GIRRAWEEN HIGH SCHOOL

TASK 1

2022 (HSC 2023)

MATHEMATICS

EXTENSION 2

Time allowed – 90 minutes

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.
- Start each question on a separate page. Each paper must show your name.

Multiple Choice (5 marks) Write the letter corresponding to the correct answer in your answer booklet.

1. Which of the following is the complex number $-2+2i$?

(A) $\sqrt{2}e^{\frac{3\pi i}{4}}$

(B) $2\sqrt{2}e^{\frac{3\pi i}{4}}$

(C) $\sqrt{2}e^{-\frac{3\pi i}{4}}$

(D) $2\sqrt{2}e^{-\frac{3\pi i}{4}}$

2. Given that $(1+i)^n = ai,$ where a is a non-zero real constant, then $(1+i)^{2n+2}$ simplifies to

(A) a^4

(B) $2a^2i$

(C) $1+a^2i$

(D) $-2a^2i$

3. In the complex plane, a circle C has diameter AB, where the pints A and B represent the complex numbers $1-5i$ and $3-i$ respectively. What is the equation of C?

(A) $|z-2+3i| = 2\sqrt{5}$

(B) $|z-2+3i| = \sqrt{5}$

(C) $|z+2-3i| = 2\sqrt{5}$

(D) $|z+2-3i| = \sqrt{5}$

4. Given that $z = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$, which expression is equal to $\left(\bar{z}\right)^{-1}$?

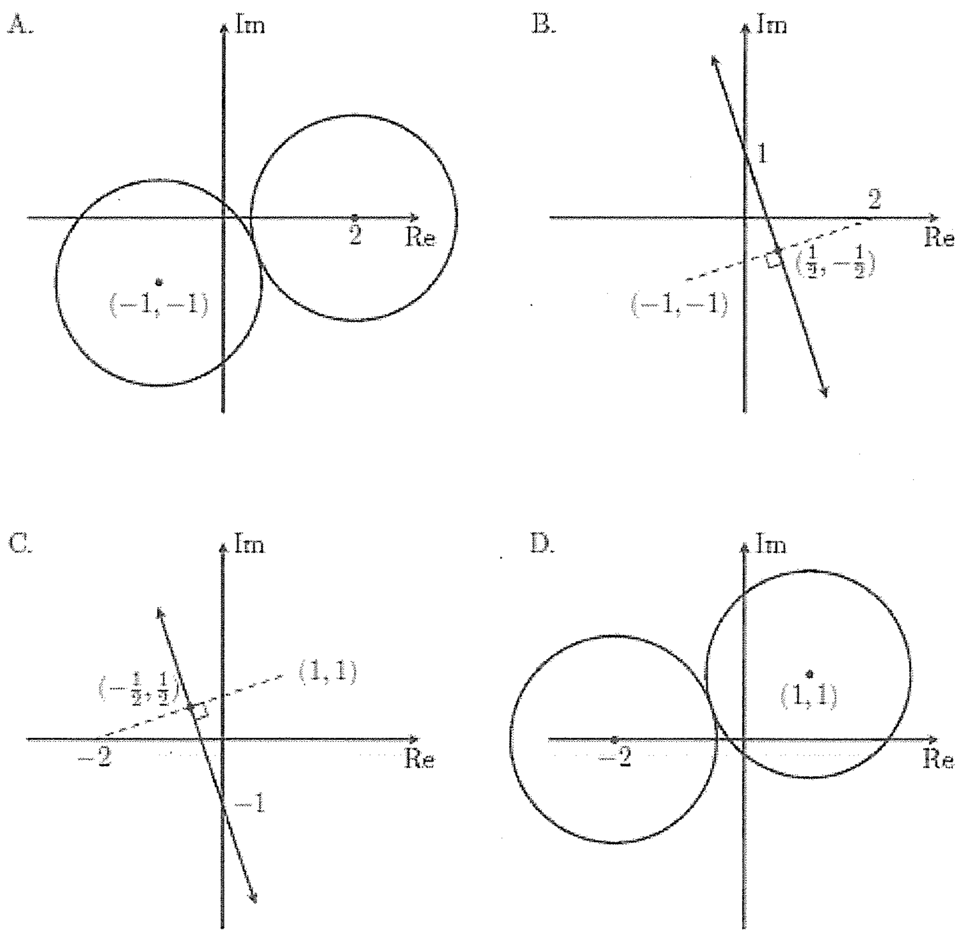
(A) $\frac{1}{2}\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$

(B) $2\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$

(C) $\frac{1}{2}\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

(D) $2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

5. Which diagram best represents the solution to the equation $|z+2| = |z-1-i|$?



Question 6 (17 marks) Marks

(a) If $z=-\sqrt{3}+i$ and $w=1-i$

(i) Find $\frac{z}{w}$ in Cartesian form. 2

(ii) Convert z and w to modulus-argument form. 4

(iii) Find $\frac{z}{w}$ in modulus-argument form. 3

(iv) Hence find the exact value of $\sin\frac{11\pi}{12}$. 2

(b) (i) By using $(x+iy)^2=-24-10i$, find the square roots of $-24-10i$ in

cartesian form. 3

(ii) Hence solve: $z^2-(1-i)z+6+2i=0$ 3

Question 7 (17 marks)

(a) (i) Use de Moivre's theorem to show that $\sin 5\theta=16\sin^5\theta-20\sin^3\theta+5\sin\theta$ 4

(ii) Hence solve $32x^5-40x^3+10x-1=0$ (Leave your answers in terms of $\sin\theta$). 3

(iii) Hence find the exact value of $\sin\frac{\pi}{30}\sin\frac{13\pi}{30}\sin\frac{37\pi}{30}\sin\frac{41\pi}{30}$. 3

(b) (i) Using Euler's theorem, show that $\cos n\theta=\frac{e^{in\theta}+e^{-in\theta}}{2}$ 2

(ii) Hence show that $\cos^6\theta=\frac{1}{32}\left[\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10\right]$ 3

(iii) Hence find the exact value of $\cos\frac{3\pi}{8}$ (answer as a sixth root) 2

Question 8 (21 marks)

(i) Express the roots of the equation $z^5 + 1 = 0$ in modulus-argument form. 3

(ii) Show the roots of $z^5 + 1 = 0$ in an Argand diagram . 3

(iii) Show that $z^4 - z^3 + z^2 - z + 1 = \left(z^2 - 2\cos\frac{\pi}{5} z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5} + 1\right)$ 4

(iv) Hence find the value of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$. 4

(v) Form a quadratic equation with roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$. 3

(vi) Hence find the exact value of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ in simplest surd form. 4

Question 9 (17 marks)

(a) Sketch the following regions on separate complex planes.

(i) $2 \le |z| \le 5$ and $\text{Re}(z) \ge 1$ 3

(ii) $-\frac{5\pi}{6} < \arg(z) \le \frac{\pi}{3}$ 3

(b) z is a variable point which satisfy the equation $\left(\bar{z}\right)^2 - z^2 = 20i$.

(i) Find the cartesian equation of the locus of z. 2

(ii) Sketch the locus. 2

(c) The locus of $\arg\left(\frac{z-2}{z+4}\right) = \frac{3\pi}{4}$, represents part of a circle.

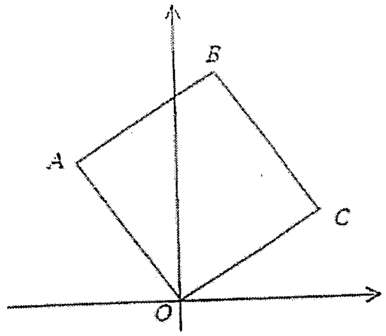
(i) Find the centre and radius of this circle (show diagram and working

out with reasons). 5

(ii) Write the cartesian equation of the locus. 2

Question 10 (19 marks)

(a) In the Argand diagram below, $OABC$ is a square. The point A represents the complex number $-3+4i$.



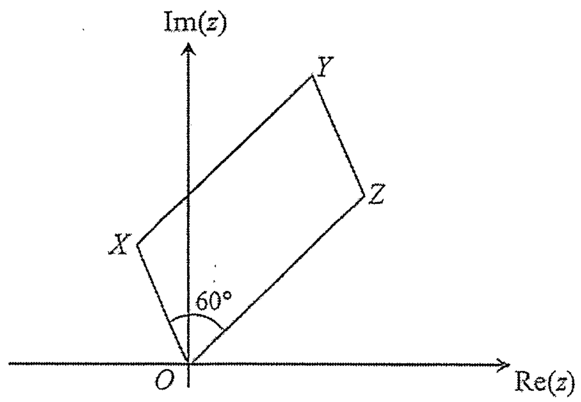
Find in the form $a+ib$:

(i) The complex number represented by C. 2

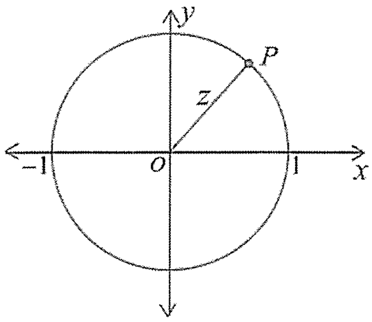
(ii) The complex number represented by B. 2

(iii) The complex number represented by the vector CA. 2

(b) In the diagram below $OXYZ$ is a parallelogram with $OX=\frac{1}{2}OZ$. The point X represents the complex number $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$. If $\angle XOZ=60°$, what complex number does Z represent? 4



(c) P represents the complex number z such that $|z| = 1$.Copy the diagram onto your answer sheet and prove that



(i) $\arg(z+1) = \dfrac{1}{2}\arg z$ 3

(ii) $\arg(z-1) = \arg(z+1) + \dfrac{\pi}{2}$ 3

(iii) $\left|\dfrac{z-1}{z+1}\right| = \tan\left(\dfrac{1}{2}\arg z\right)$ 3

Examination continues on next page

Question 11 (14 marks)

(a) $2+i$ is a root of $x^3-7x^2+Ax+B=0$ where A and B are real.

(i) Find the other roots. 2

(ii) Find the values of A and B. 2

(iii) Hence write $P(x)$ as a product of a linear factor and a quadratic factor,

both with real coefficients. 2

(b) Given that $P(x)=x^3-x^2+mx+n$, where both m and n are integers.

(i) Show that $P(-i)=(1+n)+i(1-m)$ 2

(ii) When $P(x)$ is divided by x^2+1, the remainder is $6x-3$. Find the values

of m and n. 2

(c) Find the four fourth roots of $-8-8\sqrt{3}i$. Write answers in

modulus- argument form. 4

END OF EXAMINATION

Extension 2 Task 1 2022 (HSC 2023) Solutions

Multiple choice (5 marks)

1. $z = -2+2i$

$|z| = 2\sqrt{2}$

$\tan\alpha = 1 \quad \alpha = \frac{\pi}{4}$

$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$-2+2i = 2\sqrt{2}\, e^{i\frac{3\pi}{4}}$ B

2. $(1+i)^{2n+2} = (1+i)^{2n} \times (1+i)^2$

$= (ai)^2 \times (1+2i-1)$

$= -a^2 \times 2i$

$= -2a^2 i$ D

3. $(1,-5) \quad (3,-1)$

centre $(2,-3)$

distance $= \sqrt{1^2+2^2}$

$= \sqrt{5}$

$|z-(2-3i)| = \sqrt{5}$

$|z-2+3i| = \sqrt{5}$ B

4. $z = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

$\bar{z} = 2\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$

$= 2\left(\cos\left(\frac{-\pi}{5}\right) + i\sin\left(\frac{-\pi}{5}\right)\right)$

$(\bar{z})^{-1} = 2^{-1}\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

C

5. Perpendicular bisector of $(-2,0)$ and $(1,1)$

C

Question 6 (17 marks)

(i) $\frac{z}{w} = \frac{-\sqrt{3}+i}{1-i} \times \frac{1+i}{1+i}$

$= \frac{(-\sqrt{3}+i)(1+i)}{1-i^2}$

$= \frac{-\sqrt{3}-i\sqrt{3}+i+i^2}{2}$ (2)

$= \frac{-\sqrt{3}-1+i(1-\sqrt{3})}{2}$

$= \frac{-\sqrt{3}-1}{2} + i\frac{(1-\sqrt{3})}{2}$

(ii) $z = -\sqrt{3}+i$

$|z| = \sqrt{3+1} = 2$

$\tan\alpha = \frac{1}{\sqrt{3}} \quad \alpha = \frac{\pi}{6}$

$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ (4)

$-\sqrt{3}+i = 2\operatorname{cis}\frac{5\pi}{6}$

$w = 1-i$

$|w| = \sqrt{2}, \quad \tan\alpha = 1, \quad \alpha = \frac{\pi}{4}, \theta = \frac{-\pi}{4}$

$1-i = \sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)$

(iii) $\frac{z}{w} = \frac{2\operatorname{cis}\frac{5\pi}{6}}{\sqrt{2}\operatorname{cis}(-\frac{\pi}{4})}$

$= \sqrt{2}\operatorname{cis}\left(\frac{5\pi}{6}+\frac{\pi}{4}\right)$

$= \sqrt{2}\operatorname{cis}\frac{13\pi}{12}$ (3)

$= \sqrt{2}\operatorname{cis}\left(\frac{-11\pi}{12}\right)$

(iv) $\sqrt{2}\left(\cos\frac{11\pi}{12} - i\sin\frac{11\pi}{12}\right) = \frac{-\sqrt{3}-1}{2} + \frac{i(1-\sqrt{3})}{2}$

$\cos\frac{11\pi}{12} - i\sin\frac{11\pi}{12} = \frac{-\sqrt{3}-1}{2\sqrt{2}} + \frac{i(1-\sqrt{3})}{2\sqrt{2}}$

Equating imaginary parts

$-\sin\frac{11\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$ (2)

$\sin\frac{11\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(b) (i) $(x+iy)^2 = -24-10i$

$x^2+2ixy-y^2 = -24-10i$

$x^2-y^2 = -24$

$2xy = -10$

$xy = -5$

$x^2 - \frac{25}{x^2} = -24$

$x^4 - 25 = -24x^2$

$x^4+24x^2-25=0$

$(x^2-1)(x^2+25)=0$

$x^2=1$ or $x^2=-25$

$x = \pm 1$ ($\because x$ is real)

$x=1,\ y=-5$

$x=-1,\ y=5$ (3)

$\therefore$ the square roots are

$1-5i$ or $-1+5i$

$= \pm(1-5i)$

(ii) $z^2-(1-i)z+6+2i=0$

$$z=\frac{(1-i)\pm\sqrt{(1-i)^2-4\times1(6+2i)}}{2}$$

$$=\frac{(1-i)\pm\sqrt{1-2i-1-24-8i}}{2}$$

$$=\frac{(1-i)\pm\sqrt{-10i-24}}{2}$$

3

$$=\frac{(1-i)\pm(1-5i)}{2}$$

$$=\frac{1-i+1-5i}{2} \text{ or } \frac{1-i-1+5i}{2}$$

$$=\underline{\underline{1-3i \text{ or } 2i}}$$

<u>Question 7 (17 marks)</u>

(a) (i) $(\cos\theta+i\sin\theta)^5=\cos5\theta+i\sin5\theta$ (by De Moivre's theorems) — 1

Applying Binomial theorem

$(\cos\theta+i\sin\theta)^5=\cos^5\theta+{}^5C_1\cos^4\theta\cdot i\sin\theta+{}^5C_2\cos^3\theta(i\sin\theta)^2$

$+{}^5C_3\cos^2\theta\cdot i^3\sin^3\theta+{}^5C_4\cos\theta\, i^4\sin^4\theta+i\sin^5\theta$

$=\cos^5\theta+5i\cos^4\theta\sin\theta+10\cos^3\theta\times-\sin^2\theta+$

$10\cos^2\theta\times-i\sin^3\theta+5\cos\theta\sin^4\theta+i\sin^5\theta$ — 2

Equaling imaginary on the RHS of 1 and 2

$\sin5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$

$=5(1-\sin^2\theta)^2\sin\theta-10(1-\sin^2\theta)\sin^3\theta+\sin^5\theta$

$= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10\sin^3\theta(1-\sin^2\theta)+\sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ 4

(ii) The solutions are given by distinct values of $x=\sin\theta$ where $\sin 5\theta = \frac{1}{2}$

$$5\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \frac{41\pi}{6}, \frac{49\pi}{6}, \frac{53\pi}{6}$$

$$\theta = \frac{\pi}{30}, \frac{\pi}{6}, \frac{13\pi}{30}, \frac{17\pi}{30}, \frac{5\pi}{6}, \frac{29\pi}{30}, \frac{37\pi}{30}, \frac{41\pi}{30}, \frac{49\pi}{30}, \frac{53\pi}{30}$$

$\sin\frac{17\pi}{30} = \sin\frac{13\pi}{30}, \sin\frac{5\pi}{6} = \sin\frac{\pi}{6}, \sin\frac{29\pi}{30} = \sin\frac{\pi}{30}$ (since $\sin(180-\theta)=\sin\theta$) 3

$\therefore$ the roots of $32x^5 - 40x^3 + 10x - 1 = 0$ are

$$x=\sin\frac{\pi}{30}, x=\sin\frac{\pi}{6}, x=\sin\frac{13\pi}{30}, x=\sin\frac{37\pi}{30}, x=\sin\frac{41\pi}{30}$$

(iii) By taking the product of the roots we have

$$\sin\frac{\pi}{30}\times\sin\frac{\pi}{6}\times\sin\frac{13\pi}{30}\times\sin\frac{37\pi}{30}\times\sin\frac{41\pi}{30} = -\left(\frac{-1}{32}\right) = \frac{1}{32}$$ 3

$$\sin\frac{\pi}{30}\times\frac{1}{2}\times\sin\frac{13\pi}{30}\times\sin\frac{37\pi}{30}\times\sin\frac{41\pi}{30} = \frac{1}{32}$$

$$\sin\frac{\pi}{30}\times\sin\frac{13\pi}{30}\times\sin\frac{37\pi}{30}\times\sin\frac{41\pi}{30} = \frac{1}{32}\times 2 = \frac{1}{16}$$

(b) (i) $e^{in\theta} = \cos n\theta + i\sin n\theta$

$e^{-in\theta} = \cos n\theta - i\sin n\theta$ (by De Moivre's theorem)

$e^{in\theta} + e^{-in\theta} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$ (2)

$\cos n\theta = \dfrac{e^{in\theta} + e^{-in\theta}}{2}$

(ii) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$; $\cos^6\theta = \dfrac{(e^{i\theta} + e^{-i\theta})^6}{64}$

$\cos^6\theta = \frac{1}{64}\Big[(e^{i\theta})^6 + {}^6C_1 (e^{i\theta})^5 (e^{-i\theta}) + {}^6C_2 (e^{i\theta})^4 (e^{-i\theta})^2$

$+ {}^6C_3 (e^{i\theta})^3 (e^{-i\theta})^3 + {}^6C_4 (e^{i\theta})^2 (e^{-i\theta})^4$

$+ {}^6C_5 (e^{i\theta})(e^{-i\theta})^5 + (e^{-i\theta})^6$

$= \frac{1}{64}\Big[e^{i6\theta} + 6e^{i5\theta} e^{-i\theta} + 15 e^{i4\theta} e^{-i2\theta} + 20 e^{i3\theta} \times e^{-i3\theta}$

$+ 15 e^{i2\theta} \times e^{-i4\theta} + 6e^{i\theta} e^{-i5\theta} + e^{-i6\theta}\Big]$

$= \frac{1}{64}\Big[e^{i6\theta} + 6e^{i4\theta} + 15e^{i2\theta} + 20e^{0} + 15e^{-i2\theta} + 6e^{-i4\theta}$

$+ e^{-i6\theta}\Big]$

$= \frac{1}{64}\Big[e^{i6\theta} + e^{-i6\theta} + 6(e^{i4\theta} + e^{-i4\theta}) + 15(e^{i2\theta} + e^{-i2\theta}) + 20\Big]$

$= \frac{1}{64}\Big[2\cos 6\theta + 6 \times 2\cos 4\theta + 15 \times 2\cos 2\theta + 20\Big]$

$= \frac{1}{32}\Big[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\Big]$ (3)

(iii) $\cos^6 \frac{3\pi}{8} = \frac{1}{32}\left[\cos 6\times\frac{3\pi}{8} + 6\cos 4\times\frac{3\pi}{8} + 15\cos 2\times\frac{3\pi}{8} + 10\right]$

$= \frac{1}{32}\left[\frac{1}{\sqrt{2}} + 6\times 0 + 15\times -\frac{1}{\sqrt{2}} + 10\right]$

$= \frac{1}{32}\left[\frac{1}{\sqrt{2}} - \frac{15}{\sqrt{2}} + 10\right] = \frac{1}{32}\left[10 - \frac{14}{\sqrt{2}}\right]$

$= \frac{1}{32}\left[\frac{10\sqrt{2}-14}{\sqrt{2}}\right] = \frac{2}{32}\left[\frac{5\sqrt{2}-7}{\sqrt{2}}\right]$

$= \frac{1}{16}\left(\frac{5\sqrt{2}-7}{\sqrt{2}}\right) = \frac{5\sqrt{2}-7}{16\sqrt{2}} = \frac{\sqrt{2}(5\sqrt{2}-7)}{32} = \frac{10-7\sqrt{2}}{32}$

$\cos\frac{3\pi}{8} = \sqrt[6]{\frac{10-7\sqrt{2}}{32}}$ (2)

Question 8 (21 marks)

(i) $z^5 = -1$ Let $z = \cos\theta + i\sin\theta$

$\text{Cis}\,5\theta = \text{cis}(\pi + 2k\pi) = \text{Cis}(2k+1)\pi$ where k is any integers. (de Moivre's theorem)

$5\theta = (2k+1)\pi$

$\theta = \frac{(2k+1)\pi}{5}$

$-\pi < \theta \le \pi$

$-\pi < \frac{(2k+1)\pi}{5} \le \pi$

$-1 < \frac{(2k+1)}{5} \le 1$

$-5 < (2k+1) \le 5$

$-6 < 2k \le 4$

$-3 < k \le 2$

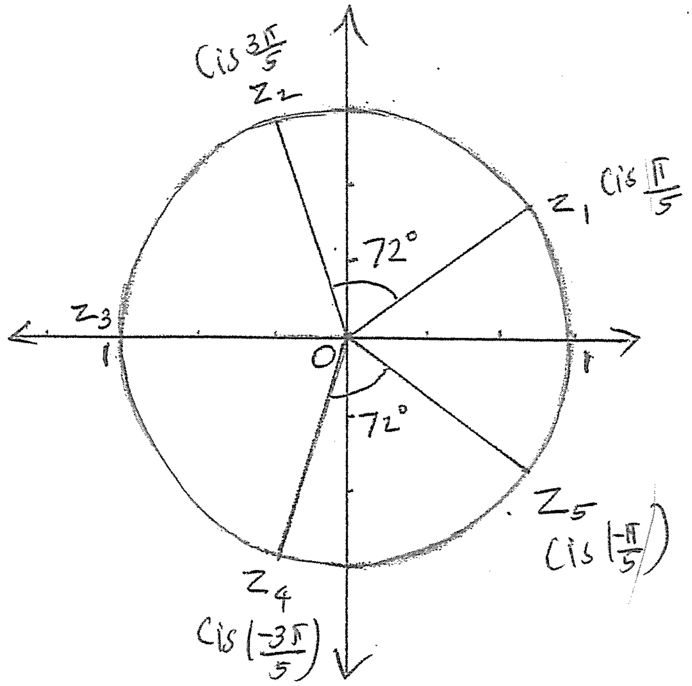
$k = -2, -1, 0, 1, 2$ (3)

$\theta = \frac{-3\pi}{5}, \frac{-\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}, \pi$

The roots are

$\text{Cis}\left(\frac{-3\pi}{5}\right), \text{cis}\left(\frac{-\pi}{5}\right), \text{Cis}\left(\frac{\pi}{5}\right), \text{Cis}\left(\frac{3\pi}{5}\right), -1$

(ii)



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(iii) From the above figure

$$z_5 = \bar{z}_1 \quad \text{and} \quad z_4 = \bar{z}_2$$

$$z_1 + z_5 = z_1 + \bar{z}_1 = 2\cos\frac{\pi}{5}$$

$$z_2 + z_4 = z_2 + \bar{z}_2 = 2\cos\frac{3\pi}{5}$$

$$z_1 z_5 = z_1\bar{z}_1 = |z_1|^2 = 1$$

$$z_2 z_4 = z_2\bar{z}_2 = |z_2|^2 = 1$$

$$z^5 + 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)$$

$$= (z+1)(z - z_1)(z - z_5)(z - z_2)(z - z_4)$$

$$= (z+1)\left(z^2 - z(z_1 + z_5) + z_1 z_5\right)\left(z^2 - z(z_2 + z_4) + z_2 z_4\right)$$

$$= (z+1)\left(z^2 - 2\cos\frac{\pi}{5}\, z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5}\, z + 1\right) \quad \text{(1)}$$

$$z^5 + 1 = (z+1)\left(z^4 - z^3 + z^2 - z + 1\right) \quad \text{(2)}$$

From ① and ② we get

$$z^4 - z^3 + z^2 - z + 1$$

$$= \left(z^2 - 2\cos\frac{\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5}z + 1\right)$$ ④

(iv) $z^4 - z^3 + z^2 - z + 1 = z^4 - 2\cos\frac{3\pi}{5}z^3 + z^2 - 2\cos\frac{\pi}{5}z^3$

$+ 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}z^2 - 2\cos\frac{\pi}{5}z + z^2 - 2\cos\frac{3\pi}{5}z + 1$

Equating coefficients of z^2

$$1 = 1 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} + 1$$

$$1 = 2 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$$

$$-1 = 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$$

$$\underline{\underline{\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}}}$$

Equating coefficients of z^3

$$-1 = -2\cos\frac{3\pi}{5} - 2\cos\frac{\pi}{5}$$

$$-1 = -2\left(\cos\frac{3\pi}{5} + \cos\frac{\pi}{5}\right)$$ ④

$$\underline{\underline{\cos\frac{3\pi}{5} + \cos\frac{\pi}{5} = \frac{1}{2}}}$$

(v) The quadratic equation with roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ is given by $x^2 - \frac{1}{2}x - \frac{1}{4} = 0$

$$\underline{\underline{4x^2 - 2x - 1 = 0}}$$ ③

(vi) $4x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{4 - 4 \times 4 \times -1}}{8}$$

$$= \frac{2 \pm 2\sqrt{5}}{8} = \frac{1 \pm \sqrt{5}}{4}$$

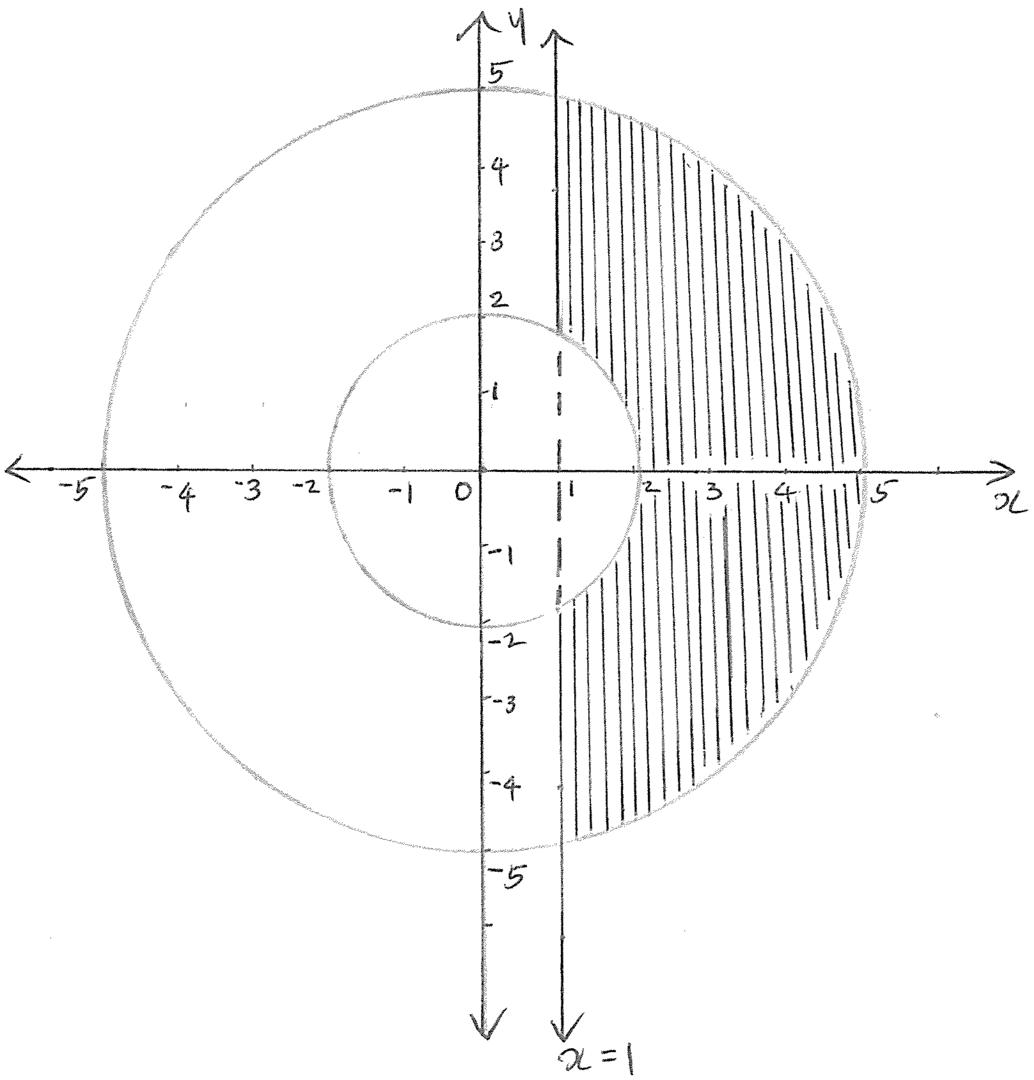
$\cos\frac{\pi}{5}$ is positive

$\cos\frac{3\pi}{5}$ is negative (4)

$\therefore \cos\frac{\pi}{5} = \underline{\underline{\frac{1+\sqrt{5}}{4}}}$ and $\cos\frac{3\pi}{5} = \underline{\underline{\frac{1-\sqrt{5}}{4}}}$

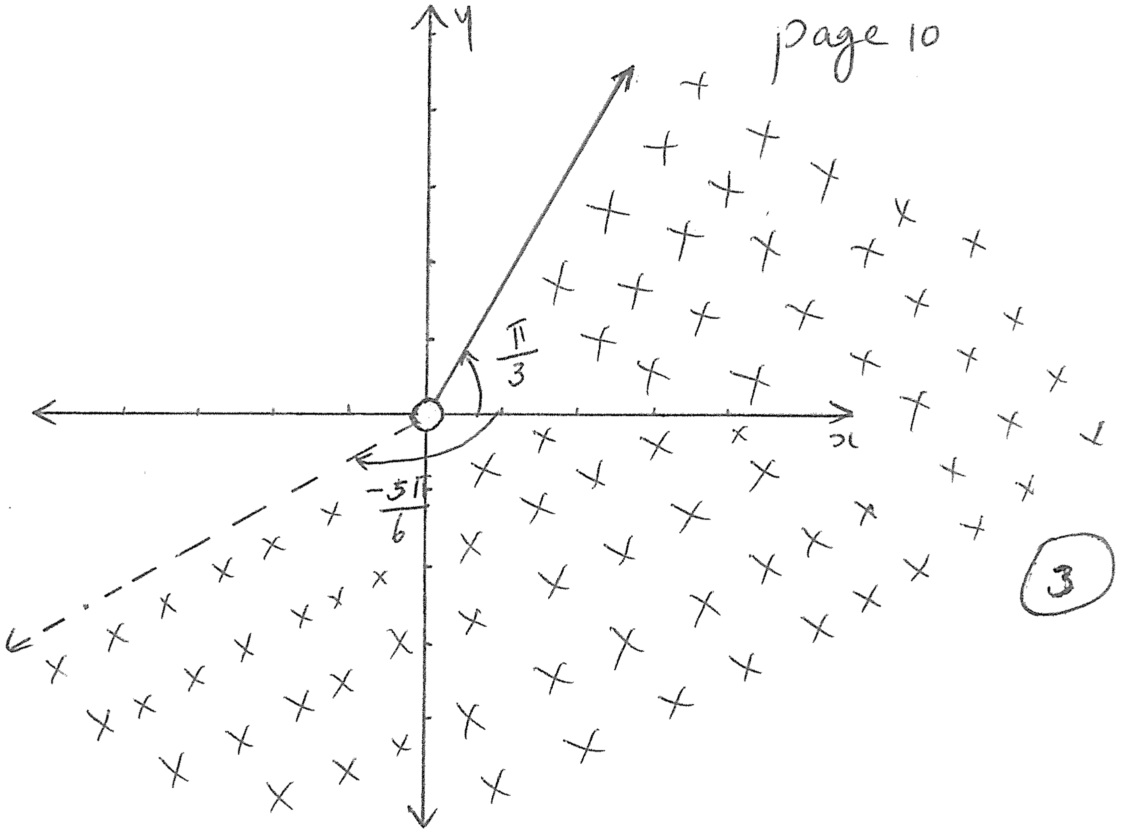
Question 9 (17 marks)

(a)(i) $2 \le |z| \le 5$ and $Re(z) \ge 1$



(3)

(ii)



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(b) (i) $(\bar{z})^2 - z^2 = 20i$ Let $z = x + iy$

$(x - iy)^2 - (x + iy)^2 = 20i$

$(x^2 - 2ixy - y^2) - (x^2 + 2ixy - y^2) = 20i$

$-4ixy = 20i$

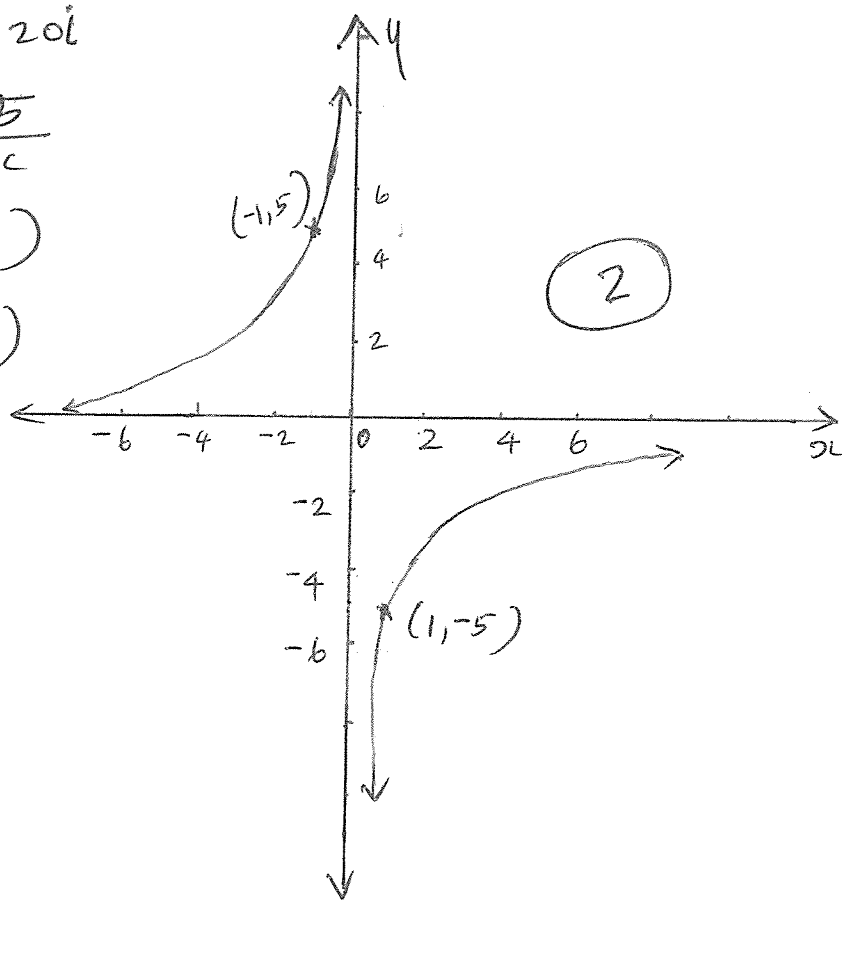
2

$y = \frac{-5}{x}$

$x = 1, \ y = -5 \quad (1, -5)$

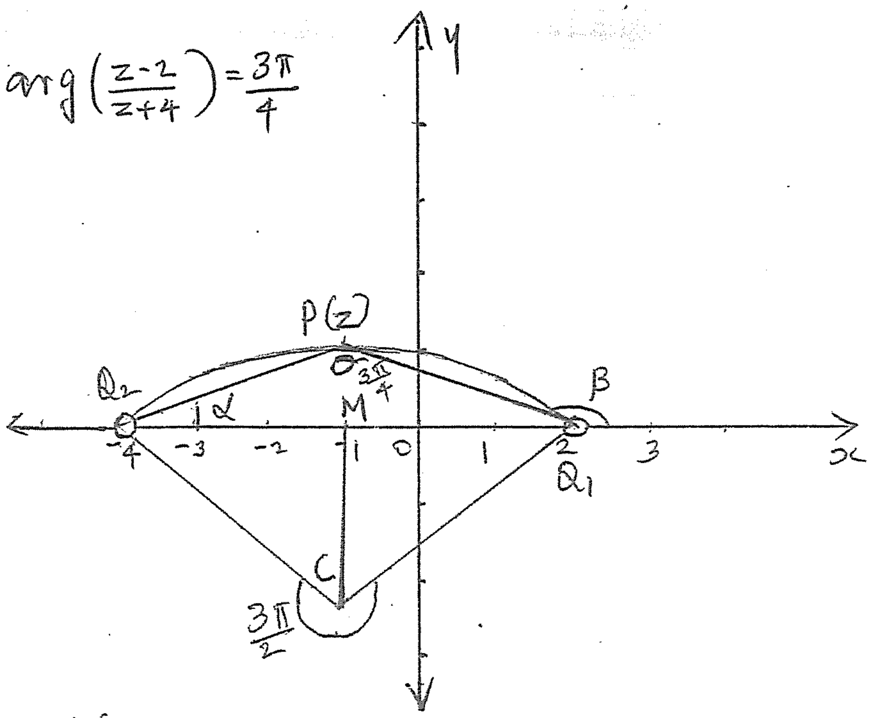
$x = -1, \ y = 5 \quad (-1, 5)$

(ii)



2

(c) (i) $\arg\left(\frac{z-2}{z+4}\right)=\frac{3\pi}{4}$



$\arg(z-2)=\angle PQ_1X=\beta$

$\arg(z+4)=\angle PQ_2X=\alpha$

$\alpha+\theta=\beta$ (exterior angle of ΔQ_2PQ_1)

$\theta=\beta-\alpha$

$=\arg(z-2)-\arg(z+4)$

$=\arg\left(\frac{z-2}{z+4}\right)=\frac{3\pi}{4}$

(ii) Let C be the centre of the circle. Draw $CM\perp Q_1Q_2$

M is the mid point of Q_1Q_2 since the perpendicular from the centre of a circle to a chord bisects the chord. $M=(-1,0)$

Reflex $\angle Q_2CQ_1=\frac{3\pi}{2}$ (angle at the centre is twice angle at the circumference standing on the same arc)

ΔQ_2CQ_1 is isosceles ($Q_2C=Q_1C$ = radii)

$\angle Q_2CQ_1 = 2\pi - \frac{3\pi}{2} = \frac{\pi}{2}$ (angles at a point)

$\angle CQ_2Q_1 = \dfrac{\pi - \frac{\pi}{2}}{2}$ (angles opposite equal sides in an isosceles Δ, angle sum of Δ)

$= \frac{\pi}{4}$

In ΔQ_2CM

$\tan\frac{\pi}{4} = \frac{CM}{Q_2M} = \frac{CM}{3}$

$\frac{CM}{3} = 1$; $CM = 3$

Centre $(-1, -3)$ 5

Apply Pythagoras' theorem in ΔQ_2CM

$Q_2C = \sqrt{9+9} = \sqrt{18}$
$= 3\sqrt{2}$

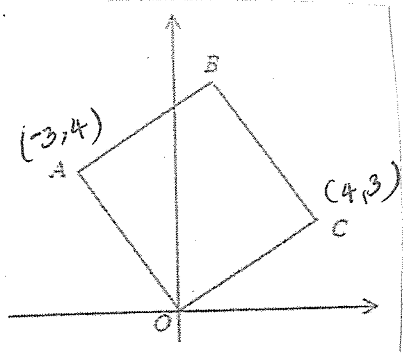
locus is the minor arc of the circle

$(x+1)^2 + (y+3)^2 = 18, y > 0$

excluding the points $(-4,0)$ and $(2,0)$ 2

Question 10 (19 marks)

(a) (i)



(i) $A = -3+4i$

$OC = -i \times OA$ 2

$= -i(-3+4i)$

$= 3i - 4i^2 = 4+3i$

(ii) $\vec{OB} = \vec{OC} + \vec{CB}$

$= 4+3i + -3+4i$

$= 1+7i$ 2

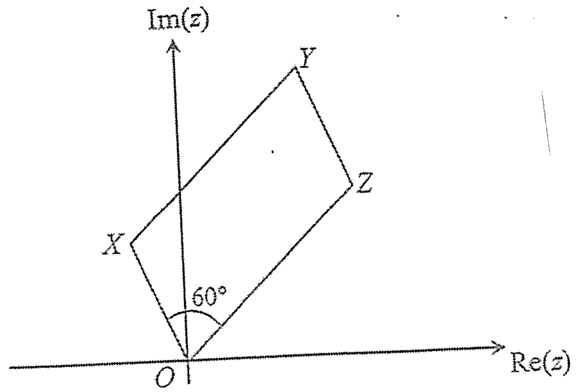
(iii) $\vec{CA} = \vec{OA} - \vec{OC}$

$= -3+4i - (4+3i)$

$= -3+4i-4-3i$

$= -7+i$ 2

(b)



$$Z = 2\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right) \operatorname{cis} \frac{-\pi}{3}$$

$$= \left(-1 + \sqrt{3} i\right) \operatorname{cis} \frac{-\pi}{3}$$

$$= \left(-1 + i\sqrt{3}\right)\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) \quad 4$$

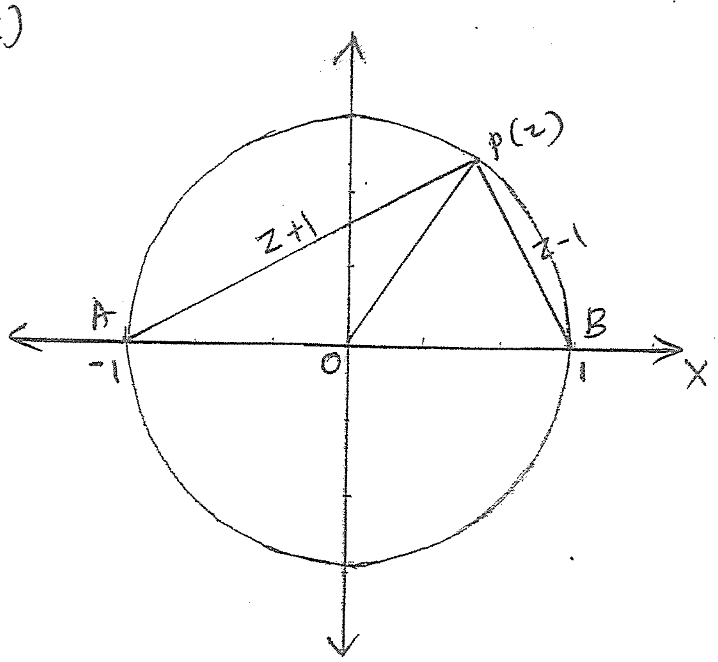
$$= \left(-1 + i\sqrt{3}\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= \frac{-1}{2} + \frac{i\sqrt{3}}{2} + \frac{i\sqrt{3}}{2} + \frac{3}{2}$$

$$= \frac{3}{2} - \frac{1}{2} + \frac{2i\sqrt{3}}{2}$$

$$= \underline{\underline{1 + i\sqrt{3}}}$$

(c)



(i) $\angle PAO + \angle APO = \angle POB$ (exterior $\angle$ of ΔAOP)

$\arg(z+1) + \arg(z+1) = \arg z$ ($\because$ ΔAOP is isosceles $OA = OP = 1$, angles opposite equal sides in isosceles Δ)

$2\arg(z+1) = \arg z$

$\arg(z+1) = \frac{1}{2}\arg z$ 3

(ii) $\angle APB = 90°$ (angle in a semicircle)

$\angle PAB + \angle APB = \angle PBX$ 3

$\arg(z+1) + \frac{\pi}{2} = \arg(z-1)$

$$\underline{\underline{\arg(z-1) = \arg(z+1) + \frac{\pi}{2}}}$$

(iii) $\tan \angle PAB = \frac{PB}{PA} = \frac{|z-1|}{|z+1|} = \left|\frac{z-1}{z+1}\right|$

$\tan(\arg(z+1)) = \left|\frac{z-1}{z+1}\right| = \tan\left(\frac{1}{2}\arg z\right)$ 3

From (i)

Question 11 (14 marks)

(a) (i) $2-i$ is a root. Let α be the third root.

Sum of roots $= 2+i+2-i+\alpha = 7$

$4+\alpha = 7$

$\alpha = 3$

$\therefore$ the roots are $2-i, 2+i, 3$

2

(ii) $\sum \alpha\beta = (2+i)(2-i)+3(2+i)+3(2-i) = A$

$4-i^2+6+3i+6-3i = A$

$4+1+6+3i+6-3i = A$

$A = 17$

2

Product of the roots $(2+i)(2-i)\times 3$

$15 = -B$

$B = -15$

(iii) $P(x) = (x-3)(x-(2+i))(x-(2-i))$

$= (x-3)(x-2-i)(x-2+i)$

$= (x-3)((x-2)^2-i^2)$

$= (x-3)(x^2-4x+4+1)$

$= (x-3)(x^2-4x+5)$

2

(b) (i) $P(-i) = (-i)^3-(-i)^2+m(-i)+n$

$= -1\times -i-(-1)-mi+n$

$= i+1-mi+n$

$= (n+1)+i(1-m)$

2

(ii) $P(x) = (x^2+1)Q(x) + 6 \quad -3$

$P(-i) = 0 + 6 \times -i - 3$

$= -6i - 3$

$(n+1) + i(1-m) = -6i - 3$

$n + 1 = -3 \qquad 1 - m = -6$

$\underline{\underline{n = -4}} \qquad \underline{\underline{m = 7}} \qquad 2$

Solution continues on next page

(c) $z^4 = -8-8\sqrt{3}i$

$= 8(-1-\sqrt{3}i)$

$w = -1-\sqrt{3}i$

$|w| = \sqrt{1+3} = 2$

$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

$\theta = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$

$-1-i\sqrt{3} = 2\,\text{cis}\left(\frac{-2\pi}{3}\right)$

$z^4 = 16\,\text{cis}\left(\frac{-2\pi}{3}\right)$

Let $z = r\,\text{cis}\,\theta$

$z^4 = r^4\,\text{cis}4\theta$

$r^4\,\text{cis}\,4\theta = 16\,\text{cis}\left(\frac{-2\pi}{3}\right)$

$= 16\,\text{cis}\left(2k\pi - \frac{2\pi}{3}\right)$

$= 16\,\text{cis}\left(\frac{(6k-2)\pi}{3}\right)$

$r^4 = 16 \quad \therefore r = 2$

$4\theta = \frac{(6k-2)\pi}{3}$

$\theta = \frac{(6k-2)\pi}{12}$

$-\pi < \theta \leq \pi$

$-\pi < \frac{(6k-2)\pi}{12} \leq \pi$

$-1 < \frac{(6k-2)}{12} \leq 1$

$-12 < (6k-2) \leq 12$

$-10 < 6k \leq 14$

$-1.7 < k \leq 2.3$

$\therefore k = -1, 0, 1, 2$

$\theta = -\frac{8\pi}{12} = -\frac{2\pi}{3}, \quad -\frac{2\pi}{12} = -\frac{\pi}{6},$

$\frac{4\pi}{12} = \frac{\pi}{3}, \quad \frac{10\pi}{12} = \frac{5\pi}{6}$

4

The roots are

$2\,\text{cis}\left(-\frac{2\pi}{3}\right), \quad 2\,\text{cis}\left(\frac{-\pi}{6}\right)$

$2\,\text{cis}\left(\frac{\pi}{3}\right), \quad 2\,\text{cis}\left(\frac{5\pi}{6}\right)$