FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 2
2019
HSC TASK 1
ANSWERS COVER SHEET

Student Number: ______________________________

QUESTION	MARK	MEX12-1	MEX12-2	MEX12-3	MEX12-4	MEX12-5	MEX12-6	MEX12-7	MEX12-8
1-5	/5	✓			✓			✓	✓
6	/23	✓			✓			✓	✓
7	/21	✓			✓			✓	✓
8	/14	✓			✓			✓	✓
9	/12	✓			✓			✓	✓
TOTAL									
	/75	/75			/75			/75	/75

Year 12 Mathematics Extension 2 outcomes

A student:

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

MEX12-3 uses vectors to model and solve problems in two and three dimensions

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

MEX12-5 applies techniques of integration to structured and unstructured problems

MEX12-6 uses mechanics to model and solve practical problems

MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2

HSC Task 1

- **Instructions**

    - Working time – 90 minutes
    - Write using black pen
    - Calculators approved by NESA may be used
    - A reference sheet is provided
    - For Section I, use the multiple-choice answer sheet provided.
    - For questions in section II, show relevant mathematical reasoning and /or calculations. Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate the numbers of the questions you are answering.
    - Marks may be deducted for careless or badly arranged work

Section I (5 marks)
Questions 1 – 5

Use the multiple-choice answer sheet for questions 1 – 5

---

1. If $z = 4 - i$ and $\omega = 3 + 5i$, what is the value of $z - \overline{\omega}$?

   A. $-1 - 4i$ B. $-1 + 4i$ C. $1 - 4i$ D. $1 + 4i$

2. If ω is a complex cube root of unity, which one of the following is not correct?

   A. $\omega^2 = -\omega - 1$

   B. ω^2 is the other complex root

   C. $\omega^6 = \omega^2 + \omega$

   D. $\overline{\omega}$ is also a root

3. The locus of a point P representing the complex number z on the Argand diagram, described by $|z - i| = Im(z)$ can be expressed as:

   A. $x^2 = 2y - 1$

   B. $x^2 + (y - 1)^2 = y$

   C. $x^2 + y^2 = 3y - 1$

   D. $x^2 + (y - 1)^2 = 1$

4. Given that x is a negative integer, which of the following is equivalent to i^{4x-1}?

   A. i B. $-i$ C. 1 D. -1

5. Which rotation about the origin is the result of multiplying a non – zero complex number by $\dfrac{1+i}{1-i}$?

   A. Clockwise by $\dfrac{\pi}{2}$ radians

   B. Clockwise by $\dfrac{\pi}{4}$ radians

   C. Anticlockwise by $\dfrac{\pi}{2}$ radians

   D. Anticlockwise by $\dfrac{\pi}{4}$ radians

Select the alternative A, B, C or D that best answers the question.

1.	A	○	B	○	C	○	D	○
2.	A	○	B	○	C	○	D	○
3.	A	○	B	○	C	○	D	○
4.	A	○	B	○	C	○	D	○
5.	A	○	B	○	C	○	D	○

Section II (70 marks)

- Answer questions in the spaces provided.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate the numbers of the questions you are answering.

Question 6 (23 marks)

a. (i) Express $1 - \sqrt{3}\,i$ and $1 + i$ in modulus – argument form. [2]

(ii) Evaluate $\dfrac{(1 - \sqrt{3}\,i)^2}{(1 + i)^3}$ in both modulus-argument form and Cartesian form. [4]

(iii) Hence, show that $\cos\dfrac{7\pi}{12} = \dfrac{\sqrt{2} - \sqrt{6}}{4}$ [1]

b. (i) Find real numbers a and b such that $(a + ib)^2 = -3 - 4i$ [3]

(ii) Hence, solve $z^2 - 5z + (7 + i) = 0$ [2]

c. (i) Given that $e^{i\theta} = \cos\theta + i\sin\theta$, write expressions for $e^{2i\theta}$ and $(e^{i\theta})^2$ [2]

(ii) Hence, write expressions for $\cos 2\theta$ and $\sin 2\theta$. [2]

d. Given that $(\sqrt{3} - i)(-\sqrt{3} + i) = re^{i\theta}$, $r > 0$, for $-\pi < \theta \leq \pi$,
find the values of r and θ. [3]

e. The polynomial $P(x) = 3x^4 - 4x^3 - 6x^2 + 20x - 16$ has a zero at $1 + i$.
Factorise $P(x)$ into linear factors. [4]

Question 7 (21 marks)

a. (i) Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^3$. [1]

(ii) Use De Moivre's theorem and your result from part (i)

to prove that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$. [3]

(iii) Hence, or otherwise, find the smallest positive solution of

$$4\cos^3\theta - 3\cos\theta = 1 \quad .$$ [2]

b. (i) If $z = \cos\theta + i\sin\theta$, show that

$z - \frac{1}{z} = 2i\sin\theta$ and $z^n + \frac{1}{z}n = 2\cos n\theta$ [2]

(ii) Hence, express $\sin^6\theta$ in terms of $\cos 6\theta$, $\cos 4\theta$ and $\cos 2\theta$. [3]

c. If ω is a non-real zero of $z^7 - 1 = 0$ with the smallest possible argument,

(i) Show that $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ and that $\omega^2, \omega^3, \omega^4, \omega^5$ and ω^6 are the other non-real roots of $z^7 - 1 = 0$ [3]

(ii) Show that $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = -1$ [1]

(iii) Show that $(\omega + \omega^6)(\omega^2 + \omega^5)(\omega^3 + \omega^4) = 1$ [3]

(iv) Hence , show that $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$ [3]

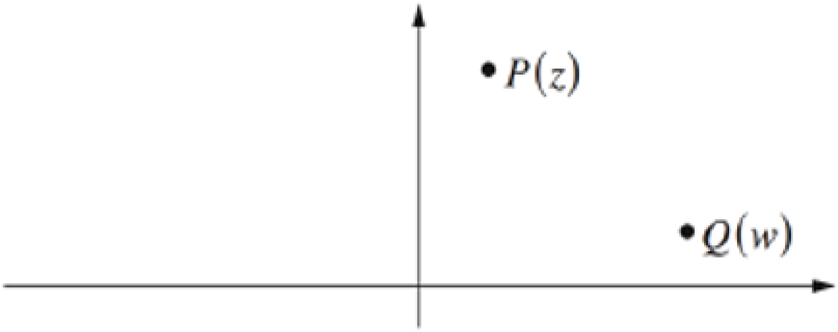
Question 8 (16 marks)

a. The points P and Q on the Argand diagram below represent the complex numbers z and w.
Show the following on the same Argand diagram.

(i) the point R representing iz [1]

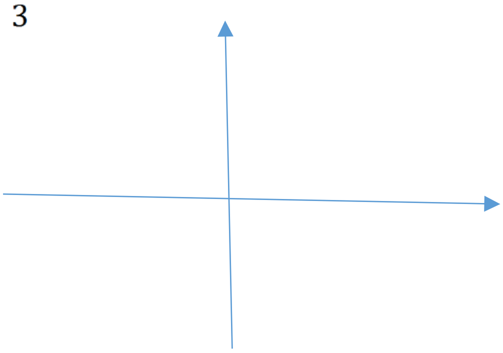
(ii) the point S representing $\overline{w}$ [1]

(iii) the point T representing $z + w$ [1]

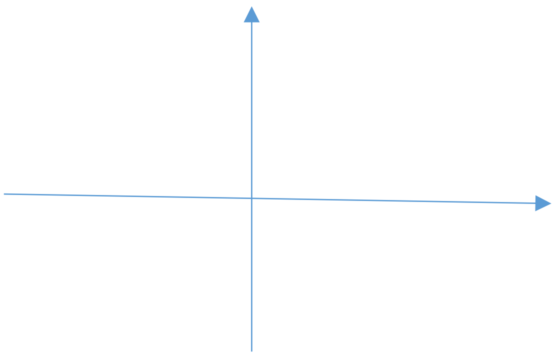


b. Show the following on Argand diagrams.

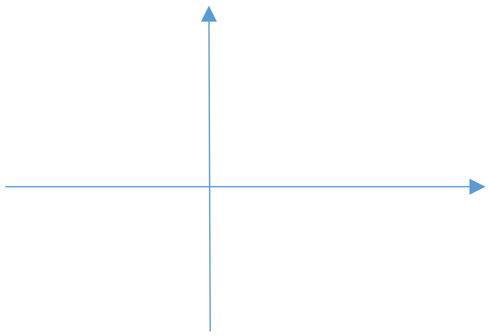
(i) $|z| < 3$ [2]



(ii) $2 \le Im\, z \le 4$ and $\frac{\pi}{6} \le \arg z \le \frac{\pi}{4}$ [3]

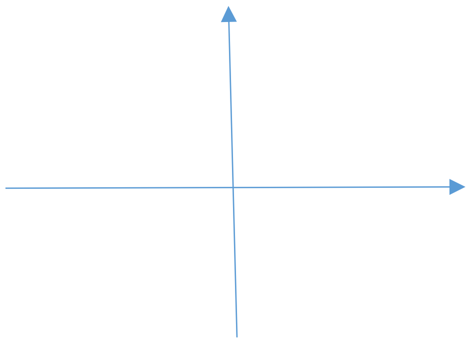


(iii) $\arg z = \arg(z + 2 + 2i)$ [2]



(iv) $|z + 3| = |z - 5|$ [2]

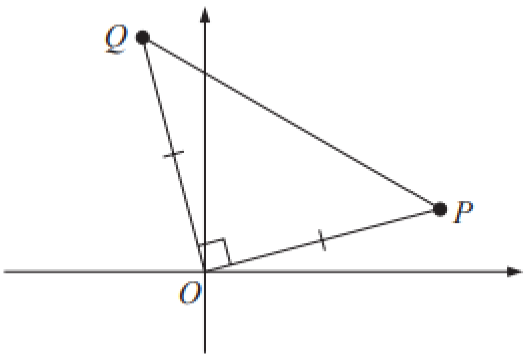
Describe the locus.



c. The points P and Q in the complex field correspond to the complex numbers z and ω respectively.

The triangle OPQ is right-angled and isosceles, with the right angle at O.

Show that $z^2 + \omega^2 = 0$ [2]



Question 9 (12 marks)

a. Resolve $z^8 + 1$ into real quadratic factors. [3]

(ii) Hence, show that [4]

$$\cos 4\theta = 8\left(\cos\theta - \cos\frac{\pi}{8}\right)\left(\cos\theta + \cos\frac{\pi}{8}\right)\left(\cos\theta - \cos\frac{3\pi}{8}\right)\left(\cos\theta + \cos\frac{3\pi}{8}\right)$$

b. The locus of $\arg\left(\frac{z-3}{z+1}\right) = \frac{\pi}{3}$ represents an arc of a circle. [5]

Draw a diagram of this locus and describe it, clearly stating the centre, radius and equation. Clearly show all necessary steps and justifications for your answer.

Extra Working out paper. Clearly indicate question numbers.

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9	/12	✓			✓			✓	✓
TOTAL									
	/75	/75			/75			/75	/75

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Section I (5 marks)
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   A. $-1 - 4i$ B. $-1 + 4i$ C. $1 - 4i$ D. $1 + 4i$

$z - \overline{\omega}$
$= 4 - i - (3 - 5i)$
$= 1 + 4i$

2. If ω is a complex cube root of unity, which one of the following is not correct?

   A. $\omega^2 = -\omega - 1$ ✓

   B. ω^2 is the other complex root ✓

   C. $\omega^6 = \omega^2 + \omega$

   D. $\overline{\omega}$ is also a root ✓

$\omega^6 = (\omega^3)^2 = 1$
$\omega^2 + \omega = \omega^2 + \omega + 1 (-1) = 0 - 1$
$= -1$

3. The locus of a point P representing the complex number z on the Argand diagram, described by $|z - i| = Im(z)$ can be expressed as:

$|(x + iy) - i| = Im(x + iy)$

   A. $x^2 = 2y - 1$

   B. $x^2 + (y - 1)^2 = y$

   C. $x^2 + y^2 = 3y - 1$

   D. $x^2 + (y - 1)^2 = 1$

$\sqrt{x^2 + (y-1)^2} = y$
$x^2 + y^2 - 2y + 1 = y^2.$
$x^2 = 2y - 1$

4. Given that x is a negative integer, which of the following is equivalent to $i^{4x - 1}$?

   A. i B. $-i$ C. 1 D. -1

$i^{4x} \cdot i^{-1}$
$= (i^4)^x \cdot \frac{1}{i} \times \frac{i}{i}$
$= 1 \cdot \frac{i}{-1}$
$= -i$

5. Which rotation about the origin is the result of multiplying a non – zero complex number by $\dfrac{1 + i}{1 - i}$?

   A. Clockwise by $\dfrac{\pi}{2}$ radians

   B. Clockwise by $\dfrac{\pi}{4}$ radians

   C. Anticlockwise by $\dfrac{\pi}{2}$ radians

   D. Anticlockwise by $\dfrac{\pi}{4}$ radians

$\dfrac{\sqrt{2} \operatorname{cis} \frac{\pi}{4}}{\sqrt{2} \operatorname{cis} -\pi/4}$
$= \operatorname{cis} \frac{\pi}{2}$

Select the alternative A, B, C or D that best answers the question.

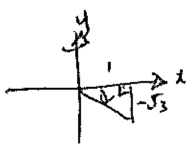
1.	A	○	B	○	C	○	D	●
2.	A	○	B	○	C	●	D	○
3.	A	●	B	○	C	○	D	○
4.	A	○	B	●	C	○	D	○
5.	A	○	B	○	C	●	D	○

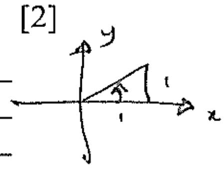
Section II (70 marks)

- Answer questions in the spaces provided.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate the numbers of the questions you are answering.

Question 6 (23 marks)

a. (i) Express $1 - \sqrt{3}\,i$ and $1 + i$ in modulus – argument form. [2]





$|1-\sqrt{3}\,i| = \sqrt{1^2+(-\sqrt{3})^2}$; $|1+i| = \sqrt{2}$

$= 2$

$\arg(1+i) = \frac{\pi}{4}$

$\arg(1-\sqrt{3}\,i) = -\frac{\pi}{3}$

$\therefore 1-\sqrt{3}\,i = 2\,\text{cis}\,\frac{-\pi}{3}$ $\qquad 1+i = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

(ii) Evaluate $\dfrac{(1-\sqrt{3}\,i)^2}{(1+i)^3}$ in both modulus-argument form and Cartesian form. [4]

$$\frac{(1-\sqrt{3}\,i)^2}{(1+i)^3} = \frac{\left(2\,\text{cis}\,\frac{-\pi}{3}\right)^2}{\left(\sqrt{2}\,\text{cis}\,\frac{\pi}{4}\right)^3}$$

$$= \frac{4\,\text{cis}\,\frac{-2\pi}{3}}{2\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}} \quad \text{(De Moivre's Thm)}$$

①

$$= \sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) \quad \text{(In Mod-Arg form)}$$

In Cartesian form : $\dfrac{(1-\sqrt{3}\,i)^2}{(1+i)^3} = \dfrac{1-2\sqrt{3}\,i-3}{1+3i-3-i}$ (Binomial Expansion)

$$= \frac{-2(1+\sqrt{3}\,i)}{-2(1-i)} \times \frac{1+i}{1+i}$$

$$= \frac{1+i+\sqrt{3}\,i-\sqrt{3}}{2}$$

$$= \frac{1-\sqrt{3}}{2} + \frac{(1+\sqrt{3})}{2}\,i$$

②

(iii) Hence, show that $\cos\frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4}$ [1]

Equating real parts of ① & ②

$\sqrt{2}\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2}$

$\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

b. (i) Find real numbers a and b such that $(a+ib)^2 = -3-4i$ [3]

$a^2 + 2iab - b^2 = -3-4i$

Equating real & imaginary parts;

$a^2 - b^2 = -3$; $2ab = -4$

$a = \frac{-2}{b}$

substitute $a = \frac{-2}{b}$ into $a^2 - b^2 = -3$

$\left(-\frac{2}{b}\right)^2 - b^2 = -3$

$4 - b^4 = -3b^2 \Rightarrow b^4 - 3b^2 - 4 = 0$

$(b^2-4)(b^2+1) = 0$ no real solution

$b^2 = 4$

$b = \pm 2$

when $b = 2$, $a = -1$

$b = -2$, $a = 1$

(ii) Hence, solve $z^2 - 5z + (7+i) = 0$ [2]

$z = \frac{5 \pm \sqrt{5^2 - 4(1)(7+i)}}{2}$

$= \frac{5 \pm \sqrt{-3-4i}}{2}$

$= \frac{5 \pm (1-2i)}{2}$ $\therefore z = 3-i,\ 2+i$

$z = \frac{6-2i}{2}$ or $\frac{4+2i}{2}$

c. (i) Given that $e^{i\theta} = \cos\theta + i\sin\theta$, write expressions for $e^{2i\theta}$ and $(e^{i\theta})^2$ [2]

$e^{2i\theta} = \cos 2\theta + i\sin 2\theta$ ——①

$(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2$

$= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$ ——②

(ii) Hence, write expressions for $\cos 2\theta$ and $\sin 2\theta$. [2]

Equating real and imaginary parts of ① & ②

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

d. Given that $(\sqrt{3} - i)(-\sqrt{3} + i) = re^{i\theta}, r > 0$, for $-\pi < \theta \leq \pi$,

find the values of r and θ. [3]

$(\sqrt{3} - i)(-\sqrt{3} + i)$

$= -3 + 2\sqrt{3}i - i^2$

$= -2 + 2\sqrt{3}i$

$= 4\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)$ $\therefore r = 4$

$= 4e^{i\frac{2\pi}{3}}$ $\theta = \frac{2\pi}{3}$

e. The polynomial $P(x) = 3x^4 - 4x^3 - 6x^2 + 20x - 16$ has a zero at $1 + i$.

Factorise $P(x)$ into linear factors. [4]

since $P(x)$ has real coefficients, $1-i$ is also a root.

For Quadratic factor, $\therefore$ sum of roots $= 2$, product of roots $= 2$

$\therefore x^2 - 2x + 2$ is a quadratic factor of $P(x)$

$$
\begin{array}{r}
3x^2 + 2x - 8 \\
x^2 - 2x + 2 \,\big)\, 3x^4 - 4x^3 - 6x^2 + 20x - 16 \\
-\; 3x^4 - 6x^3 + 6x^2 \\
\hline
2x^3 - 12x^2 + 20x \\
-\; 2x^3 - 4x^2 + 4x \\
\hline
-8x^2 + 16x - 16 \\
-\; -8x^2 + 16x - 16 \\
\hline
0
\end{array}
$$

$\therefore P(x) = (x - 1 - i)(x - 1 + i)(3x^2 + 2x - 8)$

$= (x - 1 - i)(x - 1 + i)(3x - 4)(x + 2)$

Question 7 (21 marks)

a. (i) Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^3$. [1]

$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

(ii) Use De Moivre's theorem and your result from part (i)

to prove that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$. [3]

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ (De Moivre's thm)

equating real part,

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$4\cos^3\theta = \cos 3\theta + 3\cos\theta$

$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

(iii) Hence, or otherwise, find the smallest positive solution of

$4\cos^3\theta - 3\cos\theta = 1$. [2]

From (ii) $\cos 3\theta = 1$

$3\theta = 2\pi, 4\pi, \ldots$

$\theta = \frac{2\pi}{3}$ (smallest positive solution)

b. (i) If $z = \cos\theta + i\sin\theta$, show that

$z - \frac{1}{z} = 2i\sin\theta$ and $z^n + \frac{1}{z^n} = 2\cos n\theta$ [2]

$z - \frac{1}{z} = z - z^{-1}$

$= \cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta))$

$= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$ (sin is odd, cos is even)

$= 2i\sin\theta$

$z^n + z^{-n} = \cos n\theta + i\sin n\theta + [\cos(-n\theta) + i\sin(-n\theta)]$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

(ii) Hence, express $\sin^6\theta$ in terms of $\cos 6\theta$, $\cos 4\theta$ and $\cos 2\theta$. [3]

$$\left(z-\frac{1}{z}\right)^6 = (2i\sin\theta)^6 \quad (\text{using (i)})$$
$$= -64\sin^6\theta \quad \text{—— (1)}$$

Also, $$\left(z-\frac{1}{z}\right)^6 = z^6-6z^4+15z^2-20+\frac{15}{z^2}-\frac{6}{z^4}+\frac{1}{z^6}$$
$$=\left(z^6+\frac{1}{z^6}\right)-6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)-20$$
$$=2\cos6\theta-12\cos4\theta+30\cos2\theta-20 \quad \text{—— (2)}$$

From (1) & (2)
$$-64\sin^6\theta = 2\cos6\theta-12\cos4\theta+30\cos2\theta-20$$
$$\therefore \sin^6\theta = \frac{-1}{32}\cos6\theta+\frac{3}{16}\cos4\theta-\frac{15}{32}\cos\theta+\frac{5}{16}$$

c. If ω is a non-real zero of $z^7-1=0$ with the smallest possible argument,

(i) Show that $\omega = \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$ and that $\omega^2, \omega^3, \omega^4, \omega^5$ and ω^6 are the other non-real roots of $z^7-1=0$ [3]

roots of $z^7=1 \Rightarrow z^7 = \text{cis } 2k\pi$

$$z = \text{cis}\frac{2k\pi}{7}, \quad k=0,1,2,\ldots,6$$

$z_1 = \text{cis } 0 = 1$

$z_2 = \text{cis}\frac{2\pi}{7} = \omega = \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$ ← non-real root with smallest argument.

$z_3 = \text{cis}\frac{4\pi}{7} = \omega^2$

$z_4 = \text{cis}\frac{6\pi}{7} = \omega^3$

$z_5 = \text{cis}\frac{8\pi}{7} = \text{cis}\frac{-6\pi}{7} = \omega^4$

$z_6 = \text{cis}\frac{10\pi}{7} = \text{cis}\frac{-4\pi}{7} = \omega^5$

$z_7 = \text{cis}\frac{12\pi}{7} = \text{cis}\frac{-2\pi}{7} = \omega^6$

$\therefore \omega^2, \omega^3, \omega^4, \omega^5$ and ω^6 are the other non-zero roots of $z^7-1=0$.

(ii) Show that $\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = -1$ [1]

Sum of the roots $= 1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = 0 \quad \left(\frac{-b}{a}\right)$

$$\therefore \omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = -1$$

(iii) Show that $(\omega+\omega^6)(\omega^2+\omega^5)(\omega^3+\omega^4)=1$ [3]

LHS $= (\omega+\omega^6)(\omega^2+\omega^5)(\omega^3+\omega^4)$

$= (\omega^3+\omega^6+\omega^8+\omega^{11})(\omega^3+\omega^4)$

$= \omega^6+\omega^9+\omega^{11}+\omega^{14}+\omega^7+\omega^{10}+\omega^{12}+\omega^{15}$

$= \omega^6+\omega^2+\omega^4+1+1+\omega^3+\omega^5+\omega$

$= 1+(1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6)$

$= 1+0$

$= 1 =$ RHS.

(iv) Hence, show that $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=\frac{1}{8}$ [3]

$(\omega+\omega^6) = \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\cos\frac{-2\pi}{7}+i\sin\frac{-2\pi}{7}$

$= 2\cos\frac{2\pi}{7}$

Similarly, $(\omega^2+\omega^5) = 2\cos\frac{4\pi}{7} = -2\cos\frac{3\pi}{7}$

and $(\omega^3+\omega^4) = 2\cos\frac{6\pi}{7} = -2\cos\frac{\pi}{7}$

$\therefore 2\cos\frac{2\pi}{7}\cdot -2\cos\frac{3\pi}{7}\cdot -2\cos\frac{\pi}{7} = 1$ (from (iii))

$\therefore 8\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}\cdot\cos\frac{\pi}{7} = 1$

$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$

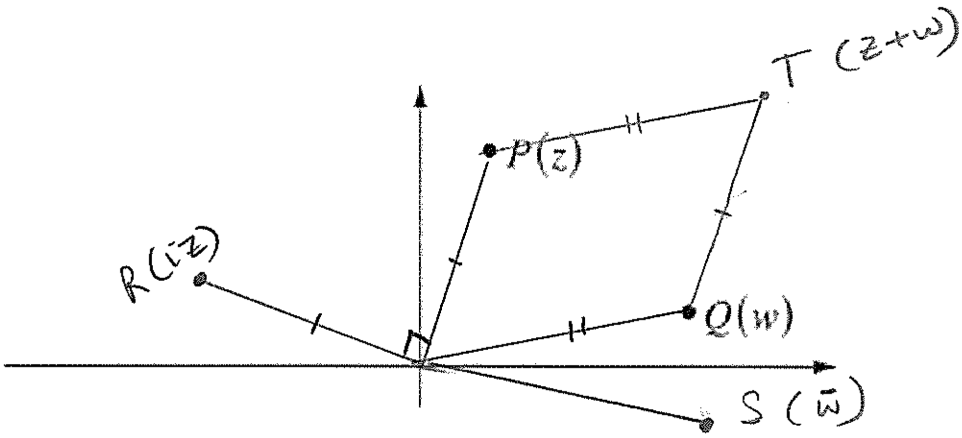
Question 8 (14 marks)

a. The points P and Q on the Argand diagram below represent the complex numbers z and w.
Show the following on the same Argand diagram.

(i) the point R representing iz [1]

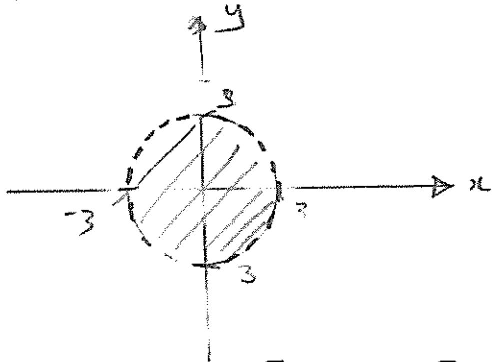
(ii) the point S representing $\overline{w}$ [1]

(iii) the point T representing $z + w$ [1]

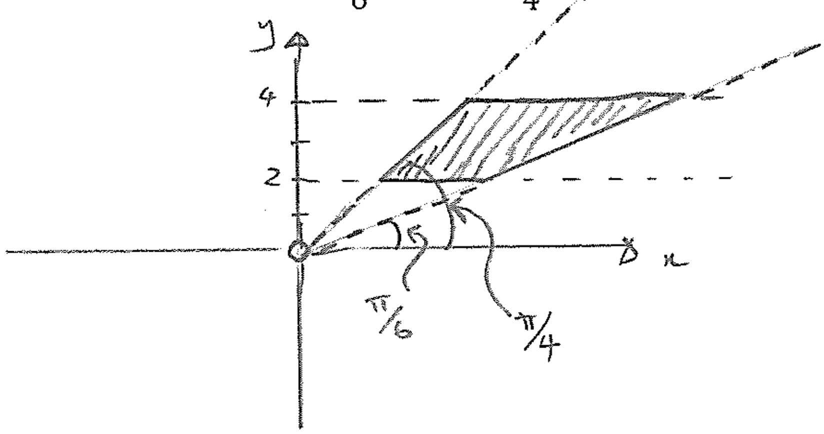


b. Show the following on Argand diagrams in the spaces provided.

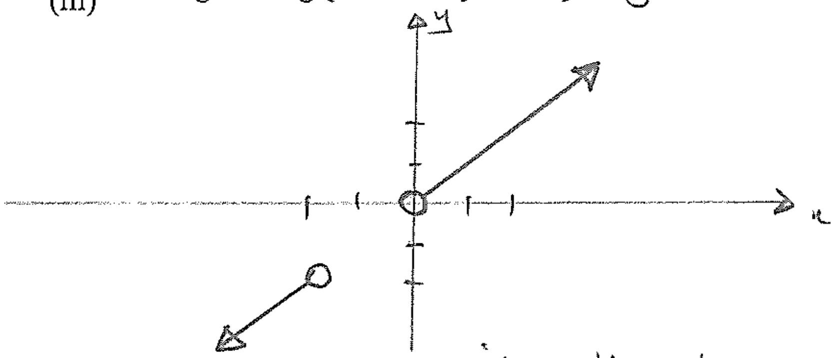
(i) $|z| < 3$ [2]



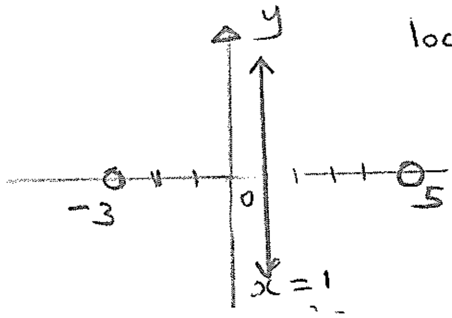
(ii) $2 \le Im\,z \le 4$ and $\frac{\pi}{6}$ $\arg z \le \frac{\pi}{4}$ [3]



(iii) $\arg z = \arg(z + 2 + 2i)$ $\Rightarrow \arg z - \arg(z - (-2-2i)) = 0$ [2]



(iv) $|z + 3| = |z - 5|$. Describe the locus. [2]



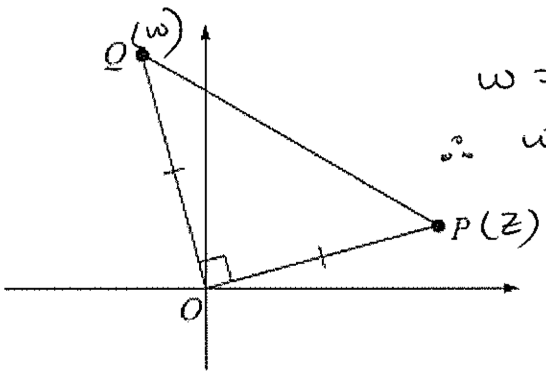
locus is the perpendicular bisector of line joining $(-3, 0)$ and $(5, 0)$

ie. $x = 1$.

c. The points P and Q in the complex field correspond to the complex numbers z ans ω respectively.

The triangle OPQ is right-angled and isosceles, with the right angle at O.

Show that $z^2 + \omega^2 = 0$ [2]



$\omega = iz$

$\therefore \omega^2 = (iz)^2$

$= -z^2$

$\therefore \omega^2 + z^2 = 0$

Question 9 (12 marks)

a. Resolve $z^8 + 1$ into real quadratic factors. [3]

$z^8 + 1 = 0$ has roots

$z^8 = -1 = \text{cis}(\pi + 2k\pi)$

$z = \text{cis}\left(\frac{\pi + 2k\pi}{8}\right)$, $k = 0, 1, 2, \ldots 7$

$z_1 = \text{cis}\,\pi/8$ $\quad z_5 = \text{cis}\,9\pi/8 = \text{cis}\,-7\pi/8$

$z_2 = \text{cis}\,3\pi/8$ $\quad z_6 = \text{cis}\,11\pi/8 = \text{cis}\,-5\pi/8$

$z_3 = \text{cis}\,5\pi/8$ $\quad z_7 = \text{cis}\,13\pi/8 = \text{cis}\,-3\pi/8$

$z_4 = \text{cis}\,7\pi/8$ $\quad z_8 = \text{cis}\,15\pi/8 = \text{cis}\,-\pi/8$

$\therefore$ Factors of $z^8 + 1$

$= (z - \text{cis}\,\pi/8)(z - \text{cis}\,-\pi/8)(z - \text{cis}\,3\pi/8)(z - \text{cis}\,-3\pi/8)$
$(z - \text{cis}\,5\pi/8)(z - \text{cis}\,-3\pi/8)(z - \text{cis}\,5\pi/8)(z - \text{cis}\,-5\pi/8)$

$z^8 + 1 = {}^{*}(z^2 - 2z\cos\pi/8 + 1)(z^2 - 2z\cos 3\pi/8 + 1)(z^2 - 2z\cos 5\pi/8 + 1)$
$(z^2 - 2z\cos 7\pi/8 + 1)$ *

Note: $^{*}\text{cis}\,\pi/8 + \text{cis}\,-\pi/8$

$= \cos\pi/8 + i\sin\pi/8 + \cos -\pi/8 + i\sin -\pi/8 = 2\cos\pi/8$ [similarly cos ea

and $\text{cis}\,\pi/8 \cdot \text{cis}\,-\pi/8 = \text{cis}\,0$

$= \cos 0 + i\sin 0 = 1$ $\quad$ Other results were obtained similarly *

(ii) Hence, show that [4]

$$\cos 4\theta = 8\left(\cos\theta - \cos\frac{\pi}{8}\right)\left(\cos\theta + \cos\frac{\pi}{8}\right)\left(\cos\theta - \cos\frac{3\pi}{8}\right)\left(\cos\theta + \cos\frac{3\pi}{8}\right)$$

since $\cos 5\pi/8 = -\cos 3\pi/8$ and $\cos 7\pi/8 = -\cos\pi/8$

$z^8 + 1 = (z^2 - 2z\cos\pi/8 + 1)(z^2 - 2z\cos 3\pi/8 + 1)(z^2 + 2z\cos\frac{3\pi}{8} + 1)$
$(z^2 + 2z\cos\pi/8 + 1)$

Dividing both sides by z^4 (Dividing each factor on RHS by z)
(Letting $z = \text{cis}\,\theta$)

$z^4 + \frac{1}{z^4} = \left(z + \frac{1}{z} - 2\cos\pi/8\right)\left(z + \frac{1}{z} - 2\cos 3\pi/8\right)\left(z + \frac{1}{z} + 2\cos 3\pi/8\right)$
$\left(z + \frac{1}{z} + 2\cos\pi/8\right)$

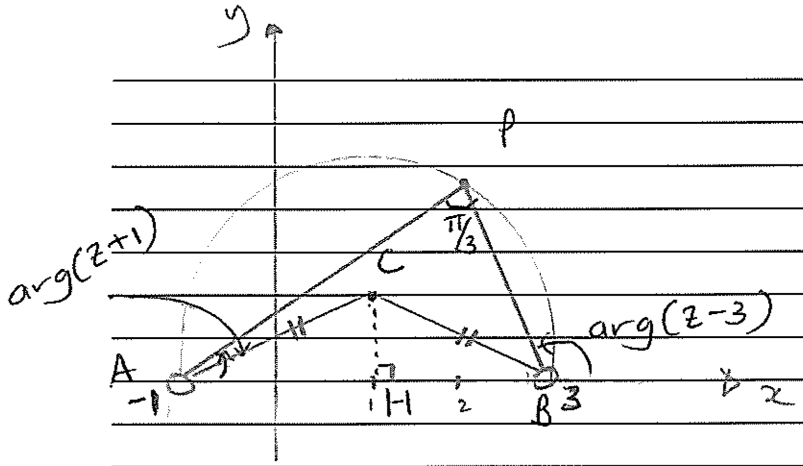
$2\cos 4\theta = (2\cos\theta - 2\cos\pi/8)(2\cos\theta - 2\cos 3\pi/8)$
$(2\cos\theta + 2\cos 3\pi/8)(2\cos\theta + 2\cos\pi/8)$

$2\cos 4\theta = 2^4(\cos\theta - \cos\pi/8)(\cos\theta + \cos\pi/8)(\cos\theta - \cos 3\pi/8)$
$(\cos\theta + \cos 3\pi/8)$

$\cos 4\theta = 8(\cos\theta - \cos\pi/8)(\cos\theta + \cos\pi/8)$
$(\cos\theta - \cos\frac{3\pi}{8})(\cos\theta + \cos 3\pi/8)$

b. The locus of $\arg\left(\frac{z-3}{z+1}\right) = \frac{\pi}{3}$ represents an arc of a circle. [5]

Draw a diagram of this locus and describe it, clearly stating the centre, radius and equation. Clearly show all necessary steps and justifications for your answer.



$\arg(z-3) - \arg(z+1) > 0$

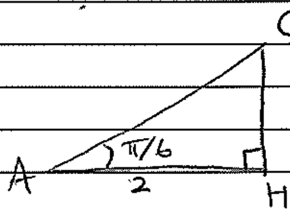
$\therefore$ correct diagram

By observation, the centre of the circle lies on the line $x = 1$

$\angle ACB = \frac{2\pi}{3}$ ($\angle$ at the centre of a circle is twice the $\angle$ at the circumference.)

$AC = CB$ (radii)

$\angle CAH = \pi/6$ ($\angle$ sum of a Δ)



$\tan \frac{\pi}{6} = \frac{CH}{2}$

$CH = \frac{2}{\sqrt{3}}$

$\therefore$ Centre $= \left(1, \frac{2}{\sqrt{3}}\right)$

radius $= AC = \sqrt{2^2 + \left(\frac{2}{\sqrt{3}}\right)^2}$

$= \sqrt{4 + \frac{4}{3}}$

$= \frac{4}{\sqrt{3}}$

$\therefore$ Locus is the major arc of the circle

$(x-1)^2 + \left(y - \frac{2}{\sqrt{3}}\right)^2 = \frac{16}{3}$, $y > 0$