+1) + b(x^2+1)$

$\therefore\; x = 2 \;\rightarrow\; 15 = 5b \;\therefore\; b = 3$ ❶

$\therefore\; x = 1 \;\rightarrow\; 3 = -(a+1) + 6 \;\therefore\; a = 2$ ❷

c)ii) $\int \frac{5x^2-3x+1}{(x^2+1)(x-2)}\,dx = \int \frac{2x+1}{x^2+1} + \frac{3}{x-2}\,dx$ ❶

$= \int \frac{2x}{x^2+1} + \frac{1}{x^2+1} + \frac{3}{x-2}\,dx$ ❷

$= \ln|x^2+1| + \tan^{-1}x + 3\ln|x-2| + C$ ❸

Q8. a) *Let* $u = \ln 2x \;\therefore\; du = \frac{2}{2x}\,dx = \frac{1}{x}\,dx$ ❶

$\therefore\; \int \frac{1}{2x\ln 2x}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$ ❷

$= \frac{1}{2}\ln|\ln 2x| + C$ ❸

b) $\int \sin^{-1}x\,dx$ $\boxed{\begin{array}{ll} u = \sin^{-1}x & \frac{dv}{dx} = 1 \\ \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} & v = x \end{array}}$ ❶

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$ ❷

$= x\sin^{-1}x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$

Let $v = 1 - x^2 \;\therefore\; dv = -2x\,dx$

$= x\sin^{-1}x + \frac{1}{2}\int \frac{1}{\sqrt{v}}\,dv$ ❸

$= x\sin^{-1}x + \frac{1}{2}\left[2\sqrt{v}\right] + C$

$= x\sin^{-1}x + \sqrt{1-x^2} + C$ ❹

Q8. c) $\int_{-2}^{-1} \frac{1}{\sqrt{-x^2-4x-3}} dx = \int_{-2}^{-1} \frac{1}{\sqrt{1-(x+2)^2}} dx$ ❶

$= \sin^{-1}(x+2) \Big]_{-2}^{-1} = \frac{\pi}{2}$ ❷ ❸

Q9. a) $\int \frac{dx}{1+sinx}$ $\therefore \sin x = \frac{2t}{1+t^2}$ $dx = \frac{2}{1+t^2} dt$

$= \int \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2} dt$

$= 2\int \frac{1}{(1+t)^2} dt$ ❶

$= \frac{-2}{1+t} = \frac{-2}{1+\tan\frac{x}{2}}$ ❷

b) $\angle FOD = \angle FOE = \frac{\pi}{3}$ (equilateral triangle)

In ΔFOD:

$$\cos \angle FOD = \frac{\overrightarrow{OF} \cdot \overrightarrow{OD}}{|\overrightarrow{OF}|\,|\overrightarrow{OD}|}$$

$$\therefore \cos\frac{\pi}{3} = \frac{\begin{bmatrix} a \\ b \\ c \end{bmatrix} . \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}{2 \times 2}$$

$$\therefore \frac{1}{2} = \frac{2a+0+0}{4}$$

$\therefore a = 1$ ❶

We have $E(1, \sqrt{3}, 0)$using exact triangles

In ΔFOE:

$$\cos \angle FOE = \frac{\overrightarrow{OF} \cdot \overrightarrow{OE}}{|\overrightarrow{OF}|\,|\overrightarrow{OE}|}$$

$$\therefore \cos\frac{\pi}{3} = \frac{\begin{bmatrix} 1 \\ b \\ c \end{bmatrix} . \begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \end{bmatrix}}{2 \times 2}$$

$$\therefore \frac{1}{2} = \frac{1+b\sqrt{3}+0}{4}$$

$\therefore b = \frac{1}{\sqrt{3}}$ ❷

As $\left|\overrightarrow{OF}\right| = 2$ $\qquad \therefore \sqrt{(1)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + c^2} = 2$

$\therefore c = \frac{2\sqrt{2}}{\sqrt{3}}$ ❸

F is $\left(1, \frac{1}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right)$ ❹

c) $I_n =$

$\int_0^1 x^n e^{\frac{x^2}{2}} dx$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^{n-1} & \frac{dv}{dx} = xe^{\frac{x^2}{2}} \\ \frac{du}{dx} = \frac{n-1}{2}x^{n-2} & v = e^{\frac{x^2}{2}} \end{array}}$$ ❶

$$= \frac{1}{2}\left[x^{n-1}e^{\frac{x^2}{2}}\right]_0^1 - \frac{n-1}{2}\int_0^1 x^{n-2}e^{\frac{x^2}{2}} dx$$ ❷

$$= \frac{1}{2}\left(e^{\frac{1}{2}} - 0\right) - \frac{n-1}{2}\int_0^1 x^{n-2}e^{\frac{x^2}{2}} dx$$

$$= \frac{e^{\frac{1}{2}}}{2} - \frac{n-1}{2} \times I_{n-2}$$ ❸

$$\therefore a = \frac{e^{\frac{1}{2}}}{2}, f(n) = \frac{n-1}{2}$$ ❹