Fort Street High School

2025

YEAR 12
MATHEMATICS EXTENSION 2

ASSESSMENT TASK 2

Syllabus Outcomes	
MEX12-1	understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts and logical argument
MEX12-2	choose appropriate strategies to construct arguments and proofs in both practical and abstract settings
MEX12-4	uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems
MEX12-7	Applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems
MEX12-8	communicates and justifies abstract ideas and relationships using appropriate language, notation

NESA Number: | | | | | | | | |

First Name: ______ **Last Name:** ______

Teacher			
	▢	▢	▢

Time allowed: 90 minutes (plus 5 minutes reading time)
Total Marks: 54

Section I – 5 marks
- Attempt Questions 1–5 on the multiple-choice answer sheet
- Allow about 10 minutes for this section.

Section II – 49 marks
- Attempt Questions 6–8
- Allow about 80 minutes for this section.

General Instructions:
- Write using a black or blue pen.
- A reference sheet containing course formulae has been provided.
- In Section II, answer each question in the relevant answer booklet.
- The marks allocated for each question are indicated.
- In all questions, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- NESA-approved calculators may be used.

Section I – Multiple Choice

5 Marks
Attempt question 1−5.
Allow about 10 minutes for this section.

Use the multiple-choice answer sheet for Questions 1−5

---

1. Given that $z = 1+2i$ is a root of the equation $z^2 - (3+i)z + k = 0$.

What is the value of k?

(A) $k = 3i$

(B) $k = 1-2i$

(C) $k = 2-i$

(D) $k = 4+3i$

2. Consider the following two statements.

$$P: \quad \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \ \ \mathrm{y}^3 = x$$

$$Q: \quad \exists \mathrm{y} \in \mathbb{R}, \forall x \in \mathbb{R}, y^3 = x$$

Which statement best represents the truth of each P and Q?

(A) P is true and Q is true.

(B) P is true and Q is false.

(C) P is false and Q is true.

(D) P is false and Q is false.

3. Which of the following is the negation of the statement:

"$\forall p \in P$, p is of the form $4m+1 \Rightarrow p$ can be written as a sum of two squares"?

(A) $\forall p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

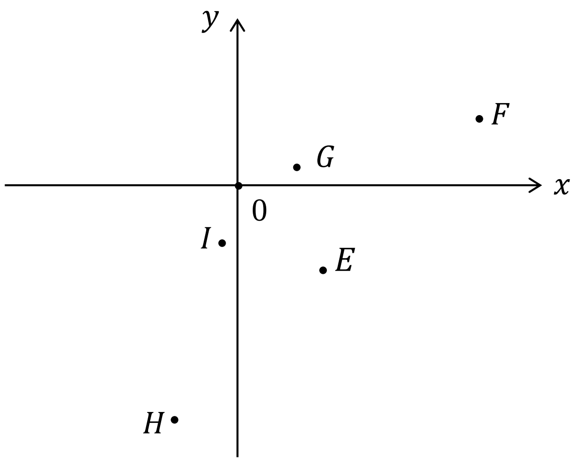
(B) $\exists p \in P$, p is not of the form $4m+1$ and p can be written as a sum of two squares.

(C) $\forall p \in P$, p is not of the form $4m+1$ or p cannot be written as a sum of two squares.

(D) $\exists p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

4. In the Argand diagram the point E represents a complex number.

When this number is multiplied by $\left(2e^{\frac{\pi}{6}i}\right)^4\left(\frac{1}{2}e^{-\frac{\pi}{3}i}\right)^3$ it gave a new complex number.



Which of the following points represents the new complex number?

(A) F

(B) G

(C) H

(D) I

5. A complex number ω satisfies

$$\omega^2 + \frac{1}{1+\omega^2} = \omega$$

Which of the following is the correct statement about ω^{2024}?

(A) $\omega^{2024} = \omega^4$

(B) $\omega^{2024} = \omega^3$

(C) $\omega^{2024} = \omega^2$

(D) $\omega^{2024} = \omega$

End of Section I

Section II

50 Marks
Attempt Questions 6, 7 and 8
Allow about 80 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing sheets are available.
In Questions 6,7 and 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (17 marks) Use a SEPARATE writing booklet.

(a) Solve $z^2 - z + (4 - 2i) = 0$. **4**

Give your answer in the form of $x + iy$,where x and y are real.

(b) Prove that $\sqrt[3]{4}$ is irrational. **3**

(c) For the polynomial $P(z) = az^4 + az^2 - 2z + b$, where a and b are real.

(i) Given that $P(1+i) = 0$, Show that $a = 1$ and $b = 6$ **3**

(ii) Show that $P(z)$ can be written as the product of two quadratic factors with real coefficients. **2**

(iii) Hence or otherwise solve the equation $P(z) = 0$ **2**

(d) Solve for x **3**

$$\frac{\sqrt{2}e^{ix}}{2+i} = \frac{1-3i}{5} \quad \text{where } -\pi < x \le \pi$$

End of Question 6

Question 7 (17 marks) Use a SEPARATE writing booklet.

(a) The points A and B represent the complex numbers **3**

$$z_1 = \ln^3 \alpha + i \ln^2 \alpha \text{ and } z_2 = -2\ln\alpha + 8i \quad \text{respectively.}$$

The point B is rotated about the origin by $\dfrac{\pi}{2}$ in a clockwise direction to a new point C.

Find out the value/s of α so that the points A and C coincide.

(b) (i) Prove that for $\alpha \in \mathbb{C}$ if $\alpha = \overline{\alpha}$, then α is real. **1**

(ii) Hence or otherwise, prove that if the complex numbers **2**

z and w are such that $|z| = |w| = 1$, then $\dfrac{z+w}{1+zw}$ is real.

(c) Prove that $x^3 + \dfrac{1}{x^3} \geq 2$, where $x > 0$. **2**

(d) (i) Find the roots of the complex equation. **3**

$$z^5 = i$$

(ii) Hence show that **3**

$$\sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = \frac{1}{2}.$$

(e) Prove, by contraposition, $\forall\, x, y \in \mathbb{R}$ **3**

if $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet

(a)

(i) Sketch, on a single Argand Diagram, the region $\mathbb{C}$ representing both of the following conditions **3**

$$-\frac{\pi}{4} \leq \arg(z-i) \leq 0 \quad \text{and} \quad |z-1| \leq 1$$

(ii) Hence, determine, $\forall z \in \mathbb{C}$, the range of values of $|z+i|$. **2**

(b) Without using induction, prove that,

$\forall n \in \mathbb{Z}, \quad n^3 - n$ is divisible by 6. **2**

Question 8 continues on next page

Question 8 (continued)

(c) A sequence of complex numbers a_0 , a_1 , a_2 , a_3 ,......., a_n is defined as

$$a_{k+1} = \frac{i(n-k)a_k}{(k+1)} \quad , \ n \text{ is an integer, } n \geq 0 \text{ and } 0 \leq k \leq n-1$$

(i) Use mathematical induction to prove that **3**

$$a_r = i^r \binom{n}{r} a_0 \qquad \text{, for all integers } r \geq 0$$

(ii) It is known that $a_0 + a_1 + a_2 + ... + a_n = 2^n$ **2**

Show that $a_0 = (1-i)^n$

(iii) It is given that $\sum_{m=0}^{n} \binom{n}{m}^2 = \binom{2n}{n}$ (Do Not Prove This) **3**

Also, it is given that if x_1 , x_2 , ..., x_n are positive real numbers,

then

$$\frac{x_1 + x_2 + ...x_n}{n} \geq \sqrt[n]{x_1 x_2 ......x_n} \qquad \text{(Do Not Prove This)}$$

Hence or otherwise show that

$$|a_0|^2 + |a_1|^2 + |a_2|^2 + ... + |a_n|^2 \leq \frac{(3n+1)^n}{n!}$$

End of Paper

Fort Street High School
YEAR 12
MATHEMATICS EXTENSION 2

2025

ASSESSMENT TASK 2

Syllabus Outcomes	
MEX12-1	understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts and logical argument
MEX12-2	choose appropriate strategies to construct arguments and proofs in both practical and abstract settings
MEX12-4	uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems
MEX12-7	Applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems
MEX12-8	communicates and justifies abstract ideas and relationships using appropriate language, notation

NESA Number: | | | | | | | | |

First Name: ____________ **Last Name:** ____________

Solutions

Time allowed: 90 minutes (plus 5 minutes reading time)
Total Marks: 54

Section I – 5 marks
- Attempt Questions 1–5 on the multiple-choice answer sheet
- Allow about 10 minutes for this section.

Section II – 49 marks
- Attempt Questions 6–8
- Allow about 80 minutes for this section.

General Instructions:
- Write using a black or blue pen.
- A reference sheet containing course formulae has been provided.
- In Section II, answer each question in the relevant answer booklet.
- The marks allocated for each question are indicated.
- In all questions, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- NESA-approved calculators may be used.

Section I – Multiple Choice

5 Marks
Attempt question 1−5.
Allow about 10 minutes for this section.

Use the multiple-choice answer sheet for Questions 1−5

---

1. Given that $z = 1 + 2i$ is a root of the equation $z^2 - (3+i)z + k = 0$.

Let the other root is $a+bi$

$1+2i+a+bi = 3+i$

$(a+1)+(b+2)i = 3+i$

$a+1=3$, $b+2=1$

$a=2$, $b=-1$

$\therefore$ The othe Root is $2-i$

Product of Roots $= (1+2i)(2-i)$

$k = (1+2i)(2-i)$

$= 2+2-i+4i$

$= 4+3i$

What is the value of k ?

(A) $k = 3i$

(B) $k = 1-2i$

(C) $k = 2-i$

(D) $k = 4+3i$

2. Consider the following two statements.

$P: \quad \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad y^3 = x$

P is true and Q is false.

$Q: \quad \exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y^3 = x$

Which statement best represents the truth of each P and Q?

(A) P is true and Q is true.

(B) P is true and Q is false.

(C) P is false and Q is true.

(D) P is false and Q is false.

3. Which of the following is the negation of the statement:

"$\forall p \in P$, p is of the form $4m+1 \Rightarrow p$ can be written as a sum of two squares"?

(A) $\forall p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

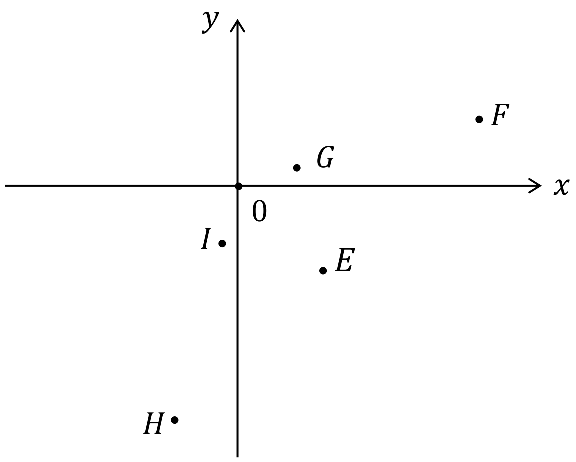
(B) $\exists p \in P$, p is not of the form $4m+1$ and p can be written as a sum of two squares.

(C) $\forall p \in P$, p is not of the form $4m+1$ or p cannot be written as a sum of two squares.

(D) $\exists p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

4. In the Argand diagram the point E represents a complex number.

When this number is multiplied by $\left(2e^{\frac{\pi}{6}i}\right)^4 \left(\frac{1}{2}e^{-\frac{\pi}{3}i}\right)^3$ it gave a new complex number.



Which of the following points represents the new complex number?

(A) F

(B) G

(C) H

(D) I

$\sqrt{3} + i = 2 \operatorname{cis} \frac{\pi}{6}$

$(\sqrt{3}+i)^4 = 2^4 \operatorname{cis} 4 \times \frac{\pi}{6} = 16 \operatorname{cis} \frac{2\pi}{3}$

$\frac{1}{4} - \frac{\sqrt{3}}{4} i = \frac{1}{2} \operatorname{cis}\left(-\frac{\pi}{3}\right)$

$\left(\frac{1}{4} - \frac{\sqrt{3}}{4} i\right)^3 = \left(\frac{1}{2}\right)^3 \operatorname{cis} 3 \times -\frac{\pi}{3} = \frac{1}{8} \operatorname{cis}(-\pi)$

$(\sqrt{3}+i)^4 \left(\frac{1}{4} - i\frac{\sqrt{3}}{4}\right)^3 = 16 \operatorname{cis} \frac{2\pi}{3} \times \frac{1}{8} \operatorname{cis}(-\pi)$

$= 2 \operatorname{cis}\left(\frac{2\pi}{3} + (-\pi)\right)$

$= 2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$

The new point is H

5. A complex number ω satisfies

$$\omega^2 + \frac{1}{1+\omega^2} = \omega$$

Which of the following is the correct statement about ω^{2024}?

(A) $\omega^{2024} = \omega^4$

(B) $\omega^{2024} = \omega^3$

(C) $\omega^{2024} = \omega^2$

(D) $\omega^{2024} = \omega$

$$\omega^2 + \frac{1}{1+\omega^2} = \omega$$

$$\frac{\omega^2 + \omega^4 + 1}{1+\omega^2} = \omega$$

$$\omega^2 + \omega^4 + 1 = \omega + \omega^3$$

$$\omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$$

$$(\omega+1)(\omega^4 - \omega^3 + \omega^2 - \omega + 1) = 0 \times (\omega+1)$$

$$\omega^5 + 1 = 0$$

$$\omega^5 = -1$$

$$\omega^{2024} = (\omega^5)^{404} \times \omega^4$$

$$= (-1)^{404} \times \omega^4$$

$$= 1 \times \omega^4$$

$$= \omega^4$$

End of Section I

Section II

50 Marks

Attempt Questions 6, 7 and 8

Allow about 80 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing sheets are available.
In Questions 6,7 and 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (17 marks) Use a SEPARATE writing booklet.

(a) Solve $z^2 - z + (4 - 2i) = 0$. **4**

Give your answer in the form of $x + iy$, where x and y are real.

Q6.
(a) $z^2 - z + (4-2i) = 0$

$\Delta = (-1)^2 - 4 \times 1 \times (4-2i)$

$= 1 - 4(4-2i)$

$= 1 - 16 + 8i$

$= -15 + 8i$ ✓

$z = \dfrac{-(-1) \pm \sqrt{-15+8i}}{2}$

Let $x + iy = \sqrt{-15+8i}$

$(x+iy)^2 = -15 + 8i$

$x^2 - y^2 = -15$ ① $\quad 2xy = 8$ ② ✓

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$(x^2+y^2)^2 = (-15)^2 + 8^2$

$= 225 + 64$

$= 289$

$x^2 + y^2 = 17$ ③ as $x^2+y^2 > 0$

① + ③ $\quad 2x^2 = 2$

$x^2 = 1$

$x = \pm 1$

③ − ① $\quad 2y^2 = 32$

$y^2 = 16$

$y = \pm 4$ ✓

Marker's comments:

Generally Answered well.

Since $xy > 0$

$x = 1, \quad y = 4$

$x = -1, \quad y = -4$

$\therefore \quad x + iy = \pm(1 + 4i)$

$$z = \frac{1 \pm (1+4i)}{2}$$

$$z = \frac{1 + (1+4i)}{2} \quad \text{or} \quad z = \frac{1 - (1+4i)}{2}$$

$$= \frac{2 + 4i}{2} \qquad\qquad = \frac{-4i}{2}$$

$$= 1 + 2i \qquad\qquad = 2i$$

$\therefore \quad \boxed{z = 1 + 2i} \quad \text{or} \quad \boxed{z = -2i}$

$\boxed{4}$ marks

(b) Prove that $\sqrt[3]{4}$ is irrational. **3**

Q6. (b) Assume $\sqrt[3]{4}$ is rational

$\therefore$ $\sqrt[3]{4} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and have no common factor except 1 ✓

$4 = \frac{p^3}{q^3}$

$p^3 = 4q^3$ ①

$\Rightarrow p^3$ is even

$\therefore p$ is even

Let $p = 2k$ where k is an integer ✓

Then from ①

$(2k)^3 = 4q^3$

$8k^3 = 4q^3$

$q^3 = 2k^3$

$\Rightarrow q^3$ is even

$\Rightarrow q$ is even

$\Rightarrow p$ and q have common factor 2

which is a contradiction as p and q have no common factors except 1

Hence $\sqrt[3]{4}$ is irrational ✓

$\boxed{3}$ marks

Marker's comments:

Many students made the incorrect statement that if b^3 has a factor of 4, then b also has a factor of 4. This resulted in loss of a mark.

(c) For the polynomial $P(z) = az^4 + az^2 - 2z + b$, where a and b are real.

(i) Given that $P(1+i) = 0$, Show that $a = 1$ and $b = 6$ **3**

Q6 (c) (i)

$P(z) = az^4 + az^2 - 2z + b$

$P(1+i) = a(1+i)^4 + a(1+i)^2 - 2(1+i) + b$

$= a(2i)^2 + a(2i) - 2 - 2i + b$ ✓

$= -4a + 2ai - 2 - 2i + b$

$= (-4a - 2 + b) + i(2a - 2)$ ✓

Given $P(1+i) = 0$

$\therefore \; -4 \cdot a - 2 + b = 0$ $\quad 2a - 2 = 0$

$\boxed{a = 1}$

$-4 \times 1 - 2 + b = 0$

$\boxed{b = 6}$

Hence Shown. ✓ $\boxed{3}$

Marker's comments:
Many students used sum and product of roots to various levels of success. The simplest solution is as shown here.

(ii) Show that $P(z)$ can be written as the product of two quadratic factors with real coefficients. **2**

Since all the Coefficients are real, and $P(1+i) = 0$

$\therefore$ $1+i$ is a root of $P(z)$

$1-i$ " also a root of $P(z)$

$P(z) = (z - (1+i))(z - (1-i))\,Q(z)$

$= (z^2 - 2z + 2)\;Q(z)$ ✓

$$
\begin{array}{r l}
 & z^2 + 2z + 3 \\
z^2 - 2z + 2 \,\big) & z^4 + z^2 - 2z + 6 \\
 & z^4 + 2z^2 \quad -2z^3 \\
\hline
 & 2z^3 - z^2 - 2z + 6 \\
 & 2z^3 - 4z^2 + 4z \\
\hline
 & 3z^2 - 6z + 6 \\
 & 3z^2 - 6z + 6 \\
\hline
 & \times
\end{array}
$$

$\therefore \; P(z) = (z^2 - 2z + 2)(z^2 + 2z + 3)$ ✓ $\boxed{2}$

Marker's comments:
Students are reminded that solution "by inspection" is not enough for a 2 mark question. This time, students were not penalised. As the previous part, many students attempt to use sum and product of roots to various levels of success.

(iii) Hence or otherwise solve the equation $P(z)=0$ **2**

(iii) $P(z) = 0$

$(z-(1+i))(z-(1-i))(z^2+2z+3) = 0$

$z = 1+i$, or $z = 1-i$ ✓

or $z^2+2z+3=0$

$z = \dfrac{-2 \pm \sqrt{4-12}}{2}$

$= \dfrac{-2 \pm 2\sqrt{2}i}{2}$

$= -1 \pm \sqrt{2}i$ ✓

$\therefore z = 1+i$, or $z = 1-i$ or $z = -1+\sqrt{2}i$

or $z = -1-\sqrt{2}i$

$\boxed{2}$ marks

Marker's comments:
Students who had successfully found the roots earlier were given full marks.

(d) Solve for x **3**

$$\frac{\sqrt{2}e^{ix}}{2+i} = \frac{1-3i}{5} \quad \text{where } -\pi < x \le \pi$$

$\dfrac{\sqrt{2}\,e^{ix}}{2+i} = \dfrac{1-3i}{5}$

$e^{ix} = \dfrac{1-3i}{5} \times \dfrac{2+i}{\sqrt{2}}$

$e^{ix} = \dfrac{(1-3i)(2+i)}{5\sqrt{2}}$

$= \dfrac{1}{5\sqrt{2}}(2+3+i-6i)$ ✓

$= \dfrac{1}{5\sqrt{2}}(5-5i)$

$= \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i$ ✓

$e^{ix} = e^{i(-\frac{\pi}{4})}$ as $-\pi < x \le \pi$

$x = -\dfrac{\pi}{4}$. ✓

$\boxed{3}$ marks

Marker's comments:

Generally answered well. Common error was ignoring the given domain.

End of Question 6

Question 7 (17 marks) Use a SEPARATE writing booklet.

(a) The points A and B represent the complex numbers **3**

$$z_1 = \ln^3\alpha + i\ln^2\alpha \text{ and } z_2 = -2\ln\alpha + 8i \quad \text{respectively.}$$

The point B is rotated about the origin by $\dfrac{\pi}{2}$ in a clockwise direction to a new point C.

Find out the value/s of α so that the points A and C coincide.

(a) $z_1 = \ln^3\alpha + i\ln^2\alpha$, $z_2 = -2\ln\alpha + 8i$

The point C corresponds to the complex number $-i(-2\ln\alpha + 8i)$

$= 2i\ln\alpha + 8$ ✓

Since A and C coincide

$\therefore \ln^3\alpha + i\ln^2\alpha = 2i\ln\alpha + 8$

$= 8 + 2i\ln\alpha$

Comparing Real and imaginary parts

$\ln^3\alpha = 8$ (1) $\qquad \ln^2\alpha = 2\ln\alpha$ (2)

$\ln\alpha = 2 \qquad \ln^2\alpha - 2\ln\alpha = 0$

$\alpha = e^2 \qquad \ln\alpha(\ln\alpha - 2) = 0$

$\ln\alpha = 0$ or $\ln\alpha = 2$

$\alpha = 1$ or $\alpha = e^2$ ✓

Since α should satisfy both conditions (1) and (2)

$\therefore \boxed{\alpha = e^2}$

✓

3

Marker's comments:

Some students did not know the meaning of coincidence and did not allow for comparison of both real and imaginary parts.

(b) (i) Prove that for $\alpha \in \mathbb{C}$ if $\alpha = \overline{\alpha}$, then α is real. **1**

(i) Let $\alpha = a + ib$ where a and b are real

$\alpha = \overline{\alpha}$

$a + ib = a - ib$

$2ib = 0$

$b = 0$

$\boxed{\alpha = a}$

$\therefore$ α is real. ✓ $\boxed{1}$

Marker's comments:

Generally answered well.

(ii) Hence or otherwise, prove that if the complex numbers z and w are such that $|z| = |w| = 1$, then $\dfrac{z+w}{1+zw}$ is real. **2**

(ii) $\overline{\left(\dfrac{z+w}{1+zw}\right)} = \dfrac{\overline{z+w}}{\overline{1+zw}}$

$= \dfrac{\overline{z} + \overline{w}}{\overline{1} + \overline{zw}}$

$= \dfrac{\overline{z} + \overline{w}}{1 + \overline{z}\,\overline{w}}$ ① ✓

Since $|z| = 1$ $\qquad$ $|w| = 1$

$|z|^2 = 1$ $\qquad$ $|w|^2 = 1$

$z\overline{z} = 1$ $\qquad$ $w\overline{w} = 1$

$\overline{z} = \dfrac{1}{z}$ $\qquad$ $\overline{w} = \dfrac{1}{z}$

From ① $\overline{\left(\dfrac{z+w}{1+zw}\right)} = \dfrac{\frac{1}{z} + \frac{1}{w}}{1 + \frac{1}{z}\frac{1}{w}} = \dfrac{\frac{z+w}{zw}}{\frac{zw+1}{zw}}$

$= \dfrac{z+w}{zw+1} = \dfrac{z+w}{1+zw}$

Since $\overline{\left(\dfrac{z+w}{1+zw}\right)} = \dfrac{z+w}{1+zw}$

$\therefore$ $\dfrac{z+w}{1+zw}$ is Real ✓ $\boxed{2}$

Marker's comments:

Not answered very well. Students missed the link with part (i) which resulted in length approaches and errors. There was also lack of recall of features of complex numbers and conjugate, making it difficult for students to arrive at a result.

(c) Prove that $x^3 + \dfrac{1}{x^3} \geq 2$, where $x > 0$. **2**

Q7. (c) To prove that

$$x^3 + \frac{1}{x^3} - 2 \geq 0 \quad \text{for } x > 0$$

Now $x^3 + \dfrac{1}{x^3} - 2$

$$= \frac{x^6 + 1 - 2x^3}{x^3}$$

$$= \frac{(x^3 - 1)^2}{x^3}$$ ✓

Since $(x^3-1)^2$ is a perfect square

$\therefore (x^3-1)^2 \geq 0$

Also $x^3 > 0$ as $x > 0$

$\therefore \dfrac{(x^3-1)^2}{x^3} > 0$

$\therefore x^3 + \dfrac{1}{x^3} - 2 \geq 0$

or $x^3 + \dfrac{1}{x^3} \geq 2$

Hence proved ✓ $\boxed{2}$

Marker's comments:

Students used a variety of approaches btu the question was generally answered well

(d) (i) Find the roots of the complex equation. **3**

$$z^5 = i$$

Q7. (d) $z^5 = i$ (12)

Let $z = r\,cis\theta$

$z^5 = r^5 cis 5\theta$ By De moivre's Theorem

$r^5 cis 5\theta = i$

$r^5 cis 5\theta = cis(4k+1)\frac{\pi}{2}$ ✓

$r^5 = 1$, $5\theta = (4k+1)\frac{\pi}{2}$

$\therefore r = 1$ $\theta = (4k+1)\frac{\pi}{10}$

Since $-\pi < \theta \leq \pi$

$-\pi < (4k+1)\frac{\pi}{10} \leq \pi$

$-10 < 4k+1 \leq 10$

$-11 < 4k \leq 9$

$-\frac{11}{4} < k \leq \frac{9}{4}$

$k = -2, -1, 0, 1, 2$ ✓

$k = -2$, $\theta = -\frac{7\pi}{10}$

$k = -1$, $\theta = -\frac{3\pi}{10}$

$k = 0$, $\theta = \frac{\pi}{10}$

$k = 1$, $\theta = \frac{5\pi}{10} = \frac{\pi}{2}$

$k = 2$, $\theta = \frac{9\pi}{10}$

3

$\therefore$ The Roots are

$cis\left(-\frac{7\pi}{10}\right)$, $cis\left(-\frac{3\pi}{10}\right)$, $cis\left(\frac{\pi}{10}\right)$, $cis\left(\frac{\pi}{2}\right)$, $cis\left(\frac{9\pi}{10}\right)$

or $e^{-\frac{7i\pi}{10}}$, $e^{-\frac{3i\pi}{10}}$, $e^{\frac{i\pi}{10}}$, $e^{\frac{i\pi}{2}}$, $e^{\frac{i9\pi}{10}}$ ✓

Marker's comments:

Often students did not define z as part of their working out (e.g. let $z = rcis\theta$).

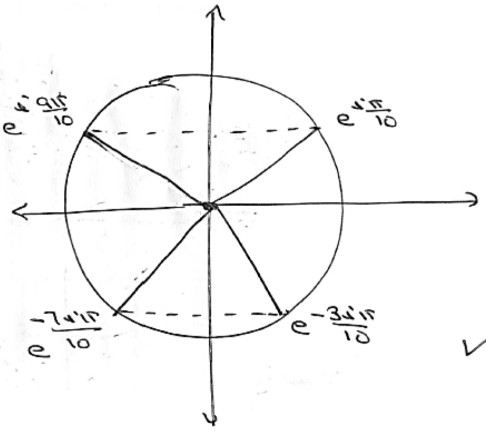
(ii) Hence show that **3**

$$\sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = \frac{1}{2}.$$

Q7 (d) (ii) Sum of the roots of the equation $z^5 - i = 0$

is $e^{-i\frac{7\pi}{10}} + e^{-i\frac{3\pi}{10}} + e^{i\frac{\pi}{10}} + e^{i\frac{\pi}{2}} + e^{i\frac{9\pi}{10}} = 0$

$\left(e^{i\frac{\pi}{10}} + e^{i\frac{9\pi}{10}}\right) + \left(e^{-i\frac{3\pi}{10}} + e^{-i\frac{7\pi}{10}}\right) + e^{i\frac{\pi}{2}} = 0$



$2i\sin\frac{\pi}{10} + 2i\sin\left(\frac{-3\pi}{10}\right) + i = 0$

$2i\sin\frac{\pi}{10} - 2i\sin\frac{3\pi}{10} = -i$

$\sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = \frac{1}{2}$

3

Marker's comments:

Not answered very well, students did not realise they could use sum of roots and equate imaginary parts.

(e) Prove, by contraposition, $\forall\, x, y \in \mathbb{R}$ **3**

if $y^3 + yx^2 \le x^3 + xy^2$, then $y \le x$

(e) Contrapositive of the statement is

if $y > x$, then $y^3 + yx^2 > x^3 + xy^2$

or if $y - x > 0$ then $y^3 + yx^2 - x^3 - xy^2 > 0$

Now $y^3 + yx^2 - x^3 - xy^2$

$= y^3 - x^3 + yx^2 - xy^2$

$= (y^3 - x^3) + yx(x - y)$

$= (y-x)(y^2 + xy + x^2) + yx(x-y)$

$= (y-x)(y^2 + xy + x^2) - yx(y-x)$

$= (y-x)(y^2 + \cancel{xy} + x^2 - \cancel{yx})$

$= (y-x)(y^2 + x^2)$

Since $y - x > 0$ and $y^2 + x^2 > 0$

$\therefore (y-x)(y^2+x^2) > 0$

Hence $y^3 + yx^2 - x^3 - xy^2 = (y-x)(y^2+x^2) > 0$

OR $y^3 + yx^2 > x^3 + xy^2$

$\therefore$ if $y > x$ then $y^3 + yx^2 > x^3 + xy^2$ is true.

$\therefore$ By contraposition the given statement

if $y^3 + yx^2 \le x^3 + xy^2$ then $y \le x$ is true $\boxed{3}$

Marker's comments:

Generally answered well but marks were deducted when steps in the proof were left out

End of Question 7

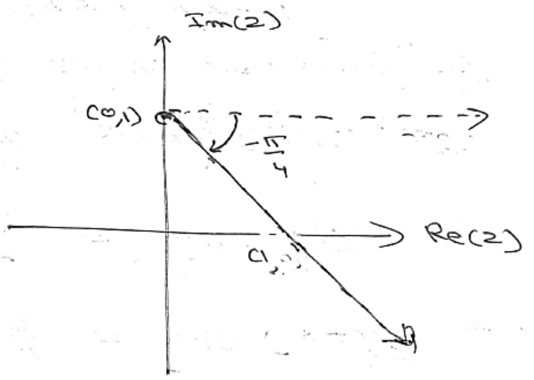
Question 8 (15 marks) Use a SEPARATE writing booklet

(a)

(i) Sketch, on a single Argand Diagram, the region $\mathbb{C}$ representing both of the following conditions **3**

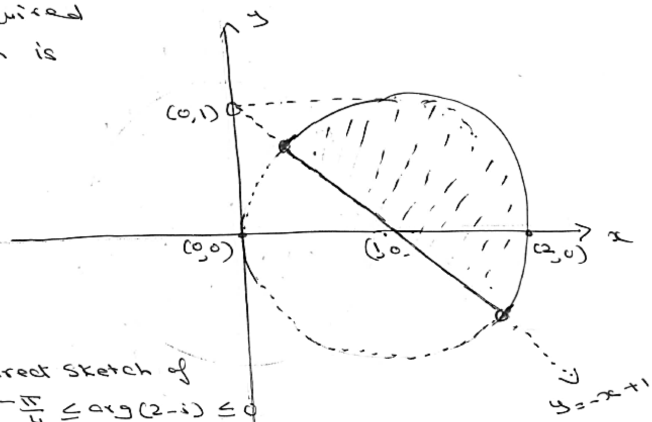
$$-\frac{\pi}{4} \le \arg(z-i) \le 0 \quad \text{and} \quad |z-1| \le 1$$

i) $-\frac{\pi}{4} \le \arg(z-i) \le 0$



$|z-1| \le 1$ is a circle with centre $(1,0)$ and radius 1

The Required Region is



✓ For Correct Sketch of $-\frac{\pi}{4} \le \arg(z-i) \le 0$

✓ For Correct sketch of $|z-1| \le 1$

✓ Correct region with open circle at $(0,1)$ $\boxed{3}$

Marking guide:

3 - Correct solution

2 - Both Correct circle drawn and correct two rays drawn or correct circle drawn and correct region shaded but one of the ray did not cross the x-axis at 1.

1 - Correct circle drawn or correct two rays drawn

Marker's comments:

Majority of candidates were able to score full marks.

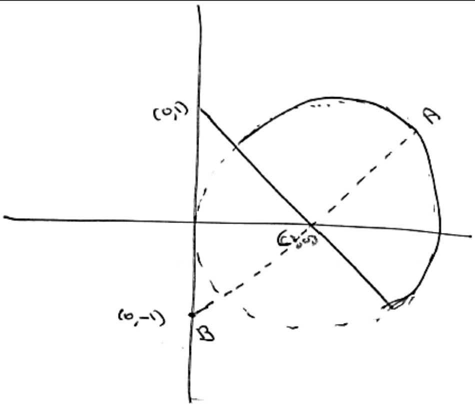
Some lost 1 mark due to incorrect region or left the part below the x-axis unshaded.

Some lost 1 mark if one of the ray did not cross the x axis at 1.

Some candidates drew the two rays intersecting at the origin instead of (0,1), despite correctly shading the region, it made the question easier so they can only score 1 mark under the marking guidelines.

(ii) Hence, determine, $\forall z \in \mathbb{C}$, the range of values of $|z+i|$. **2**

Q8 (a) (ii)



The minimum distance Z will have from (0, -1) is the distance between (0, -1) and (0, 1)

$= \sqrt{2}$

maximum distance $BA = BC + CA$

$= \sqrt{2} + 1$

So the range of values is

$\sqrt{2} \le |z+i| \le 1 + \sqrt{2}$

$\boxed{2}$

✓ For correct minimum value $\sqrt{2}$

✓ " " maximum value $1+\sqrt{2}$

Marking guide:

2 - Correct solution

1 - Correct min value or correct maximum value

Marker's comments:
Many scored full marks. Some found the minimum distance to the circle instead of the ray. Quite a number of students attempted to solve the problem algebraically instead of using geometry which also made the question harder to solve, they were able to find a range of values for the real part and imaginary part but the question asked for the range of value for the modulus. No candidates were able to gain part or full mark using this method.

(b) Without using induction, prove that,

$\forall n \in \mathbb{Z}, \quad n^3 - n$ is divisible by 6. **2**

Q8 (b) $n^3 - n$, n is an integer

$= n(n^2-1)$

$= (n-1)\,n\,(n+1)$

$(n-1)$, n and $(n+1)$ are three consecutive integers. ✓

Since $(n-1)$ and n are consecutive, one of them must be multiple of 2

Since $(n-1)$, n and $(n+1)$ are consecutive integers, one of them must be a multiple of 3.

$\therefore$ $(n-1)n(n+1)$ is divisible by both 2 and 3

$\Rightarrow$ $(n-1)n(n+1)$ is divisible by 6
or n^3-n " " " "

✓

$\boxed{2}$

Marking guide:

2 - Correct solution

1 - Prove the expression is even

Marker's comments:

Poorly done, only a small proportion of the cohort was able to gain full or 1 mark.

A few students was able to get full mark by considering 6 cases, n =6k, 6k+1, …6k+5 and showed algebraically that all 6 cases resulting the expression involving k has a factor of 6.

Many candidates only considered 2 cases, n = odd and even, but was not able to infer for both cases, the expression is even.

Some candidate realised it is a product of 3 consecutive after substituting n = 2k+1, and made inference about the factors, was only able to score 1 mark as they only prove it is divisible by 6 for n odd.

Most of the successful candidates realised after factorisation, the expression is a product of 3 consecutive integer, and explain one of the factors must be even, and one must be a multiple of three.

Quite a few candidates made no attempt on this question.

(c) A sequence of complex numbers a_0 , a_1 , a_2 , a_3 ,......., a_n is defined as

$$a_{k+1} = \frac{i(n-k)a_k}{(k+1)} \quad , n \text{ is an integer, } n \geq 0 \text{ and } 0 \leq k \leq n-1$$

(i) Use mathematical induction to prove that **3**

$$a_r = i^r \binom{n}{r} a_0 \quad , \text{for all integers } r \geq 0$$

Given: $a_{k+1} = \frac{i(n-k)a_k}{(k+1)}$, $\underline{\text{Statement}}$ $a_r = i^r \binom{n}{r} a_0 \quad \forall r \geq 0$

(i) Step 1 :- To prove the statement is true for $r = 0$

L.H.S. $= a_0$

R.H.S. $= i^0 \binom{n}{0} a_0$

$= 1 \times 1 \times a_0$

$= a_0$

L.H.S. = R.H.S.

$\therefore$ the statement is true for $r = 0$ ✓

Step 2 :- Assume the statement is true for $r = k$ where k is an integer; $k > 0$

i.e. $a_k = i^k \binom{n}{k} a_0 \quad (*)$

Step 3 :- To prove that the statement is true for $r = k+1$

i.e. To prove $a_{k+1} = i^{k+1} \binom{n}{k+1} a_0$

Now $a_{k+1} = \frac{i(n-k)}{k+1} \times a_k$

$= \frac{i(n-k)}{k+1} \times i^k \binom{n}{k} a_0$ using *

$= i^{k+1} \frac{(n-k)}{k+1} \binom{n}{k} a_0$

Marking guide:

3 - Correct solution

2 - Prove n = 0 is true, and make a correct substitution using the inductive assumption into the recursive formula when n = k+1 or equivalent merit

1 - Prove n = 0 is true or equivalent merit

Q8 (c)

(ii) $z_0 + z_1 + z_2 + \cdots + z_n = 2^n$ Given

$i^0\binom{n}{0}a_0 + i^1\binom{n}{1}a_0 + i^2\binom{n}{2}a_0 + \cdots + i^n\binom{n}{n}a_0 = 2^n$

$\left[\binom{n}{0} + \binom{n}{1}i + \binom{n}{2}i^2 + \cdots + \binom{n}{n}i^n\right]a_0 = 2^n$

$(1+i)^n a_0 = 2^n$

$a_0 = \frac{2^n}{(1+i)^n}$

$= 2^n \times \left(\frac{1}{1+i} \times \frac{1-i}{1-i}\right)^n$

$= 2^n \times \frac{(1-i)^n}{(1^2+1^2)^n}$

$= \frac{\cancel{2^n}(1-i)^n}{\cancel{2^n}}$

$\therefore a_0 = (1-i)^n$

Marker's comments:
A number of candidates correctly proved n = 1 is true and correctly proved n = k+1 is true given n = k is true, the question states prove the result is true for non negative values of n, so they can only score 2 marks as a result.
Many candidates were not able to correctly show n = k +1 is true given n = k is true by omitting the last few steps that involve simplifying factorial expression.
Many students assumed the formula (n-k)/(k+1) nCk = nCk+1, this needs to be proven as it is not a formula stated that can be used in the syllabus.
Quite a few candidates made no attempt on this question.

(ii) It is known that $a_0 + a_1 + a_2 + ... + a_n = 2^n$ **2**
Show that $a_0 = (1-i)^n$

Q8 (c) (i) Continued

$a_{k+1} = i^{k+1}\frac{n-k}{k+1} \times \frac{n!}{k!\,(n-k)!}a_0$

$= \frac{i^{k+1}}{(k+1)k!}\cancel{(n-k)} \times \frac{n!}{\cancel{(n-k)}(n-k-1)!}a_0$

$= \frac{i^{k+1}}{(k+1)!} \times \frac{n!}{(n-(k+1))!}a_0$

$= i^{k+1}\frac{n!}{(k+1)!\,(n-(k+1))!}a_0$

$= i^{k+1}\binom{n}{k+1}a_0$

the statement is true for $n = k+1$

Hence by mathematical induction the statement is true for all integers $r \geq 0$

Marking guide:

2 - correct solution

1 - Showed the LHS = $a_0(1+i)^n$.

Marker's comments:

Many candidates failed to make the connection between part i and part ii.
Candidates were able to score full mark when they used the formula in part i into the LHS and realised it is the expanded form of $(1+i)^n$ and subsequently divide it on the RHS to make a_0 the subject.
Quite a few candidates attempted to use the sum of GP formula.
Many candidates made no attempt on this question.

(iii) It is given that $\sum_{m=0}^{n}\binom{n}{m}^2=\binom{2n}{n}$ (Do Not Prove This) **3**

Also, it is given that if x_1 , x_2 , ..., x_n are positive real numbers,

then $\frac{x_1+x_2+.........x_n}{n}\geq\sqrt[n]{x_1x_2......x_n}$ (Do Not Prove This)

Hence or otherwise show that

$|a_0|^2+|a_1|^2+|a_2|^2+.........+|a_n|^2\leq\frac{(3n+1)^n}{n!}$

Q8. (c) (iii) $\sum_{m=0}^{n}\binom{n}{m}^2=\binom{2n}{n}$ * Given

$|a_0|^2+|a_1|^2+|a_2|^2+\cdots+|a_n|^2$

$=|a_0|^2+|i^1\binom{n}{1}a_0|^2+|i^2\binom{n}{2}a_0|^2+\cdots+|i^n\binom{n}{n}a_0|^2$

$=|a_0|^2+\binom{n}{1}^2|a_0|^2+\binom{n}{2}^2|a_0|^2+\cdots+\binom{n}{n}^2|a_0|^2$

$=|a_0|^2\left(1+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2\right)$

$=|a_0|^2\left(\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2\right)$

$=|a_0|^2\sum_{m=0}^{n}\binom{n}{m}^2$

$=|a_0|^2\binom{2n}{n}$ ✓ (using Given *)

$=|(1-i)^n|^2\binom{2n}{n}$ using (ii)

$=|1-i|^{2n}\binom{2n}{n}$

$=(\sqrt{2})^{2n}\binom{2n}{n}$

$=2^n\binom{2n}{n}$ ✓

Marking guide:

3 - correct solution

2 - Simplified the expression (2nCn) to an expression with n! in the denominator and cancelled the other n!

1- Correctly substituted the formula from part i into the LHS and used the formula to simplify it to 2^n (2nCn)

Marker's comments:

A lot candidates made no attempt on this question.
Only 1 candidate scored full mark in this question.
A number of candidates was able to score 1 mark.
Many candidates incorrectly deduce that they needed to substitute the LHS of the inequality into the given AM GM inequality.
Again many candidates failed to make the connection between subsequent part which made it very difficult for them to make progress on this question.

$$= 2^n \times \frac{2n!}{n!\,n!}$$

$$= \frac{2^n}{n!} \times \frac{2n(2n-1)(2n-2)\ldots\ldots(n+1)\cancel{n(n-1)}(n-2)\ldots1}{\cancel{n!}}$$

$$= \frac{2^n}{n!}\,2n(2n-1)(2n-2)\ldots\ldots(n+1)$$

$$\leq \frac{2^n}{n!}\left[\frac{(2n)+(2n-1)+(2n-2)+\ldots+(n+1)}{n}\right]^n$$

$$[G.M \leq A.M.]$$

$$= \frac{2^n}{n!\,n^n}\Big[\underbrace{2n+(2n-1)+(2n-2)+\ldots+(n+1)}_{\text{A.P. with } n \text{ terms and first term } 2n, \text{ Common difference } = -1}\Big]^n$$

$$= \frac{2^n}{n!\,n^n}\left[\frac{n}{2}\left[2n+(n-1)\times(-1)\right]\right]^n$$

$$= \frac{2^n}{n!\,n^n} \times \left[\frac{n}{2}\times(3n+1)\right]^n$$

$$= \frac{\cancel{2^n}}{n!\,\cancel{n^n}} \times \frac{\cancel{n^n}}{\cancel{2^n}} \times (3n+1)^n$$

$$= \frac{(3n+1)^n}{n!}$$

As required $\boxed{3}$

End of Paper