FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 2
TASK 2 2019 – Polynomials and Conic Sections
ANSWERS COVER SHEET

Name: ______________________________

QUESTION	MARK	E2	E3	E4	E5	E6	E7	E8	E9
MC	/5			✔					✔
Q6	/18			✔					✔
Q7	/16		✔	✔					✔
Q8	/9		✔	✔					✔
Q9	/8		✔	✔					✔
TOTAL									
	/56	/0	/33	/56	/0	/0	/0	/0	/56

HSC Outcomes **Mathematics Extension 2**

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings.

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials.

E5 uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.

E7 uses the techniques of slicing and cylindrical shells to determine volumes.

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.

E9 communicates abstract ideas and relationships using appropriate notation and logical argument.

GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2
2019 HSC Task 2

MULTIPLE CHOICE ANSWER SHEET

Select the alternative A, B, C or D that best answers the question.

1.	A	○	B	○	C	○	D	○
2.	A	○	B	○	C	○	D	○
3.	A	○	B	○	C	○	D	○
4.	A	○	B	○	C	○	D	○
5.	A	○	B	○	C	○	D	○

GIRRAWEEN HIGH SCHOOL

Task 2

2019

MATHEMATICS

EXTENSION 2

Time Allowed: 90 minutes

DIRECTIONS TO CANDIDATES

- There are 9 questions in this paper. All questions are compulsory.
- Use blue or black pen.
- Write all your answers in the Answer Booklets provided.
- For Questions 1 – 5, fill in the circle corresponding to the correct answer in your answer booklet.
- For Questions 6 – 9, start each question on a new page.
- Write on both sides of the paper.
- Show all necessary working.
- Board-approved calculators may be used.
- Mathematics reference sheets are provided.
- Marks may be deducted for careless or badly arranged work.

Questions 1 - 5 (5 marks)

Colour in the circle corresponding to the correct answer on the answer sheet provided.

1. What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3+pz+q=0$?

   (A) $p=-2$ and $q=-4$
   (B) $p=-2$ and $q=4$
   (C) $p=2$ and $q=4$
   (D) $p=2$ and $q=-4$

2. What is the double root of the equation $x^3-5x^2+8x-4=0$?

   (A) $x=-2$
   (B) $x=-1$
   (C) $x=1$
   (D) $x=2$

3. For the ellipse with the equation $\dfrac{(x-2)^2}{5}+\dfrac{(y+3)^2}{3}=1$, what is the eccentricity?

   (A) $\sqrt{\dfrac{2}{5}}$

   (B) $\sqrt{\dfrac{2}{3}}$

   (C) $\sqrt{\dfrac{5}{3}}$

   (D) $\sqrt{\dfrac{5}{2}}$

4. The equations of the asymptotes of the hyperbola $9x^2-4y^2=36$ are

   (A) $x=\pm\dfrac{3}{2}y$
   (B) $y=\pm\dfrac{3}{2}x$
   (C) $x=\pm\dfrac{9}{4}y$
   (D) $y=\pm\dfrac{9}{4}x$

5. The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x.

Which of the following is the expression for $\frac{dy}{dx}$?

(A) $\frac{y^2 - x}{x^2 + y}$

(B) $\frac{y^2 + x}{x^2 - y}$

(C) $\frac{x^2 + y}{y^2 - x}$

(D) $\frac{x^2 - y}{y^2 + x}$

Question 6 (18 marks)

a. If α, β and γ are the roots of the cubic equation $x^3 + mx + n = 0$,

(i) find, in terms of m and n, the values of:

I. $\alpha + \beta + \gamma$ [1]

II. $\alpha\beta + \beta\gamma + \alpha\gamma$ [1]

III. $\alpha\beta\gamma$ [1]

IV. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ [2]

V. $\alpha^3 + \beta^3 + \gamma^3$ [2]

(ii) Determine the cubic equation whose roots are α^2, β^2 and γ^2 [3]

b. The polynomial $P(x) = x^4 + 7x^3 + 9x^2 - 27x + C$ has a triple zero.

(i) Determine the value of the triple zero. [2]

(ii) Hence, find the value of C. [1]

(iii) Factorise $P(x)$. [1]

c. If α is a non-real double root of $P(x) = x^4 - 4x^3 + 14x^2 - 20x + 25$, factorise $P(x)$ completely into linear factors. [4]

Question 7 (16 marks)

a. The equation of an ellipse is given by $\frac{x^2}{4} + \frac{y^2}{3} = 1$

(i) Find the eccentricity. [1]

(ii) Find the coordinates of the foci, S and S'. [1]

(iii) Find the equations of the directrices. [1]

(iv) Sketch the ellipse, clearly showing the important features. [2]

(v) Let P be any point on the ellipse. Show that the sum of the distances SP and $S'P$ is independent of P. [2]

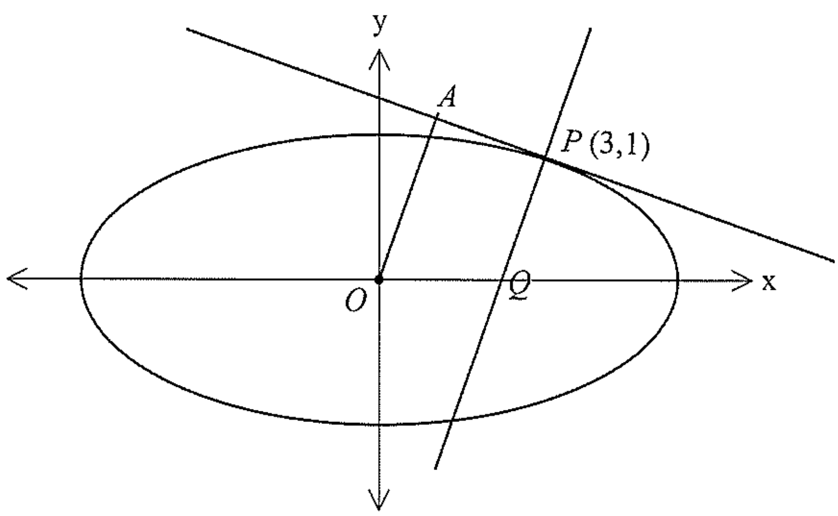
(vi) Find the equation of the chord of contact from (3, 2) [2]

[Equation of chord of contact: $\frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$]

b. An ellipse has the equation $x^2 + 16y^2 = 25$

(i) Find the equations of the **tangent** and **normal** to the ellipse at the point P (3, 1). [4]

(ii) The normal in (i) meets the major axis at Q.
A line from the origin, O, meets the tangent in (i) at right angles at A.
Show that the value of $PQ \times OA$ is equal to the square of the semi-minor axis. [3]



Question 8 (9 marks)

Consider the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

a. Sketch the graph of the hyperbola, clearly showing any intercepts with the coordinate axes, the coordinates of the foci and the equations of the directrices and asymptotes. [3]

b. $T\ (x_1 , y_1)$ is a point on the hyperbola.
Show that the equation of the tangent to the hyperbola at T is
$\frac{x\,x_1}{4} - \frac{y\,y_1}{12} = 1$. [3]

c. $P(4 , 6)$ and $Q(14 , 24)$ are two points on the hyperbola.
M is the midpoint of PQ and $O(0, 0)$ is the origin.
The tangents at P and Q intersect at R.
Show that the points R, O and M are collinear. [3]

Question 9 (8 marks)

Consider the hyperbola with the equation $xy = 25$.

a. $P\left(5p , \frac{5}{p}\right)$ and $Q\left(5q , \frac{5}{q}\right)$ are two distinct points on the hyperbola and p and q are both greater than zero.

Find the equation of the chord PQ . [2]

b. Prove that the equation of the tangent at P is $x + p^2 y = 10p$. [2]

c. The tangents at P and Q intersect at T.

Find the coordinates of T . [2]

d. The chord PQ produced passes through the point $N(0, 10)$.

Find and describe the locus of T . [2]

End of Examination

2019 GHS EXT 2 TASK 2 SOLUTIONS.

MC (5 marks)

1. real coefficients $\therefore$ $1+i$ is also a root

roots : $1 \pm i$, α

sum of roots $\Rightarrow$ $2+\alpha=0$

$\alpha=-2$

Product of roots $\Rightarrow$ $(1+i)(1-i)\times -2 = -q$

$q=4$

$\sum \alpha\beta = (1+i)(1-i) + -2(1+i) + -2(1-i) = p$

$p=-2$ $\boxed{B}$

2. $f'(x) = 3x^2-10x+8$

$(3x-4)(x-2)=0$

$x=\frac{4}{3}, 2$ $\boxed{D}$

3. $b^2=a^2(1-e^2)$

$3=5(1-e^2)$

$e^2=\frac{2}{5}$ $\boxed{A}$

$e=\sqrt{\frac{2}{5}}$

4. $\frac{x^2}{4}-\frac{y^2}{9}=1$ $a=2, b=3$

Asymptotes : $y=\pm\frac{b}{a}x$

$=\pm\frac{3}{2}x$ $\boxed{B}$

5. $3x^2-3y^2\frac{dy}{dx}+3x\frac{dy}{dx}+3y=0$

$\frac{dy}{dx}(3x-3y^2) = -3x^2-3y$

$\frac{dy}{dx} = \frac{-3(x^2+y)}{-3(y^2-x)} = \frac{x^2+y}{y^2-x}$ $\boxed{C}$

Question 6 (18 marks)

a) $x^3 + mx + n = 0$

Roots : α, β, γ

i) I. $\alpha + \beta + \gamma = 0$

II. $\alpha\beta + \beta\gamma + \alpha\gamma = m$

III. $\alpha\beta\gamma = -n$

IV. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$= \frac{-m}{n}$

V. $\alpha^3 + \beta^3 + \gamma^3$

If α, β, γ are roots, then

$\alpha^3 + m\alpha + n = 0$ — ①

$\beta^3 + m\beta + n = 0$ — ②

$\gamma^3 + m\gamma + n = 0$ — ③

① + ② + ③

$\alpha^3 + \beta^3 + \gamma^3 + m(\alpha + \beta + \gamma) + 3n = 0$

$\alpha^3 + \beta^3 + \gamma^3 = -3n$

ii) $y = x^2 \Rightarrow x = \sqrt{y}$

substituting,

$(\sqrt{y})^3 + m\sqrt{y} + n = 0$

$y\sqrt{y} + m\sqrt{y} + n = 0$

$\sqrt{y}(y + m) = -n$

$y(y^2 + 2my + m^2) = n^2$ squaring both sides

$y^3 + 2my^2 + m^2y - n^2 = 0$

or $x^3 + 2mx^2 + m^2x - n^2 = 0$

6 b) $P(x) = x^4 + 7x^3 + 9x^2 - 27x + C$ ← triple zero

i) $P'(x) = 4x^3 + 21x^2 + 18x - 27$ ← double of same.

$P''(x) = 12x^2 + 42x + 18$ ← single

$P''(x) = 0$

$2x^2 + 7x + 3 = 0$

$(2x+1)(x+3) = 0$

$x = -\frac{1}{2}, -3$

$P'(-3) = 4(-3)^3 + 21(-3)^2 + 18(-3) - 27$

$= 0$

$\therefore$ triple zero at $x = -3$

ii) $P(-3) = 0$

$(-3)^4 + 7(-3)^3 + 9(-3)^2 - 27(-3) + c = 0$

$c = -54$

iii) $P(x) = (x+3)^3 \cdot Q(x)$

$P(x) = (x+3)^3(x-2)$ [By observation] ~~(c = -54)~~

c) $P(x) = x^4 - 4x^3 + 14x^2 - 20x + 25$,

If α is a double root, then $\bar{\alpha}$ is also double root.

$\therefore$ roots : $\alpha, \alpha, \bar{\alpha}, \bar{\alpha}$

Sum of roots : $\alpha + \alpha + \bar{\alpha} + \bar{\alpha} = 4$

let $\alpha = a + ib \Rightarrow$ $4a = 4$

$a = 1$

Product of roots : $\alpha\alpha\bar{\alpha}\bar{\alpha} = 25$ and $a^2 + b^2 = 5$

$(\alpha\bar{\alpha})^2 = 25$ $1 + b^2 = 5$

$\alpha\bar{\alpha} = 5$ $b = \pm 2$

$\therefore P(x) = (x - 1 + 2i)^2 (x - 1 - 2i)^2$

Question 7 (16 marks)

a) $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$a^2 = 4, \ b^2 = 3$

i) $b^2 = a^2(1-e^2)$

$3 = 4(1-e^2)$

$e^2 = \frac{1}{4}$

$e = \frac{1}{2}$

(ii) Foci $(\pm ae, 0)$

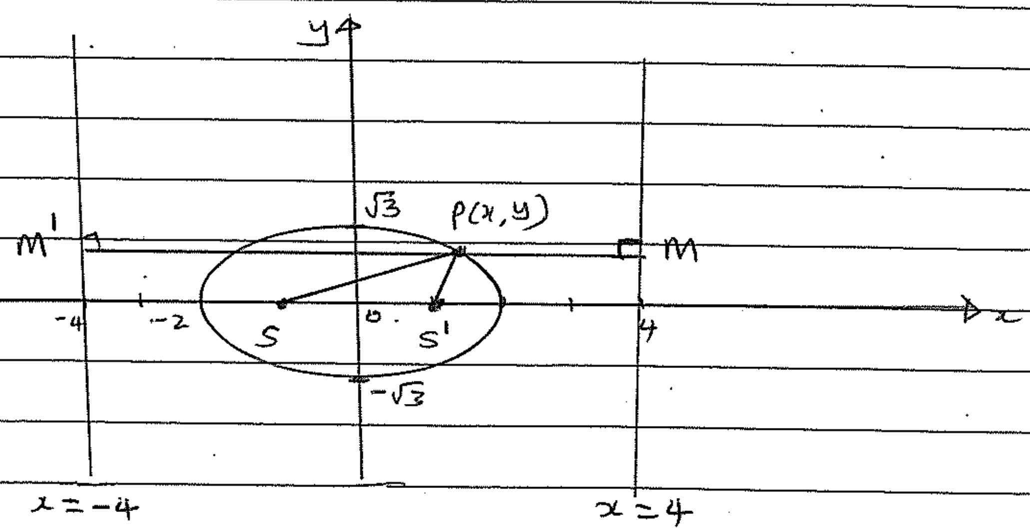
$S(1,0), \ S'(-1,0)$

iii) Directrices : $x = \pm \frac{a}{e}$

$= \pm \frac{2}{1/2}$

$x = \pm 4$

iv)



v) $PS = e\,PM$ —① } definition of ellipse
$PS' = e\,PM'$ —②

① + ②

$PS + PS' = e(PM + PM')$

$= e(8)$

$= \frac{1}{2}(8) = 4$

$\therefore \ PS + PS' = 4$ which is indepent of P.

7a) vi) $E_{\text{CHORD OF CONTACT}}$

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1 \qquad (x_0, y_0) = (3, 2)$$

$$\frac{3x}{4} + \frac{2y}{3} = 1$$

$$9x + 8y - 12 = 0$$

7(b) $x^2 + 16y^2 = 25$

i) $2x + 32y \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-x}{16y}$$

At $P(3,1)$, $m_{\text{tangent}} = \frac{-3}{16}$; $m_{\text{normal}} = \frac{16}{3}$

E_{tangent} : $y - 1 = \frac{-3}{16}(x-3)$

$$16y - 16 = -3x + 9$$

$$\underline{3x + 16y - 25 = 0}$$

E_{normal} : $y - 1 = \frac{16}{3}(x-3)$

$$\underline{16x - 3y - 45 = 0}$$

7b (ii) $\frac{x^2}{25} + \frac{16y^2}{25} = 1$

square of semi-minor ~~axis~~

$b^2 = \frac{25}{16}$

$OA = \frac{|3(0) + 4(0) - 25|}{\sqrt{3^2 + 16^2}}$

$P(3,1)$

$= \frac{25}{\sqrt{265}}$

PQ meets the major axis when $y = 0$

$\therefore \quad 16x = 45$

$x = \frac{45}{16} \qquad Q\left(\frac{45}{16}, 0\right)$

$PQ = \sqrt{\left(3 - \frac{45}{16}\right)^2 + (1-0)^2}$

$= \frac{\sqrt{265}}{16}$

$\therefore \quad PQ \times OA = \frac{\sqrt{265}}{16} \times \frac{25}{\sqrt{265}}$

$= \frac{25}{16}$

$= b^2$ (square of semi-minor axis)

Q8 (9 marks)

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

$a^2 = 4$; $b^2 = 12$

a) $12 = 4(e^2 - 1)$

$e^2 = 4$

$e = 2$

Foci $(\pm ae, 0)$

$(\pm 4, 0)$

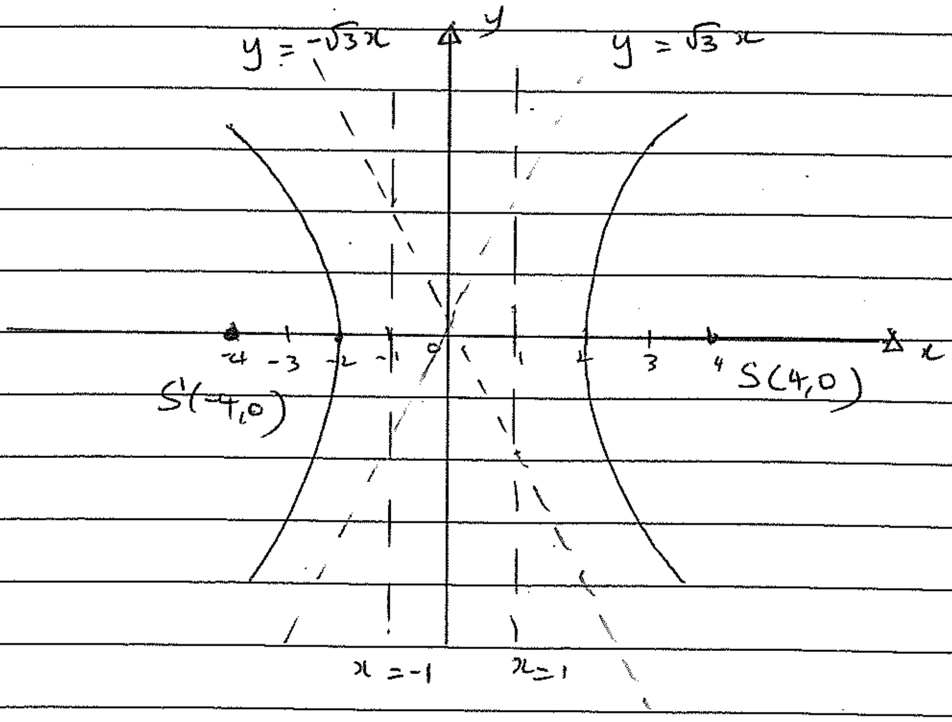
Directrices : $x = \frac{\pm a}{e}$

$x = \pm 1$

Asymptotes : $y = \frac{\pm b}{a} x$

$= \pm \frac{\sqrt{12}}{2} x$

$y = \pm \sqrt{3} x$



8 b) $\frac{x^2}{4} - \frac{y^2}{12} = 1$

$\frac{x}{2} - \frac{y}{6}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{3x}{y}$

At $T(x_1, y_1)$, $m_{tangent} = \frac{3x_1}{y_1}$

$E_{tangent}$: $y - y_1 = \frac{3x_1}{y_1}(x - x_1)$

$yy_1 - y_1^2 = 3xx_1 - 3x_1^2$

$3xx_1 - yy_1 = 3x_1^2 - y_1^2$ $(\div 12)$

$\frac{xx_1}{4} - \frac{yy_1}{12} = \frac{x_1^2}{4} - \frac{y_1^2}{12}$ (T lies on hyperbola $\therefore \frac{x_1^2}{4} - \frac{y_1^2}{12} = 1$)

$\therefore \frac{xx_1}{4} - \frac{yy_1}{12} = 1$

c) $P(4,6)$, $Q(14,24)$

$M = (9,15)$

Tangent at P : $\frac{4x}{4} - \frac{6y}{12} = 1 \Rightarrow x - \frac{y}{2} = 1$ — ①

Tangent at Q : $\frac{14x}{4} - \frac{24y}{12} = 1 \Rightarrow \frac{7x}{2} - 2y = 1$ — ②

Tangents intersect at R

Solving ① and ② simultaneously $\Rightarrow 4 \times ① - ②$

$4x - 2y = 4$

$- \quad \frac{7}{2}x - 2y = 1$

$\frac{x}{2} = 3$

$x = 6$

$y = 10$

$\therefore R = (6,10)$

$m_{OM} = \frac{15}{9}$; $m_{OR} = \frac{5}{3}$

$= \frac{5}{3}$

$\therefore$ R, O, M are collinear.

8

Q9 (8 marks)

a) $P(5p, \frac{5}{p})$; $Q(5q, \frac{5}{q})$

$$m_{PQ} = \frac{\frac{5}{q} - \frac{5}{p}}{5q - 5p} = \frac{5\left(\frac{p-q}{qp}\right)}{5(q-p)}$$

$$= -\frac{1}{pq}$$

$$E_{PQ} : y - \frac{5}{p} = -\frac{1}{pq}(x - 5p)$$

$$pqy - 5q = -x + 5p$$

$$x + pqy - 5(p+q) = 0$$

b) $xy = 25$

$$y = \frac{25}{x}$$

$$\frac{dy}{dx} = \frac{-25}{x^2}$$

At $P(5p, \frac{5}{p})$, $m_{tangent} = \frac{-1}{p^2}$

$$E_{tangent} : y - \frac{5}{p} = \frac{-1}{p^2}(x - 5p)$$

$$p^2y - 5p = -x + 5p$$

$$x^2 + p^2y = 10p$$

c) $E_{\text{tangent at } P}$: $x + p^2 y = 10p$

$$\Rightarrow x = 10p - p^2 y \quad \text{—— (1)}$$

$E_{\text{tangent at } Q}$: $x + q^2 y = 10q$

$$\Rightarrow x = 10q - q^2 y \quad \text{—— (2)}$$

Solving (1) & (2) simultaneously,

$$10p - p^2 y = 10q - q^2 y$$
$$q^2 y - p^2 y = 10q - 10p$$
$$y(q^2 - p^2) = 10(q - p)$$
$$y = \frac{10}{p+q}$$

$$x = 10p - p^2\left(\frac{10}{p+q}\right)$$
$$= \frac{10p(p+q) - 10p^2}{p+q}$$
$$= \frac{10p^2 + 10pq - 10p^2}{p+q}$$
$$x = \frac{10pq}{p+q}$$

$$\therefore T = \left(\frac{10pq}{p+q}, \frac{10}{p+q}\right)$$

d) PQ passes through $N(0, 10$

$\therefore$ N must satisfy the equation of chord PQ

$$x + pqy - 5(p+q) = 0$$
$$\therefore 10pq = 5(p+q)$$
$$pq = \frac{p+q}{2} \quad \text{—— (1)}$$

For T,

$$X = \frac{10pq}{p+q}$$
$$= \frac{10\left(\frac{p+q}{2}\right)}{p+q} \quad (\text{from (1)})$$
$$= 5$$

$\therefore$ Locus is the vertical line $x = 5$

10