Name: ____________________

Teacher: ____________________

Class: ____________________

FORT STREET HIGH SCHOOL

2020

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 2

Mathematics Extension 2

Time allowed: 60 minutes (plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description
MEX12-3	Uses vectors to model and solve problems in two and three dimensions
MEX12-5	Applies techniques of integration to structured and unstructured problems
MEX12-7	Applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems
MEX12-8	Communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

Total Marks: 41

Question	Total	Marks
Q1	/15	
Q2	/15	
Q3	/11	
Total	/41	
		%

General Instructions:

- Questions 1-3 are to be started in a new booklet
- The marks allocated for each question are indicated
- Show relevant mathematical reasoning and/or calculations
- Marks may be deducted for careless or badly arranged work
- NESA-approved calculators may be used

Question 1. **(15 marks)** *Start a separate booklet*

(a) Find k given the unit vector $\begin{pmatrix} 3/4 \\ -1/2 \\ k \end{pmatrix}$ **1**

(b) Find the angle between the vectors $\underset{\sim}{\mathbf{a}} = 3\underset{\sim}{\mathbf{i}} + \underset{\sim}{\mathbf{j}} - 2\underset{\sim}{\mathbf{k}}$ and $\underset{\sim}{\mathbf{b}} = 2\underset{\sim}{\mathbf{i}} + 5\underset{\sim}{\mathbf{j}} + \underset{\sim}{\mathbf{k}}$ **2**

(c) Find t given that $\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix}$ and $\begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix}$ are perpendicular **2**

(d) Evaluate the definite integral $\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)\, dx$ **3**

(e) Evaluate $\int_0^{1/2} xe^{2x}\, dx$ **3**

(f) By splitting $\dfrac{2x+2}{(x-1)\left(x^2+1\right)}$ into partial fractions, or otherwise, find $\int \dfrac{2x+2}{(x-1)\left(x^2+1\right)}dx$ **4**

Question 2. (15 marks) *Start a separate booklet*

(a) (i) If $t=\tan\frac{\theta}{2}$, show that $d\theta=\frac{2}{1+t^2}dt$ **1**

(ii) By using the substitution $t=\tan\frac{\theta}{2}$, show that $\int_0^{\frac{\pi}{3}}\sec\theta\ d\theta=\ln\left(2+\sqrt{3}\right)$ **3**

(b) Find $\int\frac{x}{\sqrt{1+x}}\,dx$ using the substitution $x=u^2-1$ **3**

(c) Find $\int\cos^5 x\ dx$ **3**

(d)

(i) Show that $\int_{-a}^{a}f\left(x\right)dx=\int_0^a f\left(x\right)dx+\int_0^a f\left(-x\right)dx$ **2**

(ii) Hence, or otherwise, evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{1-\sin x}$ **3**

Question 3. (11 marks) *Start a separate booklet*

(a) Use the substitution $x = \tan\theta$ to evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx$ **3**

(b) Let $\underset{\sim}{\mathbf{u}} = 3\underset{\sim}{\mathbf{i}} - \underset{\sim}{\mathbf{j}} - \underset{\sim}{\mathbf{k}}$, $\underset{\sim}{\mathbf{v}} = -\underset{\sim}{\mathbf{i}} + 3\underset{\sim}{\mathbf{j}} + \underset{\sim}{\mathbf{k}}$ and $\underset{\sim}{\mathbf{w}} = 2\underset{\sim}{\mathbf{i}} - 3\underset{\sim}{\mathbf{k}}$. Find $(\underset{\sim}{\mathbf{u}} \bullet \underset{\sim}{\mathbf{v}})\underset{\sim}{\mathbf{w}}$ **2**

(c)

(i) If $I_n = \displaystyle\int_0^1 \left(1-x^2\right)^{\frac{n}{2}}\,dx$, where n is a positive integer, show that $I_n = \dfrac{n}{n+1} I_{n-2}$ **4**

(ii) Hence, evaluate I_5 **2**

End of examination.

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Solutions

Question 1. (14 marks) *Start a separate booklet*

(a) Find k given the unit vector $\begin{pmatrix} 3/4 \\ -1/2 \\ k \end{pmatrix}$ **1**

Solution

$$\sqrt{\left(\frac{3}{4}\right)^2+\left(-\frac{1}{2}\right)^2+k^2}=1$$

$$1=k^2+\frac{13}{16}$$

$$\therefore\ k=\frac{\pm\sqrt{3}}{4}$$

Marking guideline

1 Correct values

Marker's comments

Take care with square roots to include answer. A mark was not deducted for this error.

(b) Find the angle between the vectors $\underset{\sim}{\mathbf{a}}=3\underset{\sim}{\mathbf{i}}+\underset{\sim}{\mathbf{j}}-2\underset{\sim}{\mathbf{k}}$ and $\underset{\sim}{\mathbf{b}}=2\underset{\sim}{\mathbf{i}}+5\underset{\sim}{\mathbf{j}}+\underset{\sim}{\mathbf{k}}$ to the nearest degree **2**

Solution

$$\cos\theta=\frac{\underset{\sim}{\mathbf{a}}\bullet\underset{\sim}{\mathbf{b}}}{|\mathbf{a}||\mathbf{b}|}$$

$$=\frac{3\times2+5-2}{\sqrt{14}\times\sqrt{30}}$$

$$=\frac{9}{\sqrt{420}}$$

$$\therefore\ \theta=\cos^{-1}\frac{9}{\sqrt{420}}=64^\circ$$

Marking guideline

2 Correct substitution into the formula
1 Correct calculation of to the nearest degree

Marker's comments

Ensure calculator is not in radians

(c) Find t given that $\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix}$ and $\begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix}$ are perpendicular **2**

Solution

$$(2-t)t+12+t(t+1)=0$$
$$2t-t^2+12+t^2+t=0$$
$$3t=-12$$
$$\therefore t=-4$$

Marking guideline

2 Correct use of dot product and equating to
1 Correct calculation of

Marker's comments

Mostly well done

(d) Evaluate the definite integral $\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)\,dx$ **3**

Solution

$$\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)\,dx=\frac{1}{2}\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\frac{d}{dx}\left(x^2\right)dx$$
$$=\frac{1}{2}\left[-\cos x^2\right]_0^{\sqrt{\pi}}$$
$$=\left[-\cos\pi+\cos 0\right]\times\frac{1}{2}$$
$$=(1+1)\frac{1}{2}$$
$$=1$$

Marking guideline

3 Correct response
2 Finding correct integral
1 Correctly re-writing integral using reverse chain rule

Marker's comments
Mostly well done

(e) Evaluate $\int_0^{1/2} xe^{2x}\, dx$ **3**

Solution

$u = x \qquad v' = e^{2x}$

$u' = 1 \qquad v = \frac{1}{2}e^{2x}$

$\int uv'\,dx = uv - \int u'v\,dx$

$\int_0^{1/2} xe^{2x}\, dx = \left[\frac{x}{2}e^{2x}\right]_0^{\frac{1}{2}} - \int_0^{1/2} \frac{1}{2}e^{2x}\, dx$

$= \left(\frac{1}{4}e^1 - 0\right) - \left[\frac{1}{4}e^{2x}\right]_0^{1/2}$

$= \frac{1}{4}e - 0 - \left(\frac{1}{4}e^1 - \frac{1}{4}\right) = \frac{1}{4}$

Marking guideline

3	**Correct response**
2	**Finding correct integral**
1	**Correctly re-writing integral using integration by parts**

Marker's comments

Take care in choosing and to not make the integral more or just as complicated as the original question

(f) By splitting $\frac{2x+2}{(x-1)(x^2+1)}$ into partial fractions, or otherwise, find $\int \frac{2x+2}{(x-1)(x^2+1)}dx$ **3**

Solution

$\frac{2x+2}{(x-1)(x^2+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+1}$

Hence $a(x^2+1) + (bx+c)(x-1) = 2x+2$

If $x = 1$

$2a = 4$

$a = 2$

If $x = 0$

$a + c \times -1 = 2$

$a - c = 2$

$c = 0$

If $x = 2$

$5a+2b+c=6$

$5\times 2+2b=6$

$2b=-4$

$b=-2$

$$\int \frac{2}{x-1}+\frac{-2x}{x^2+1}\,dx=2\log_e|x-1|-\log_e\left(x^2+1\right)+c$$

Marking guideline

3	**Correct response**
2	**Finding correct values of , and**
1	**Correctly re-writing integral using partial fractions**

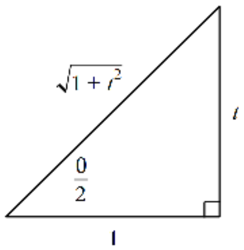
Marker's comments

Take care with indefinite integral to include . A mark was not deducted for this error.

Question 2. **(15 marks)** *Start a separate booklet*

(a) (i) If $t = \tan\dfrac{\theta}{2}$, show that $d\theta = \dfrac{2}{1+t^2}dt$ **1**

Solution



if $t = \tan\dfrac{\theta}{2}$

$$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$$

$$\begin{aligned}2\,dt &= \sec^2\frac{\theta}{2}d\theta \\ &= 1+\tan^2\frac{\theta}{2}\ \ d\theta \\ &= 1+t^2\ \ d\theta\end{aligned}$$

$$\therefore d\theta = \frac{2}{1+t^2}dt$$

Marking guideline

1 **Correct response**

Marker's comments

Students should be aware that in a show that question, especially involving trigonometry, you are required to demonstrate steps using course identities such as . This distinguishes you from a solution that simply quotes the answer.

In the solution below, the student has correctly differentiated but has not demonstrated or communicated why the second line can progress to the given solution.

(ii)

$$\int_0^{\frac{\pi}{3}} \sec\theta\, d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{\cos\theta} d\theta \qquad t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \text{ and } t = \tan 0 = 0$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2} dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(1-t)(1+t)} dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+t} + \frac{1}{1-t} dt$$

$$= \left[\ln\left(\frac{1+t}{1-t}\right)\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \ln\left(\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\right)$$

$$= \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}} \div \frac{\sqrt{3}-1}{\sqrt{3}}\right)$$

$$= \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

$$= \ln\left(\frac{3+2\sqrt{3}+1}{2}\right)$$

$$= \ln\left(\frac{4+2\sqrt{3}}{2}\right)$$

$$= \ln\left(2+\sqrt{3}\right)$$

Marking guideline

3	**Correct response**
2	**Finding correct integral**
1	**Correctly re-writing integral in terms of *t* (including correct limits)**

Marker's comments

Many students had difficulty with this question. Students should be mindful to change out the limits when using integration by substitution. Many students additionally found it difficult to separate the integral into .

The solution below illustrates a common error in incorrectly maintaining the limits.

(b) Find $\int \frac{x}{\sqrt{1+x}}\,dx$ using the substitution $x = u^2 - 1$ **3**

$$x = u^2 - 1$$
$$dx = 2u\,du$$
$$\frac{1}{2}dx = u\,du$$

$$\int \frac{x}{\sqrt{1+x}}\,dx = 2\int \frac{u^2-1}{\sqrt{1+u^2-1}} \times u\,du$$
$$= 2\int u^2 - 1\,du$$
$$= \frac{2u^3}{3} - 2u + c$$
$$= \frac{2}{3}\sqrt{x+1}^{\,3} - 2\sqrt{x+1} + c$$

Marking guideline

3	**Correct response**
2	**Finding correct integral**
1	**Some progress towards the answer**

Marker's comments
Mainly well answered with some students forgetting to convert *u* back into terms of *x*.

The solution below illustrates a common error.

(c) Find $\int \cos^5 x \; dx$ **3**

$$\int \cos^5 x \; dx = \int \cos^4 x \cos x \, dx$$
$$= \int \left(1-\sin^2 x\right)^2 \cos x \, dx$$
$$= \int \left(1-2\sin^2 x+\sin^4 x\right)\cos x \, dx$$
$$= \int \cos x - 2\sin^2 x \cos x + \sin^4 x \cos x \, dx$$
$$= \int \cos x - 2\int \sin^2 x \frac{d}{dx}\sin x + \int \sin^4 x \frac{d \sin x}{sx} dx$$
$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$$

Marking guideline

3	**Correct response**
2	**Some correct terms of the answer**
1	**Correctly re-writing integral in terms of *t* (including correct limits)**

Marker's comments
Many students had difficulty with this question. Students had difficulty in simplifying the terms so that they could apply the reverse chain rule.

(d)

(i) Show that $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx$ **2**

$$\int_{0}^{a} f(-x)\,dx = \int_{0}^{-a} -f(u)\,du = \int_{-a}^{0} f(u)\,du$$

Where $\begin{matrix} u=-x \\ du=-dx \end{matrix}$

Replacing u with x

$$\int_{0}^{a} f(-x)\,dx = \int_{-a}^{0} f(u)\,du$$

Therefore

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx\end{aligned}$$

(ii) Hence, or otherwise, evaluate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1-\sin x}$ **3**

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_{0}^{\frac{\pi}{4}} \frac{1}{1-\sin x} + \frac{1}{1-\sin(-x)}\,dx$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{1}{1-\sin x} + \frac{1}{1+\sin x}\,dx$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{1+\sin x+1-\sin x}{1-\sin^2 x}\,dx$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{2}{\cos^2 x}\,dx$$

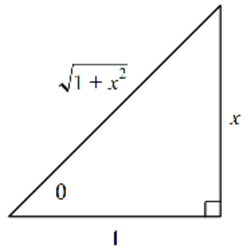
$$= 2\int_{0}^{\frac{\pi}{4}} \sec^2 x\,dx$$

$$= \left[2\tan x\right]_{0}^{\pi/4}$$

$$= 2$$

Question 3. (11 marks) *Start a separate booklet*

(a) Use the substitution $x = \tan\theta$ to evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx$ **3**



Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ $\qquad \cos\theta = \dfrac{1}{\sqrt{1+x^2}}$

Let
$x = 1$, $\tan\theta = 1$, $\theta = \dfrac{\pi}{4}$ $\qquad x = \sqrt{3}$, $\tan\theta = \sqrt{3}$, $\theta = \dfrac{\pi}{3}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta} \times \cos\theta \times \sec^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos^2\theta}{\sin^2\theta} \times \cos\theta \times \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot\theta\,\mathrm{cosec}\,\theta\,d\theta$$

$$= \left[-\mathrm{cosec}\,\theta\right]_{\pi/4}^{\pi/3}$$

$$= -\mathrm{cosec}\frac{\pi}{3} + \mathrm{cosec}\frac{\pi}{4}$$

$$= \frac{-1}{\sin\frac{\pi}{3}} + \frac{1}{\sin\frac{\pi}{4}}$$

$$= \frac{-2\sqrt{3}}{3} + \sqrt{2}$$

Marking guideline

3 **Correct response**
2 **Correct calculation of integral**
1 **Correctly rewriting integral in terms of θ**

Marker's comments:
Many students had difficulty with the calculation of the integral . they left the solution upto

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta} \times \sec\theta\, d\theta$$

(b) Let $\underset{\sim}{\mathbf{u}} = 3\underset{\sim}{\mathbf{i}} - \underset{\sim}{\mathbf{j}} - \underset{\sim}{\mathbf{k}}$, $\underset{\sim}{\mathbf{v}} = -\underset{\sim}{\mathbf{i}} + 3\underset{\sim}{\mathbf{j}} + \underset{\sim}{\mathbf{k}}$ and $\underset{\sim}{\mathbf{w}} = 2\underset{\sim}{\mathbf{i}} - 3\underset{\sim}{\mathbf{k}}$. Find $\left(\underset{\sim}{\mathbf{u}} \bullet \underset{\sim}{\mathbf{v}}\right)\underset{\sim}{\mathbf{w}}$ **2**

$3\times -1 + -1\times 3 + -1 = -7$

$-14\underset{\sim}{\mathbf{i}} + 21\underset{\sim}{\mathbf{k}}$

Marking guideline

2 **Correct response**
1 **Correct calculation of dot product**

Marker's comments
Mostly well done

(c)

(i) If $I_n = \int_0^1 \left(1-x^2\right)^{\frac{n}{2}}\, dx$, where n is a positive integer, show that $I_n = \dfrac{n}{n+1} I_{n-2}$ **4**

Let

$u = \left(1-x^2\right)^{\frac{n}{2}} \qquad v' = 1$

$du = \dfrac{n}{2}\left(1-x^2\right)^{\frac{n}{2}-1} \times -2x\, dx \qquad v = x$

$$I_n = \left[x\left(1-x^2\right)^{\frac{n}{2}}\right]_0^1 - \int x \times \frac{n}{2}\left(1-x^2\right)^{\frac{n}{2}-1} \times -2x\ dx$$

$$= 0 + n\int_0^1 x^2\left(1-x^2\right)^{\frac{n}{2}-1} dx$$

$$= n\int_0^1 x^2\left(1-x^2\right)^{\frac{n-2}{2}} dx$$

Integrating by parts again does not lead to the solution. Instead consider re-writing x^2 as $1-\left(1-x^2\right)$

$$n\int_0^1 x^2\left(1-x^2\right)^{\frac{n-2}{2}}\,dx = n\int_0^1\left(1-\left(1-x^2\right)\right)\left(1-x^2\right)^{\frac{n-2}{2}}\,dx$$

$$= n\int_0^1\left(1-x^2\right)^{\frac{n-2}{2}} - n\int_0^1\left(1-x^2\right)\left(1-x^2\right)^{\frac{n-2}{2}}\,dx$$

$$= n\int_0^1\left(1-x^2\right)^{\frac{n-2}{2}} - n\int_0^1\left(1-x^2\right)^{\frac{n-2}{2}} + n\int_0^1 x^2\left(1-x^2\right)^{\frac{n-2}{2}}\,dx$$

$$= nI_{n-2} + n\int_0^1\left(x^2-1\right)\left(1-x^2\right)^{\frac{n-2}{2}}\,dx + n\int_0^1\left(x^2-1\right)\left(1-x^2\right)^{\frac{n}{2}}\left(1-x^2\right)^{-2}\,dx$$

Marking guideline

4 **Correct response**
3 **Replacing x^2 with $1-\left(1-x^2\right)$ in the integral**
2 **Correct calculation of integral**
1 **Correctly selecting u and dv**

Marker's comments

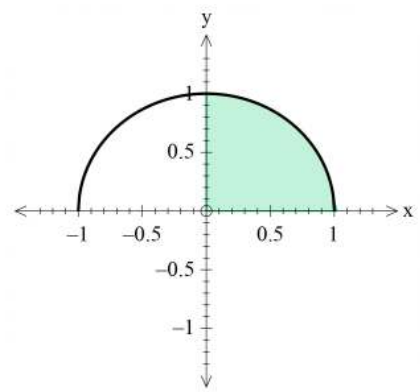
Not well done
Most of the students couldn't recognise how to change x^2 to $1-(1-x^2)$

(ii) Hence, evaluate I_5 **2**

Solution

$$I_5 = \frac{5}{6}I_3 = \frac{5}{6}\times\frac{3}{4}I_1 = \frac{2}{3}\times\frac{\pi}{4} = \frac{5\pi}{32}$$

$$I_1 = \int_0^1\left(1-x^2\right)^{\frac{1}{2}} = \frac{\pi\times 1^2}{4}\left(\text{since the area represents 1/4 a circle}\right) = \frac{\pi}{4}$$



Marking guideline

2 **Correct response**
1 **Correct substitution into recurrence formula in i)**

Marker's comments
Well done