Name:____________________

DANEBANK

An Anglican School for Girls

YEAR 12
Mathematics Extension 2

Assessment Task # 2

Assessment Week April 2007

Time Allowed: **2 hours Plus 5 Minutes Reading Time**

Outcomes tested: E2 , HE2 , E3 , E4 , E9 , HE4 , HE6 , HE7 , E7

Weighting: 20 % Maximum Marks 80

DIRECTIONS TO CANDIDATE:

- Attempt all questions.
- Use the BOOKLETS provided.
- Write using blue or black pen.
- Start each question in a New Booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All questions are of equal value. Not all parts are of equal value.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- The format of the paper does not necessarily reflect that of the HSC.

Question 1. Start a New Booklet. 16 Marks

Marks

a) Find $\int x^2 e^{1-x^3}\, dx$ **1**

b) Using the substitution $u = 6 - x$, evaluate

$$\int_1^6 x\sqrt{6-x}\ dx$$ **3**

c) (i) Differentiate $y = 7^x$ with respect to x. **2**

(ii) Hence, or otherwise, find $\int 7^x\, dx$ **1**

d) Find the gradient of the curve $x^3 + 4xy + y^2 = 13$ at the point(s) where $x = 1$. **4**

e) (i) $x = 3\sec\theta$ and $y = 5\tan\theta$ are the parametric equations of a hyperbola.

Find the cartesian equation of the hyperbola. **1**

(ii) Write the equations of the asymptotes. **1**

(iii) Draw a diagram, with clear labelling, to show the relevance of the numbers 3 and 5. **2**

(iv) State the equation of the auxiliary circle. **1**

Question 2. Start a New Booklet. 16 Marks

Marks

a) Let $z = 3 + 2i$ and $w = 1 - i$

Find, in the form $x + iy$,

(i) $z\,w$ **1**

(ii) $\dfrac{z}{\overline{w}}$ **2**

b) (i) Express $1 - i$ in modulus-argument form. **2**

(ii) Express $(1 - i)^5$ in modulus-argument form. **2**

(iii) Hence, express $(1 - i)^5$ in the form $x + iy$. **1**

c) The ellipse E has the equation $\dfrac{x^2}{100} + \dfrac{y^2}{75} = 1$.

(i) Sketch the curve E, showing on your diagram the coordinates of the foci and equation of each directrix. **3**

(ii) Find the equation of the normal to the ellipse at the point $P(5, 7.5)$ **3**

(iii) Find the equation of the circle that is tangential to the ellipse at P and $Q\ (5, -7.5)$ **2**

Question 3. Start a New Booklet. 16 Marks

Marks

a) The polynomial $A(x) = x^4 + ax^2 + bx + 36$ has a double root at $x = 2$.

Find the values of a and b. **3**

b) A is the point on the Argand Diagram which represents the complex number $z = 3 + i$ and B is the point which represents $-z$.

(i) Draw a diagram showing the positions of A and B. **1**

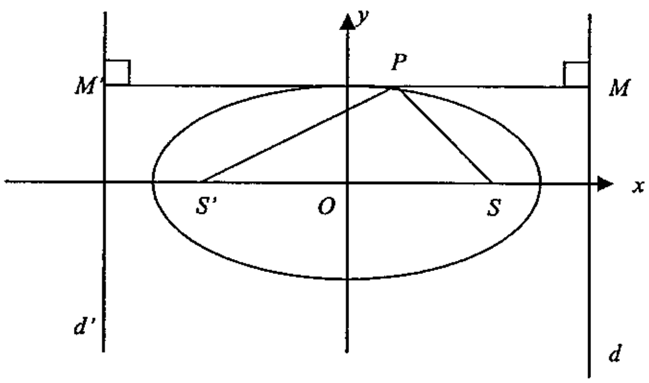
(ii) P and Q are points such that figure $APBQ$ is a rhombus whose area is 150 units^2.

Find the complex numbers represented by P and Q. **4**

(Question 3 Continued on next page)

(Question 3 continued) Marks

c) (i)



Use the diagram above, and the definition of an ellipse, to prove that

$$PS + PS' = 2a$$

2

(ii) Draw a number line to illustrate the meaning of $|x-3| = 4$ and complete the statement: " The distance of x from ……. " **1**

Use Parts (i) and (ii) to answer the following.

The equation $|z+3+i| + |z+3+9i| = 10$
represents an ellipse in the Argand Plane.

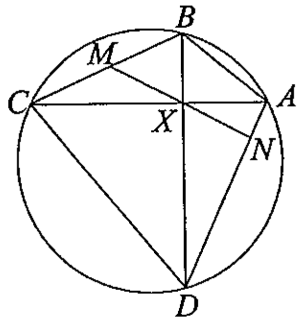
(iii) Sketch the ellipse on the Argand Plane showing:
the foci , the end-points of the major and minor axes, the centre of the ellipse. **4**

(iv) Write down the range of values of $\arg z$ for complex numbers z corresponding to points on the ellipse. **1**

Question 4. Start a New Booklet. 16 Marks

Marks

a)



$ABCD$ is a cyclic quadrilateral. The diagonals AC and BD intersect at right angles at X. M is the midpoint of BC. MX produced meets AD at N.

Copy the diagram above into your own booklet showing the above information. **1**

(i) Explain why BC is the diameter of the circle which can be drawn through points B, C and X. **2**

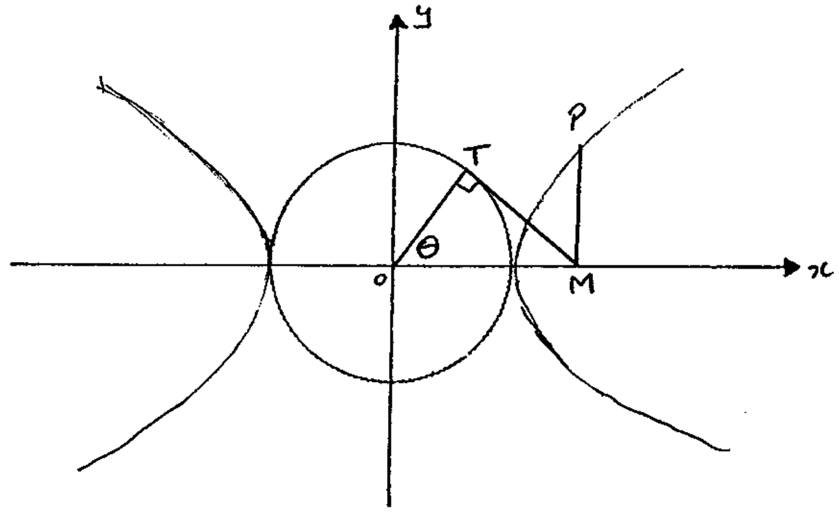
(ii) Show that $\angle MBX = \angle MXB$. **5**

(iii) Show that MN is perpendicular to AD.

(Question 4 Continued on next page)

(Question 4 continued) Marks

b)



The figure shows the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$, where $a, b > 0$.

The point T lies on the circle with $\angle TOM = \theta$, where $0 < \theta < \frac{\pi}{2}$. The tangent to the circle at T meets the x-axis at M, MP is perpendicular to the x-axis and P is a point on the hyperbola in the first quadrant.

(i) Show that P has coordinates $(a\sec\theta, \ b\tan\theta)$. **2**

(ii) Suppose that Q is a point on the hyperbola with coordinates $(a\sec\theta, b\tan\theta)$.
If $\theta + \phi = \frac{\pi}{2}$, and $\theta \neq \frac{\pi}{4}$, show that the chord PQ has equation

$$ay = b(\cos\theta + \sin\theta)x - ab$$ **3**

(iii) Show that every such chord PQ passes through a **fixed** point and find the coordinates of this point. **1**

(iv) Show that as θ approaches $\frac{\pi}{2}$ the chord PQ approaches a line parallel to an asymptote. **2**

Question 5. Start a New Booklet. 16 Marks

Marks

a) Suppose that b and d are real numbers and $d \neq 0$. Consider the polynomial

$$P(z) = z^4 + bz^2 + d$$

The polynomial has a double root α.

(i) Prove that $P'(z)$ is an odd function. **2**

(ii) Prove that $-\alpha$ is also a double root of $P(z)$. **1**

(iii) Prove that $d = \dfrac{b^2}{4}$ **2**

(iv) For what values of b does $P(z)$ have a double root equal to $\sqrt{3}\,i$? **1**

(v) For what values of b does $P(z)$ have real roots? **2**

b) (i) Show that for all values of A and B,

$$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$$ **2**

(ii) Use the method of Mathematical Induction to show that for all positive integers, n,

$$\cos x + \cos 2x + \cos 3x + \dots + \cos nx = \frac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$ **6**

---- End of Test ---

Question	Your Mk	Max Mk	Topic	Outcomes
1 a)		1	Harder ext 1-integration	HE6
1 b)		3	Harder ext 1-integration	HE6
1 c) (i)		2	Harder ext 1-diff'n integration	HE6
1 c) (ii)		1	Harder ext 1-diff'n integration	HE6
1 d)		4	Conics-implicit diff'n	E4
1 e) (i)		1	Conics-elim of parameters	E4
1 e) (ii)		1	Conics-knowledge of terms	E4
1 e) (iii)		2	Conics-understanding of a and b	E4
1 e) (iv)		1	Conics-knowledge of terms	E4
		16		
2 a) (i)		1	Complex Nos-manipulating algebra	E3
2 a) (ii)		2	Complex Nos-manipulating algebra	E3
2 b) (i)		2	Complex Nos-knowing terms	E3
2 b) (ii)		2	Complex Nos-De Moivre's rule	E3
2 b) (iii)		1	Complex Nos-mod/arg→cartesian	E3
2 c) (i)		3	Conics-knowledge of terms	E4
2 c) (ii)		3	Conics-diff'n equation normal	E4
2 c) (iii)		2	Conics-problem, language	E4
		16		
3 a)		3	Polys-finding a and b	PE3 E4
3 b) (i)		1	Complex nos-plotting pts	E3
3 b) (ii)		4	Complex nos-geometry of	E3 HE7 E9
3 c (i)		2	Conics-proof	E3
3 c) (ii)		1	Harder Ext 1- Geometry of distance	H9
3 c) (iii)		4	Conics-concept of defn & posn foci	E3 E9
3 c) (iv)		1	Complex nos-argument	E3 E9
		16		
4 a) (i)		1	Harder Ext 1-circle geometry	PE3
4 a) (ii)		2	Harder Ext 1-angles & isostriangles	PE3
4 a) (iii)		5	Harder Ext 1-angle sum triangle	PE3
4 b) (i)		2	Conics-concept aux.circle	E4
4 b) (ii)		3	Conics-equation of chord	E4
4 b) (iii)		1	Conics-concept of "fixed pt"	E3 E4
4 b) (iv)		2	Conics-asymptotes and limits	E3 E4
		16		
5 a) (i)		2	Polynomials-odd and even fns	E4
5 a) (ii)		1	Polys-concept of roots in odd fns	E4 HE7 E9
5 a) (iii)		2	Polynomials-use of data	E4 HE7 E9
5 a) (iv)		1	Polys-finding b given root	E4 HE7 E9
5 a) (v)		2	Polys-concept of real roots	E4 HE7 E9
5 b) (i)		2	Harder Ext 1-trig relnsps	HE2 E2
5 b) (ii)		6	Harder Ext 1-Mathematical Indn	E2 HE2
		16		
Total		80		

Year 12 Outcomes
Mathematics Extension 2
4th April, 2007 Assessment # 2

Name: ______________________________

Complex Numbers

2a) (i)	2 a) (ii)	2 b) (i)	2b) (ii)	2 b) (iii)	3 b) (i)	3b (ii)	3c) (iv)	Tot
1	2	2	2	1	1	4	1	14

Polynomials

3 a)	5a) (i)	5a) (ii)	5a) (iii)	5a) (iv)	5a) (v)	Tot
3	2	1	2	1	2	11

Harder Ext 1

1 a)	1 b)	1 c) (i)	1 c) (ii)	3c) (ii)	4a) (i)	4a) (ii)	4a) (iii)	5b) (i)	5b) (ii)	Tot
1	3	2	1	1	1	2	5	2	6	24

Conics

1 d)	1 e) (i	1 e) (ii)	1 e) (iii)	1 e) (iv)	2c) (i	2 c) (ii)	2c) (iii)	3 c) (i)	3c) (iii)	4b) (i)	4 b) (ii)	4 b) (ii i)	4 b) (i v)	Tot
4	1	1	2	1	3	3	2	2	4	2	3	1	2	31

2007 Mathematics Extension 2 Assessment # 2

Q1 a) $\int x^2 e^{1-x^3}\,dx = -\frac{1}{3}\int -3x^2 e^{1-x^3}\,dx$

$= -\frac{1}{3} e^{1-x^3} + K$ ($\frac{1}{2}$ off no K) 1

b) $\int_1^6 x\sqrt{6-x}\,dx = \int_5^0 (6-u)\,u^{\frac{1}{2}}(-du)$ $\frac{1}{2}$ Put $u = 6-x \Rightarrow x = 6-u$

$du = -1\,dx$

when $x=1,\ u=5$

$x=6,\ u=0$ $\frac{1}{2}$

$= -\int_5^0 6u^{\frac{1}{2}} - u^{1\frac{1}{2}}\,du$ $\frac{1}{2}$

$= -\left[\frac{6u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]_5^0$ $\frac{1}{2}$

$= -\left[(0) - \left(\frac{2\times6\times5^{3/2}}{3} - \frac{2\times5^{5/2}}{5}\right)\right]$ $\frac{1}{2}$

$= -\left[-\left(4\times\sqrt{5}^3 - \frac{2\times\sqrt{5}^5}{5}\right)\right]$

$= \left[20\sqrt{5} - \frac{2\times25\sqrt{5}}{5}\right]$

$= 20\sqrt{5} - 10\sqrt{5}$

$= 10\sqrt{5}$ $\frac{1}{2}$

c) (i) $y = 7^x$

$\therefore \log_7 y = x$ $\frac{1}{2}$

ie $x = \frac{\log_e y}{\log_e 7}$ $\frac{1}{2}$

$\frac{dx}{dy} = \frac{1}{\log_e 7}\cdot\frac{1}{y}$ $\frac{1}{2}$ $\Rightarrow \frac{dy}{dx} = y\log_e 7$

$= 7^x \log_e 7$ $\frac{1}{2}$

(ii) Hence, $\int 7^x\,dx = \frac{1}{\ln 7}\int \ln 7\; 7^x\,dx$ $\frac{1}{2}$

$= \frac{1}{\ln 7} 7^x + K$ $-\frac{1}{2}$

No need to differentiate

d) $x^3 + 4xy + y^2 = 13$

when $x = 1$, $1 + 4y + y^2 = 13$

$y^2 + 4y - 12 = 0$ $\frac{1}{2}$ $\quad y \times 6,\ y \times -2$

$(y+6)(y-2) = 0$

ie $y = -6$ or $y = 2$

Points are $(1,-6)$ or $(1,2)$ $\frac{1}{2}$

Now, $3x^2 + y \cdot 4 + 4x \cdot \frac{dy}{dx} + 2y\frac{dy}{dx} = 0.$ $1\frac{1}{2}$

$\frac{dy}{dx}(4x + 2y) = -3x^2 - 4y$

$\frac{dy}{dx} = \frac{-3x^2 - 4y}{4x + 2y}$ $\frac{1}{2}$

when $x = 1, y = -6$

$\frac{dy}{dx} = \frac{-3\times1 - 4\times-6}{4\times1 + 2\times-6}$

$= \frac{-3+24}{4-12}$

$= \frac{21}{-8}$ $\frac{1}{2}$

$= -\frac{21}{8}$

when $x = 1$, $y = 2$

$\frac{dy}{dx} = \frac{-3\times1 - 4\times2}{4\times1 + 2\times2}$

$= \frac{-3-8}{4+4}$ $\frac{1}{2}$

$= \frac{-11}{8}$

e) (i) $x = 3\sec\theta$, $y = 5\tan\theta$

$\frac{x^2}{9} = \sec^2\theta$ $\qquad \frac{y^2}{25} = \tan^2\theta$

Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore \tan^2\theta + 1 = \sec^2\theta$

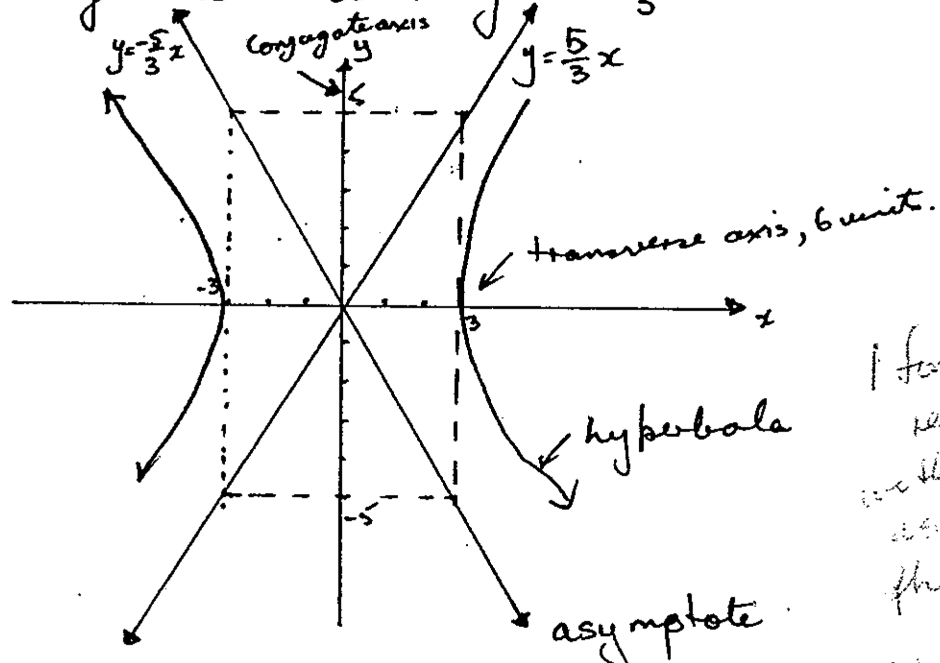
ie $\frac{y^2}{25} + 1 = \frac{x^2}{9}$

$\therefore \frac{x^2}{9} - \frac{y^2}{25} = 1$ is Cartesian form

(ii) Eqns of asymptotes are

$$y = \pm \frac{b}{a} x \quad \text{i.e.} \quad y = \pm \frac{5}{3} x$$

(iii)



1 for rectangle with the asymptotes thru corners

$\frac{1}{2}$ hyperbola

$\frac{1}{2}$ for numbers

(iv) Auxiliary circle is

$$x^2 + y^2 = 9$$

Question 2.

a) (i) $z\omega = (3+2i)(1-i)$
$= 3 - 3i + 2i - 2i^2$
$= 3 + 2 - i$
$= 5 - i$

(ii) $\dfrac{z}{\bar{\omega}} = \dfrac{3+2i}{1+i} \times \dfrac{1-i}{1-i}$
$= \dfrac{3 - 3i + 2i - 2i^2}{1 - i^2}$
$= \dfrac{5-i}{2}$
$= \dfrac{5}{2} - \dfrac{1}{2}i$

b) (i) $|1-i| = \sqrt{1^2+1^2}$
$= \sqrt{2}$

$\therefore\ 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$
$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

(ii) $(1-i)^5 = \sqrt{2}^5\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]^5$
$= 4\sqrt{2}\left[\cos\left(-\frac{5\pi}{4}\right) + i\sin\left(-\frac{5\pi}{4}\right)\right]$
$= 4\sqrt{2}\left[\cos\left(-\pi-\frac{\pi}{4}\right) + i\sin\left(-\pi-\frac{\pi}{4}\right)\right]$
$= 4\sqrt{2}\left[\cos\left(\pi-\frac{\pi}{4}\right) + i\sin\left(\pi-\frac{\pi}{4}\right)\right]$
$= 4\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

(iii) $(1-i)^5 = 4\sqrt{2}\left[\cos\left(\pi-\frac{\pi}{4}\right) + i\sin\left(\pi-\frac{\pi}{4}\right)\right]$

$= 4\sqrt{2}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$
$= -4 + 4i$

c) $\frac{x^2}{100} + \frac{y^2}{75} = 1$

(i)

$a^2 = 100 \qquad b^2 = 75$

$a = 10 \qquad b = 5\sqrt{3}$ $\frac{1}{2}$ (a, b)

$\therefore \; b^2 = a^2(1-e^2)$

$75 = 100(1-e^2)$

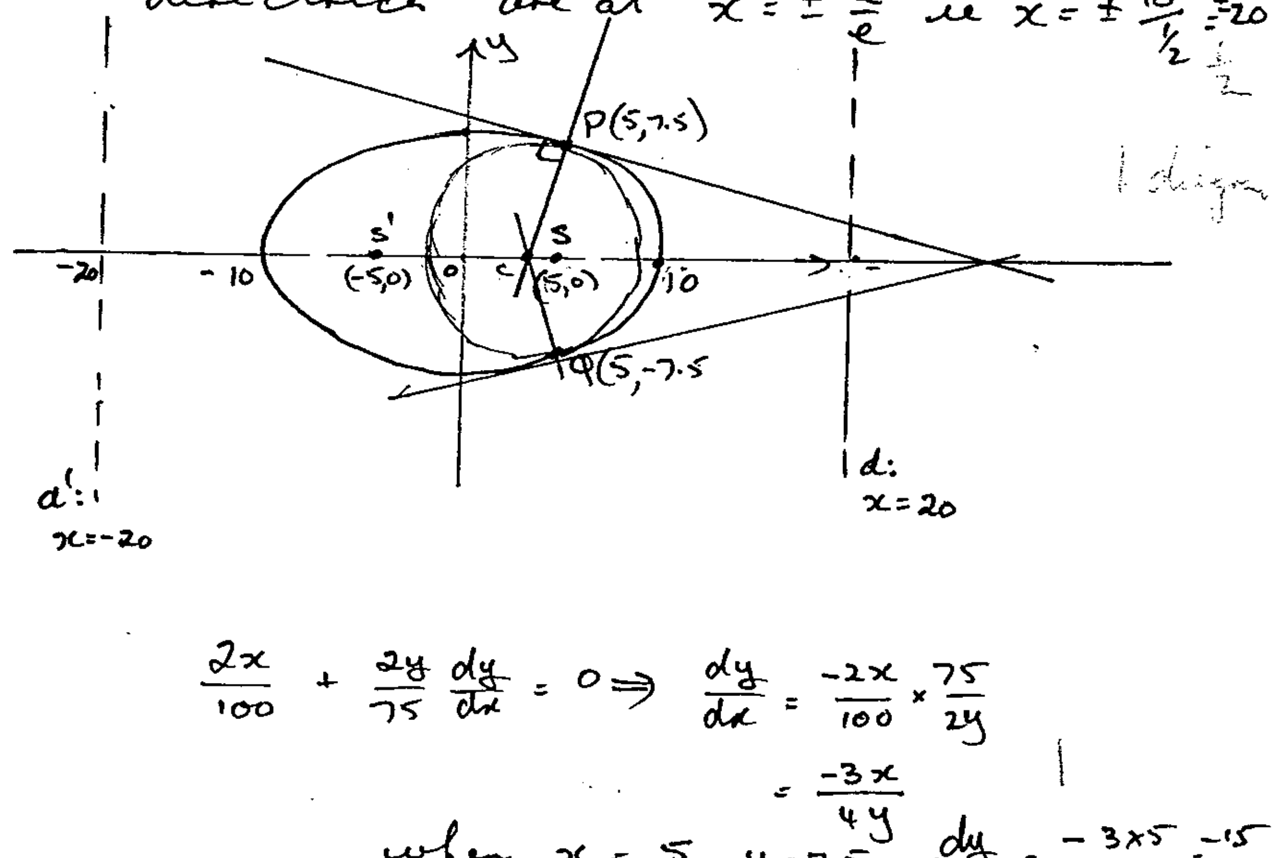
$\frac{75}{100} = 1 - e^2$ $\frac{1}{2}$ (e)

$e^2 = 1 - \frac{75}{100} = \frac{25}{100}$

$e = \frac{5}{10} = \frac{1}{2}, \quad 0 < e < 1.$

$\therefore$ Foci are at $(\pm ae, 0)$ ie $S(5,0)$ + $S'(-5,0)$ $\frac{1}{2}$

directrices are at $x = \pm \frac{a}{e}$ ie $x = \pm \frac{10}{1/2} = \pm 20$ $\frac{1}{2}$



1 diagram

(ii)

$\frac{2x}{100} + \frac{2y}{75}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{100} \times \frac{75}{2y}$ 1

$= \frac{-3x}{4y}$

when $x = 5$, $y = 7.5$, $\frac{dy}{dx} = \frac{-3\times5}{4\times7.5} = \frac{-15}{30}$

$= -\frac{1}{2}$ $\frac{1}{2}$

gradient of normal is 2 $\frac{1}{2}$

$\therefore$ Eqn of normal at P is $y - 7.5 = 2(x-5)$ $\frac{1}{2}$

$\therefore y - 7.5 = 2x - 10$

$2x - y - 2.5 = 0$

ie $4x - 2y - 5 = 0$ 1

$\frac{1}{2}$

(iii) Because of the symmetry of the ellipse & the points P & Q, the normals intersect on x-axis

ie when $y = 0$, $4x - 5 = 0$ $\frac{1}{2}$

$x = \frac{5}{4}$ ie $C\left(\frac{5}{4}, 0\right)$

distance PC is

$$\sqrt{\left(5 - \tfrac{5}{4}\right)^2 + (7.5 - 0)^2}$$

$$= \sqrt{\left(\tfrac{15}{4}\right)^2 + \ldots} \quad \tfrac{1}{2}$$

$$= \sqrt{\tfrac{225}{16} + \tfrac{225}{4}} = \sqrt{70\tfrac{5}{16}} = \sqrt{\tfrac{1125}{16}}$$

$\therefore$ Eqn circle is $\left(x - \frac{5}{4}\right)^2 + y^2 = \frac{1125}{16}$ $\frac{1}{2}$

Q3.

$A(x) = x^4 + ax^2 + bx + 36$ double root

$A(2) = 16 + 4a + 2b + 36 = 0$ $\frac{1}{2}$

$A'(x) = 4x^3 + 2ax + b$ $\frac{1}{2}$ single root

$A'(2) = 32 + 4a + b = 0$ $\frac{1}{2}$

$\therefore \quad 4a + 2b + 52 = 0$ $\frac{1}{2}$ — (1)

$4a + b + 32 = 0$ — (2)

(1) − (2) $\quad b + 20 = 0$

$b = -20$ $\frac{1}{2}$

Sub into (1): $4a + -40 + 52 = 0$

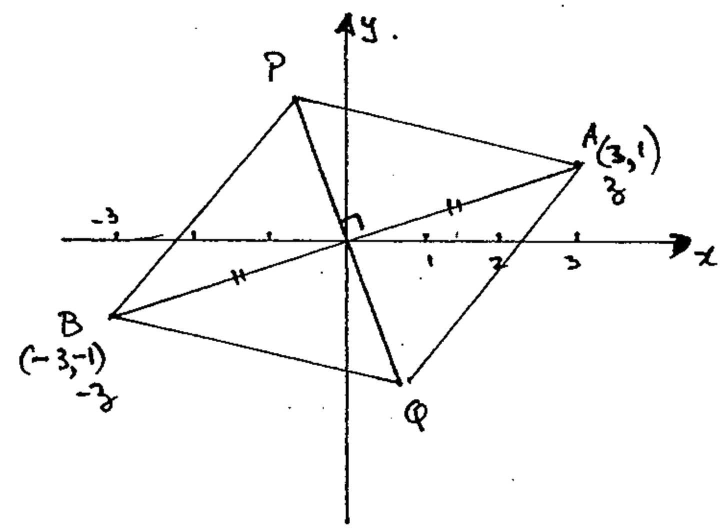
$4a + 12 = 0$

$4a = -12$

$a = -3$ $\frac{1}{2}$

$\therefore \quad a = -3, \quad b = -20$

b) (i)



(ii) Since APBQ is a rhombus then
if $P(\omega)$ ~~and~~ the $Q(-\omega)$
and $PQ \perp AB$
and $PO = OQ$, $AO = OB$ ½

now $A \to P$ by the rotation anticlockwise ½
and an enlargement by k ½

thus $ki(3+i) = 3ki + ki^2$
$= -k + 3ki$ ½

Thus P is $(-k, 3k)$ & Q is $(k, -3k)$

Now, $A = \frac{1}{2} \cdot AB \cdot PQ$

$150 = \frac{1}{2} \times \sqrt{40} \times \sqrt{40}\,k$

$300 = 40k$

$\frac{300}{40} = k$

$7.5 = k$ ½

$AB = \sqrt{(3-{-3})^2 + (1-{-1})^2}$
$= \sqrt{36+4}$
$= \sqrt{40}$ ½

$PQ = \sqrt{(k-{-k})^2 + (-3k-3k)^2}$
$= \sqrt{4k^2 + 36k^2}$
$= \sqrt{40}\,k$ ½

$\therefore$ P is $(-7.5, 22.5)$

Q is $(7.5, -22.5)$ ½

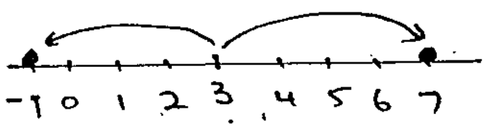
c) (i)

$\frac{PS}{PM} = e$ and $\frac{PS'}{PM'} = e$ $\frac{1}{2}$

$PS = e\,PM$ $\qquad PS' = e\,PM'$ $\frac{1}{2}$

$$PS + PS' = e(PM + PM') \quad \tfrac{1}{2}$$
$$= e\left[\left(\frac{a}{e} + \frac{a}{e}\right)\right]$$
$$= 2a \quad \tfrac{1}{2}$$

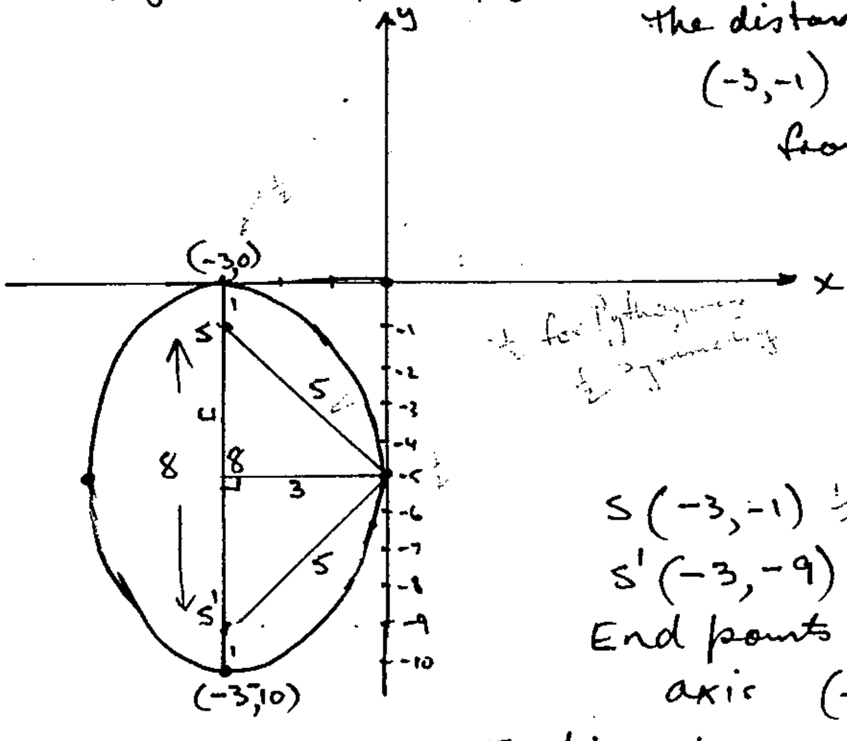
(ii)



"The distance of x from 3 is 4 units"

$$|z + 3 + i| + |z + 3 + 9i| = 10$$
$$|z - (-3 - i)| + |z - (-3 - 9i)| = 10$$

the distance of z from $(-3,-1)$ added to distance from $(-3,-9)$ is 10 units



$PS + PS' = 10$ $\frac{1}{2}$

$S(-3,-1)$ $\frac{1}{2}$

$S'(-3,-9)$ $\frac{1}{2}$

End points of major axis $(-3,0)$ and $(-3,-10)$

End point minor axis $(0,-5)$ and $(-6,-5)$

Centre of ellipse $(-3,-5)$ $\frac{1}{2}$

(iv) $-\pi \leq \arg z \leq -\frac{\pi}{2}$ is range of values

OR alternatively – although not using the method "told"

(iii) $|z+3+i| + |z+3+9i| = 10 \Rightarrow |(x+3)+i(1+y)| + |(x+3)+i(y+9)| = 10$

$$\sqrt{(x+3)^2+(1+y)^2} + \sqrt{(x+3)^2+(y+9)^2} = 10$$

$$\sqrt{(x+3)^2+(1+y)^2} = 10 - \sqrt{(x+3)^2+(y+9)^2}$$

$$(x+3)^2+(1+y)^2 = 100 + (x+3)^2+(y+9)^2 - 20\sqrt{(x+3)^2+(y+9)^2}$$

$$20\sqrt{(x+3)^2+(y+9)^2} = 100 + (y+9)^2 - (1+y)^2$$

$$20\sqrt{(x+3)^2+(y+9)^2} = 100 + [(y+9)+(1+y)][(y+9)-(1+y)]$$

$$20\sqrt{(x+3)^2+(y+9)^2} = 100 + (2y+10)(8)$$

$$5\sqrt{(x+3)^2+(y+9)^2} = 25 + 2(2y+10)$$

$$5\sqrt{(x+3)^2+(y+9)^2} = 25 + 4y + 20$$

$$5\sqrt{(x+3)^2+(y+9)^2} = 4y + 45$$

$$25[(x+3)^2+(y+9)^2] = 16y^2 + 360y + 2025$$

$$25(x+3)^2 + 25(y+9)^2 = 16y^2 + 360y + 2025$$

$$25(x+3)^2 + 25y^2 + 450y + 2025 = 16y^2 + 360y + 2025$$

$$25(x+3)^2 + 9y^2 + 90y = 0$$

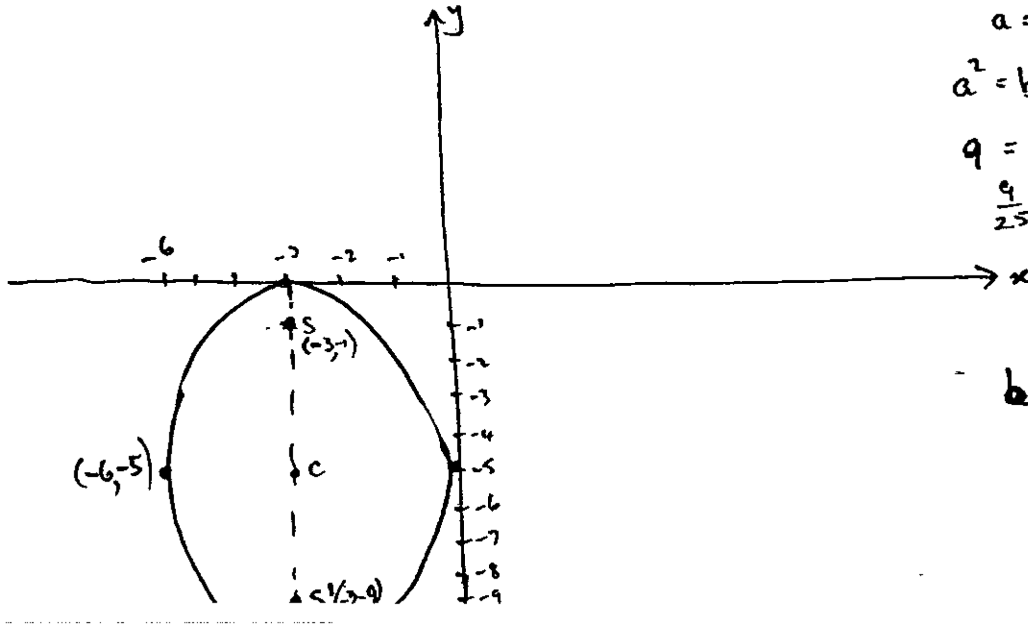
$$25(x+3)^2 + 9(y^2 + 10y + 25) = 225$$

$$25(x+3)^2 + 9(y+5)^2 = 225$$

$$\frac{25(x+3)^2}{225} + \frac{9(y+5)^2}{225} = 1$$

$$\frac{(x+3)^2}{(15/5)^2} + \frac{(y+5)^2}{(15/3)^2} = 1$$

$$\frac{(x+3)^2}{9} + \frac{(y+5)^2}{25} = 1$$



$a = 3$, $b = 5$

$a^2 = b^2(1-e^2)$

$9 = 25(1-e^2)$

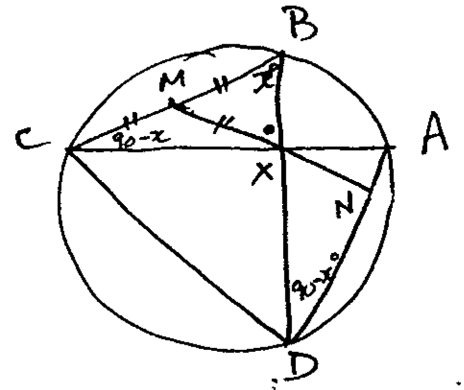
$\frac{9}{25} = 1-e^2$

$e^2 = 1 - \frac{9}{25} = \frac{16}{25}$

$e = \frac{4}{5}$

$be = 5 \times \frac{4}{5} = 4$

Q4 (a)



(i) Since $\angle CXB = 90°$ then BC is a diameter of the circle drawn through B, C + X since $\angle$ in a semi circle is $90°$.

(ii) ~~shows~~ Since M, midpt of BC is centre of circle through C, X and B.
then $MC = MB = MX$ (= radii same circle)
Then $\triangle MXB$ is isosceles.
Thus, $\angle MBX = \angle MXB$ (= base $\angle$'s opp = sides, isosceles $\triangle$)

(iii) Let $\angle MBX = x°$
Then $\angle BXM = x°$ (proved above)
$\angle BCX = 90° - x°$ ($\triangle BCX$ right $\angle$'d at X; $\angle XCB$ complementary angle to $\angle CBX$)

Also, $\angle ADB = \angle BCA = 90 - x°$ ($\angle$'s standing at circumference on same arc BA are =)

$\angle DXN = x° = \angle MXB$ (vertically opposite $\angle$'s)

Thus, in $\triangle DXN$,

$\angle XND + \angle DXN + \angle XDN = 180°$

ie $\angle XND + x° + 90° - x° = 180°$

$\angle XND = 90°$

ie $MN \perp AD$

b) (i) $\cos\theta = \dfrac{OT}{OM} = \dfrac{a}{OM}$

$\therefore \quad OM = \dfrac{a}{\cos\theta}$

ie $OM = a\sec\theta$

Now $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\therefore \quad \dfrac{a^2\sec^2\theta}{a^2} - \dfrac{y^2}{b^2} = 1$

$\sec^2\theta - 1 = \dfrac{y^2}{b^2}$

$\tan^2\theta = \dfrac{y^2}{b^2}$

$b^2\tan^2\theta = y^2$

$b\tan\theta = y$

ie $P\,(a\sec\theta,\ b\tan\theta)$

(ii) $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$

Eqn of chord PQ is

$$\frac{y - b\tan\theta}{x - a\sec\theta} = \frac{b\tan\phi - b\tan\theta}{a\sec\phi - a\sec\theta} \quad \begin{matrix}\times \cancel{\cos\theta}\cos\phi \\ \times \cos\theta.\cos\phi\end{matrix}$$

$$\frac{y\cos\theta - b\sin\theta}{x\cos\theta - a} = \frac{b(\sin\phi\cos\theta - \sin\theta\cos\phi)}{a(\cos\theta - \cos\phi)}$$

Now $\phi = \frac{\pi}{2} - \theta$,

$$\therefore \frac{y\cos\theta - b\sin\theta}{x\cos\theta - a} = \frac{b\left[\sin\left(\frac{\pi}{2}-\theta\right)\cos\theta - \sin\theta\cos\left(\frac{\pi}{2}-\theta\right)\right]}{a\left[\cos\theta - \cos\left(\frac{\pi}{2}-\theta\right)\right]}$$

$$= \frac{b}{a}\frac{[\cos\theta\cos\theta - \sin\theta\sin\theta]}{\cos\theta - \sin\theta}$$

$$= \frac{b}{a}\frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta - \sin\theta}.$$

$$\frac{y\cos\theta - b\sin\theta}{x\cos\theta - a} = \frac{b(\cos\theta + \sin\theta)}{a}$$

$$ay\cos\theta - ab\sin\theta = bx(\cos\theta + \sin\theta)\cos\theta - ab(\cos\theta + \sin\theta)$$

$$ay\cos\theta - ab\cancel{\sin\theta} = bx(\cos\theta + \sin\theta)\cos\theta - ab\cos\theta - ab\cancel{\sin\theta}$$

$$ay\cos\theta = bx(\cos\theta + \sin\theta)\cancel{\cos\theta} - ab\cos\theta$$

$$\therefore ay = bx(\cos\theta + \sin\theta) - ab$$

(iii)

$$y = \frac{b}{a}(\cos\theta + \sin\theta)x - b$$

$\therefore$ all such straight lines pass through $(0, -b)$

(iv) as $\theta \to \frac{\pi}{2}$, $y = \frac{b}{a}(1^- + 0^+)x - b$

ie $y = \frac{b}{a}x - b$

Now asymptotes are $y = \pm\frac{b}{a}x$ $\therefore$ this line is parallel to the asymptote $y = \frac{b}{a}x$ (same gradient

Q 5. a) $P(z) = z^4 + bz^2 + d$

(i) $P'(z) = 4z^3 + 2bz$

$$P'(z) = 4z^3 + 2bz = 2z(2z^2 + b)$$

$$P'(-z) = 4(-z)^3 + 2b(-z) = -4z^3 - 2zb = -2z(2z^2 + b)$$

$\therefore$ $P'(z) = -P'(-z)$

$\therefore$ fn $P'(z)$ is <u>odd</u>.

(ii) $(z - \alpha)$ is a factor of $P'(z)$

$(z-\alpha)^2$ " " " $P(z)$

Thus since $P'(z)$ is odd then $-\alpha$ is a root of $P'(z)$ to match root $x = \alpha$.

ie $(z + \alpha)$ is a factor of $P'(z)$

+ so, $(z+\alpha)^2$ is a factor of $P(z)$.

(iii)
$$P(z) = (z-\alpha)^2(z+\alpha)^2 = [(z-\alpha)(z+\alpha)]^2 = [z^2 - \alpha^2]^2$$

$$= z^4 - 2\alpha^2 z^2 + \alpha^4 \quad *$$

Thus equating coefficients.

$-2\alpha^2 = b$ and $\alpha^4 = d$

$\alpha^2 = \frac{b}{-2}$ $\quad (\alpha^2)^2 = d$

$\left(\frac{b}{-2}\right)^2 = d$

$\frac{b^2}{4} = d$

(iv) if $\alpha = \sqrt{3}\,i$, $\alpha^2 = 3i^2 = -3$

$\therefore$ $\alpha^2 = \frac{b}{-2} = -3$

$\therefore b = 6$

(v) For $P(z)$ to have real roots $\alpha^2 = \frac{b}{-2} > 0 \therefore b < 0$

b) (i) $\sin(A+B) - \sin(A-B) = \sin A\cos B + \sin B\cos A - [\sin A\cos B - \sin B\cos A]$

$= \sin A\cos B + \sin B\cos A - \sin A\cos B + \sin B\cos A$

$= 2\sin B\cos A$

$= 2\cos A\sin B$

To Prove that

(ii) $\cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \dfrac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$

Test $n=1$.

LHS $= \cos x$

RHS $= \dfrac{\sin(x+\frac{1}{2}x) - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$

$= \dfrac{\sin(x+\frac{1}{2}x) - \sin(x-\frac{1}{2}x)}{2\sin\frac{1}{2}x}$ from (i)

$= \dfrac{2\cos x\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$

$= \cos x$ $\therefore$ LHS $=$ RHS

True for $n=1$.

Assume true for $n=k$,

ie $\cos x + \cos 2x + \cdots + \cos kx = \dfrac{\sin(k+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$

Prove true for $n=k+1$,

$\cos x + \cos 2x + \cdots + \cos kx + \cos(k+1)x = \dfrac{\sin(k+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x} + \cos(k+1)x$

$= \dfrac{\sin(kx+\frac{1}{2}x) - \sin(\frac{1}{2}x) + 2\cos(k+1)x\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$

$= \dfrac{\sin(kx+\frac{1}{2}x) - \sin\frac{1}{2}x + [\sin(k+1+\frac{1}{2})x - \sin(k+1-\frac{1}{2})x]}{2\sin\frac{1}{2}x}$ from (i)

$= \dfrac{\sin(k+\frac{1}{2})x - \sin\frac{1}{2}x + \sin(k+1\frac{1}{2})x - \sin(k+\frac{1}{2})x}{2\sin\frac{1}{2}x}$

$= \dfrac{\sin(\overline{k+1}+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$

ie true for all n +ve integers.