Girraween High School

2024 Year 12 Mathematics Extension 2
Task 1, Dec-2023

General Instructions

- Working Time - 90 minutes
- Write using black or blue pen only
- Calculators approved by NESA may be used
- For questions in Section II, Show relevant mathematics reasoning and/ or calculations

Total Marks: 95

Section I - 5 marks (page 1)

- Attempt questions 1-5
- Allow about 10 minutes for this section

Section II - 87 marks (pages 3-6)

- Attempt questions 6-11
- Allow about 80 minutes for this section

Question 1 (1 mark)

Which of the following gives $\sqrt{3}+i$ in Modulus-argument form.

A. $2cis\left(\frac{-\pi}{6}\right)$

B. $2cis\left(\frac{-2\pi}{3}\right)$

C. $2cis\left(\frac{\pi}{6}\right)$

D. $2cis\left(\frac{2\pi}{3}\right)$

Question 2 (1 mark)

Which is the simplified form of i^{2011} ?

A. i

B. -1

C. $-i$

D. 1

Question 3 (1 mark)

The relation $(z+2)(\overline{z}+2)=4$, when graphed on an Argand diagram, would be

A. a circle of radius 4 with centre at (-2,0)

B. a circle of radius 4 with centre at (2,0)

C. a circle of radius 2 with centre at (-2,0)

D. a circle of radius 2 with centre at (2,0)

Question 4 (1 mark)

What are the values of real numbers p and q such that $1-i$ is a root of the equation $P(z)=z^3+pz+q$?

A. $p=-2$ and $q=-4$

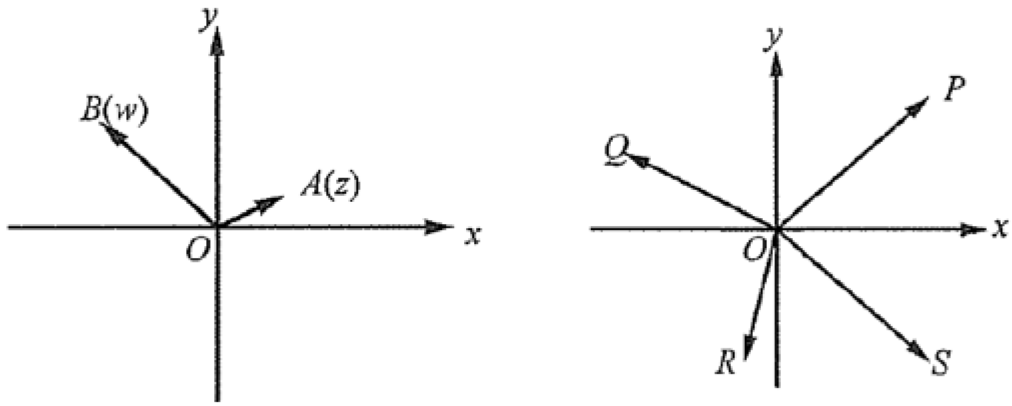
B. $p=-2$ and $q=4$

C. $p=2$ and $q=-4$

D. $p=2$ and $q=4$

Question 5 (1 mark)

The vectors OA and OB represent the complex numbers z and w respectively.



which vector represents the complex number $\overline{(z - iw)}$

A. OP

B. OQ

C. OR

D. OS

Mathematics Extension 2

Section II

87 Marks

Attempt Questions 6-11

Allow about 80 minutes for this section

Question 6 (13 marks)

(a) If $z_1 = 3 - 2i$ and $z_2 = 3 - 4i$, express the following in the form $a + ib$ where a and b are real.

i. $z_1 - z_2$ [1]

ii. ${z_2}^2$ [1]

iii. $z_1\overline{z_2}$ [1]

(b) i. Find the values of the real numbers x and y such that $(x + iy)^2 = -24 + 10i$ [3]

ii. Hence, solve th quadratic equation $z^2 - (3 + i)z + 8 - i = 0$ [2]

(c) Find the modulus of $\dfrac{(2 - i)^8}{(2 + i)^6}$ [3]

(d) Simplify $\left(e^{\frac{-i\pi}{3}}\right)^2 (1 + i\sqrt{3})^3$, writing your answer in the form $x + iy$ where x, y are real [2]

Question 7 (17 marks)

(a) Sketch the region in the Argand diagram containing the points z for which

i. $|arg\ z| \leq \dfrac{\pi}{6}$ [1]

ii. $z + \overline{z} < 4$ [1]

iii. $2 < |z| < 4$ [2]

iv. $arg\left(\dfrac{z+1}{z-3}\right) = \dfrac{\pi}{3}$ [2]

(b) On an Argand diagram, sketch the region satisfied by both inequalities: [3]
$|z+1| \leq |z-i|$ and $Im(z) < 2$

(c) i. On an Argand diagram shade the region containing all points representing complex numbers z such that both $|z-1| \leq 1$ and $0 \leq Arg(z-1) \leq \dfrac{\pi}{4}$. [2]

ii. If the point P in this region represents the complex number $z = z_1$ which has the largest principal argument, express z_1 in modulus/argument form. [2]

(d) $|z| = 1$ and $z = cos\theta + isin\theta$, where $-\pi < \theta \leq \pi$.

i. Show that $1 + z = 2cos\dfrac{\theta}{2}\left(cos\dfrac{\theta}{2} + isin\dfrac{\theta}{2}\right)$ [2]

ii. z_1 and z_2 are complex numbers such that $|z_1| = |z_2| = 1$. If z_1 and z_2 have arguments α and β respectively, where $-\pi < \alpha \leq \pi$ and $-\pi < \beta \leq \pi$, show that $\dfrac{z_1 + z_1 z_2}{z_1 + 1}$ has modulus $\left(\dfrac{cos\frac{\beta}{2}}{cos\frac{\alpha}{2}}\right)$ and argument $\left(\dfrac{\alpha+\beta}{2}\right)$ [2]

Question 8 (13 marks)

(a) i. Realise the demominator of $\dfrac{\sqrt{3}-i}{1-i}$ and express your answer in cartesian form. [2]

ii. Show that $\sqrt{3}-i$ and $1-i$ are same as $2cis\left(-\dfrac{\pi}{6}\right)$ and $\sqrt{2}cis\left(-\dfrac{\pi}{4}\right)$ respectively. [2]

iii. Hence find the exact value of $\sin\dfrac{\pi}{12}$ [2]

(b) The polynomial $P(z) = z^4 - 10z^3 + cz^2 + dz + 169$ has two zeros $\alpha = a + ib$ and $\beta = b + ia$, where a, b, c, d are real and $a \neq b$.

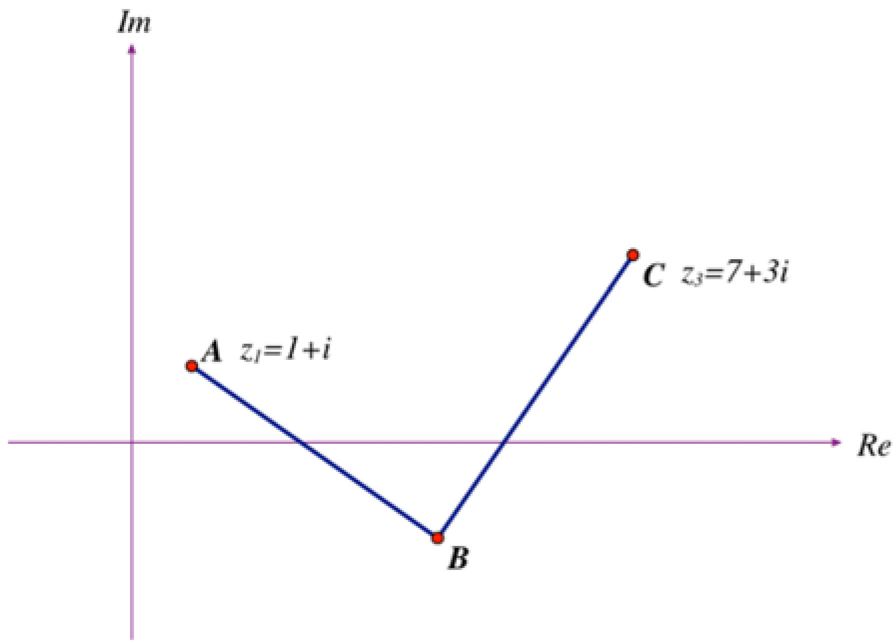
i. Express the other two zeros of $P(z)$ in terms of a and b, justifying your answer. [1]

ii. Find the values of a and b, given that they both are integers. [2]

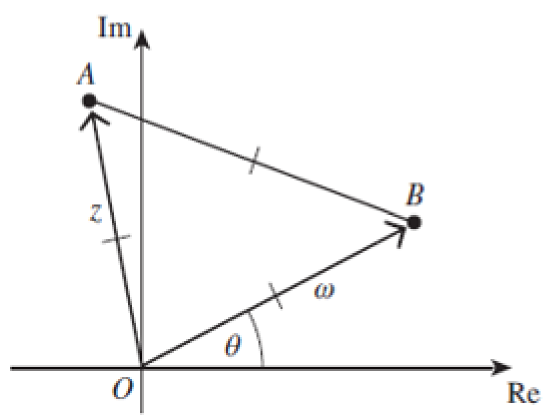
(c) Find α and β given that $z^3 + 6z - 4\sqrt{2}\,i = (z-\alpha)^2(z-\beta)$ [4]

Question 9 (15 marks)

(a) The points A and C represent the complex numbers $z_1 = 1 + i$ and $z_3 = 7 + 3i$ and B represents the complex number Z_2. Find the complex number Z_2 in the form $a + ib$ where a and b are real, such that ΔABC is isoceleses and right- angled at B. [4]



(b) Let $z = \overrightarrow{OA}$ and $\omega = \overrightarrow{OB}$ represent the two sides of a equilateral triangle OAB, as shown.



i. Find the modulus and argument of $\dfrac{z}{\omega}$ [2]

ii. Hence, find $z^3 + \omega^3$. [2]

Question 9 continued onto next page...

(c) If $z = \dfrac{1+i}{1-i}$ and $\omega = \dfrac{\sqrt{2}}{1-i}$

i. Calculate the modulus of z and ω [2]

ii. Calculate argument of z and ω [2]

On an Argand diagram the points A, B and C represent the numbers z, $z+\omega$ and ω respectively.

iii. By considering the quadrilateral $OABC$ and the argument of $z+w$, show that $tan\dfrac{3\pi}{8} = 1+\sqrt{2}$. [3]

Question 10 (16 marks)

(a) i. Show that if $z = \cos\theta + i\sin\theta$ then $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ [3]

ii. Hence, express $\cos^5\theta$ in terms of $\cos 5\theta$, $\cos 3\theta$, $\cos\theta$ [4]

(b) i. Use De Moivre's theorem to find expressions for $cos\ 5\theta$ and sin 5θ. You may leave your answers in terms of mixed powers of $\cos\theta$ and $\sin\theta$. [1]

ii. If $t = \tan\theta$ then show that $tan\ 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ [1]

iii. Use the result from (ii) and an appropriate substitution to show that $tan\dfrac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$. [3]

(c) i. Show that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = \dfrac{2\cos n\theta}{\cos^n\theta}$, where $n > 0$. [2]

ii. Hence, show that the roots of $(1+z)^2 + (1-z)^2 = 0$ are $z = \pm itan\dfrac{\pi}{4}$ when z is purely imaginary. [2]

Question 11 (13 marks)

(a) i. If ω is the complex root of $z^5 - 1 = 0$ with the smallest positive argument, show that ω^2, ω^3 and ω^4 are also roots of $z^5 - 1 = 0$. [2]

ii. Show $\omega + \omega^2 + \omega^3 + \omega^4 = -1$ [2]

(b) i. By matching pair of conjugate roots, resolve $z^5 - 1 = 0$ into real linear and quadratic factors. [2]

ii. Hence, resolve $z^4 + z^3 + z^2 + z + 1$ into real quadratic factors [1]

iii. Hence, show that [3]

$$2\cos 2\theta + 2\cos\theta + 1 = \left(2\cos\theta - 2\cos\frac{2\pi}{5}\right)\left(2\cos\theta + 2\cos\frac{2\pi}{5}\right)$$

iv. Hence, by substituting in an appropriate value for θ show that [3]

$$\left(1 - 2\cos\frac{2\pi}{5}\right)\left(1 + 2\cos\frac{\pi}{5}\right) = 1$$

End of Task 1

 Solutions & Feedback

Task: HSC Assessment Task 1	Year: 2024

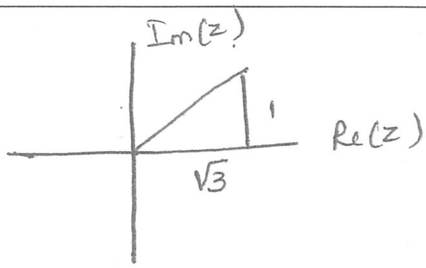
Suggested Solutions

Question 1

$z = \sqrt{3} + i$

$\arg(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= 2\text{cis}\frac{\pi}{6}$



C

Question 2

$i^{2011} = i^{3 + (502 \times 4)}$

$= i^3$

$= -i$

C

Marker's Feedback

Suggested Solutions

Question 3

$$(z+2)(\bar{z}+2) = (z\bar{z}+2z+2\bar{z}+4)$$
$$= z\bar{z}+2(z+\bar{z})+4$$

$(z+2)(\bar{z}+2) = 4$; hence $z\bar{z}+2(z+\bar{z})=0$

let $z = x+iy$

$z\bar{z} + 2(z+\bar{z})=0$

$(x+iy)(x-iy)+2(x+iy+x-iy)=0$

$x^2+y^2+4x=0$

$(x+2)^2+y^2=2^2$

$\therefore$ circle with centre $(-2,0)$ and radius 2. C

Marker's Feedback

Suggested Solutions

Question 4

All the coefficients of z^3+0z^2+pz+q are real.
Then any complex roots occur in conjugate pairs.
Since $1-i$ is a root then $1+i$ is a root.

Sum of the roots.

$$(1-i)+(1+i)+\alpha = -\frac{b}{a}$$

$$\alpha+2 = 0$$

$$\alpha = -2$$

Product of the roots

$$(1-i)\times(1+i)\times(-2) = -\frac{d}{a}$$

$$(1+1)(-2) = -q$$

$$\boxed{q = 4}$$

Sum of the roots two at a time

$$(1-i)(1+i)+(1-i)(-2)+(1+i)(-2) = \frac{c}{a}$$

$$2+(-2)+2i+(-2)-2i = \frac{p}{1}$$

$$\boxed{\therefore p = -2}$$

B

Marker's Feedback

Suggested Solutions

Question 5 : Answer D .

Question 6

a Given $z_1 = 3-2i$; $z_2 = 3-4i$

i, $z_1 - z_2 = (3-2i) - (3-4i)$
$= 2i$

ii, $(z_2)^2 = (3-4i)^2$
$= 9 - 24i + 16i^2$
$= -7 - 24i$

iii, $z_1 \overline{z_2} = (3-2i)(3+4i)$
$= 9 + 12i - 6i + 8$
$= 17 + 6i$

Marker's Feedback

Generally well done — just a few silly mistakes.

Suggested Solutions

Question 6 b,i,

$$(x+iy)^2 = -24+10i$$

$$x^2-y^2+2ixy = -24+10i$$

Equating real and imaginary parts

$$x^2-y^2 = -24 \quad (1)$$

$$xy = 5 \Rightarrow y = \frac{5}{x} \quad -(2)$$

Sub (2) in (1)

$$x^2-\left(\frac{5}{x}\right)^2 = -24$$

$$x^4 - 25 = -24x^2$$

$$x^4+24x^2-25=0$$

$$(x^2+25)(x^2-1) = 0$$

$x \neq \pm 5i$ (x real). $x = \pm 1$. Sub. $x=1$ in (2) $y=\frac{5}{1} = 5$. | Sub. $x=-1$ in (2) $y=\frac{5}{-1} = -5$.

Marker's Feedback

Generally well done – EXCEPT

→ A small number of students used an INCORRECT method [described below]:

$x^2+y^2=26$ (1) → True as $|-24+10i| = x^2+y^2$

$\underline{x^2-y^2=24}$ (2)

(1)+(2) gets $x=\pm 5$

& (1) − (2) gets $y=\pm 1$

BUT IF YOU USE THIS METHOD, YOU DON'T KNOW IF $y=1$ or -1 WHEN $x=5$.

DO NOT DO THIS!

Suggested Solutions

Question 6b,i, continued

$x=1,\ y=5$ or $x=-1;\ y=-5$

The complex square roots are $1+5i$ and $-1-5i$

b,ii, $z^2-(3+i)z+8-i=0$

$$\Delta = 9+6i-1-32+4i$$

$$= -24+10i$$

$$z = \frac{3+i \pm (1+5i)}{2} \quad \text{from i}$$

$z = 2+3i$ or $1-2i$

Marker's Feedback

(6)(b)(ii) Generally well done.

Suggested Solutions

Question 6 C

modulus of $\dfrac{(2-i)^8}{(2+i)^6}$

$|2-i| = \sqrt{5}, \quad |2+i| = \sqrt{5}$

$$\therefore \left| \frac{(2-i)^8}{(2-i)^6} \right| = \frac{|(2-i)|^8}{|(2-i)|^6} = \frac{\sqrt{5}^8}{\sqrt{5}^6} = 5$$

6d

$$\left(e^{-i\pi/3}\right)^2 \left(1+i\sqrt{3}\right)^2 = \left(e^{-i2\pi/3}\right) \cdot \left(2e^{i\pi/3}\right)^3$$

$$= e^{-i2\pi/3} \cdot 8e^{i\pi}$$

$$= 8e^{i\pi/3}$$

$$= 4 + 4\sqrt{3}i$$

Note: Could also do: $\left(\operatorname{cis}\frac{\pi}{3}\right)^2 \times \left(2\operatorname{cis}\frac{\pi}{3}\right)^3$

$= \operatorname{cis}\frac{-2\pi}{3} \times 8\operatorname{cis}\pi$

etc.

Marker's Feedback

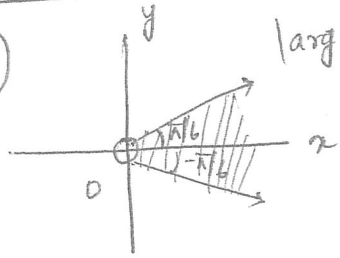
Generally well done, apart from silly mistakes.
The 2 most common were $[2e^{i\frac{\pi}{3}}]^3 = 2e^{i\pi}$
[they forgot to cube 2].
and forgetting that $-8 = 8e^{i\pi}$ and you add to add $\frac{-2\pi i}{3}$ and πi.

Suggested Solutions

Question 7

(a)

i) $|\arg z| \le \pi/6$

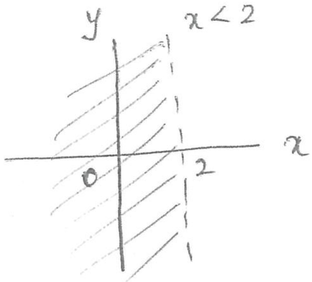


Some students forgot to put open circle at origin.

ii) $z+\bar{z} < 4$

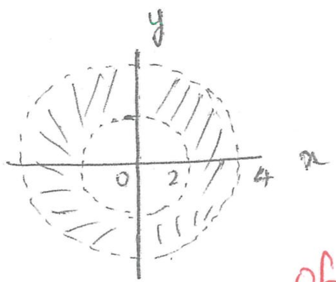
$z+\bar{z} = 2\mathrm{Re}(z) < 4$

$x < 2$



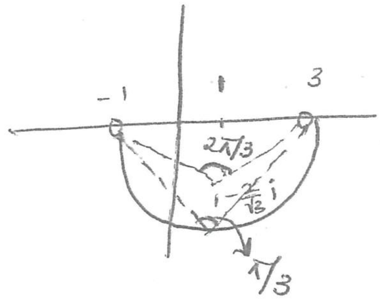
done well

iii) $2 < |z| < 4$



writing radii of circles as 4 and 16

iv) $\arg\left(\dfrac{z+1}{z-3}\right) = \dfrac{\pi}{3}$



Some students drew the arc above x axis; not showing the angle $\frac{\pi}{3}$

Marker's Feedback

Suggested Solutions

Question 7b $|z+1| \leq |z-i|$ and $\text{Im}(z) < 2$

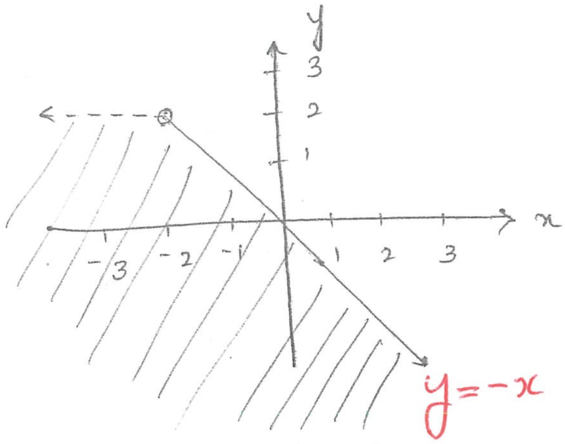
let $z = x+iy$

$|x+iy+1| \leq |x+iy-i|$

$|x+1+iy| \leq |x+i(y-1)|$

$(x+1)^2+y^2 \leq x^2+(y-1)^2$

$y \leq -x$



* Many are not writing the equation $y=-x$.

* drawing $y=-x$ as broken line

Marker's Feedback

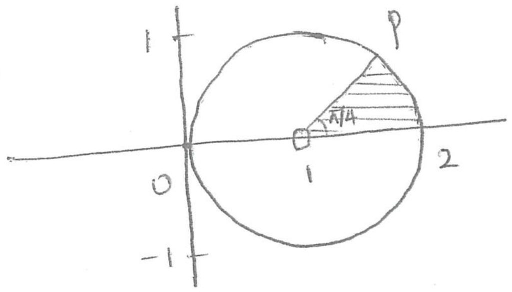
Suggested Solutions

Question 7c(i)

Sketch and shade $|z-1 \leq 1|$ and $0 \leq \arg(z-1) \leq \frac{\pi}{4}$

$|z-1 \leq 1| \Rightarrow (x-1)^2 + y^2 = 1$

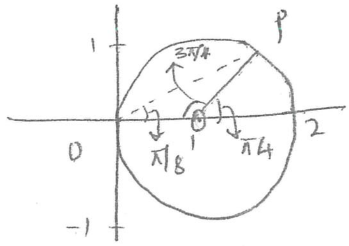
no open circle at $(1,0)$



7c(ii)

The arc bounding the region subtends an angle $\frac{\pi}{4}$ at the circle centre and hence an angle $\frac{\pi}{8}$ at O on the circumference

$\therefore \arg(z_1) = \frac{\pi}{8}$.



using the cosine rule:

$OP^2 = 1^2 + 1^2 - 2\cos\frac{3\pi}{4}$

$OP^2 = 2\left(1 - \cos\frac{3\pi}{4}\right) = 4\sin^2\frac{3\pi}{8}$

$\therefore OP^2 = 2 + \sqrt{2}$; Hence $z_1 = 2\sin\frac{3\pi}{8}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right) = \sqrt{2+\sqrt{2}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

Marker's Feedback

Students have difficulty with 7(c)(ii)

Very few got the correct answers.

Suggested Solutions

Question 7d i,

$$1+z = 1+(\cos\theta + i\sin\theta)$$

$$= \left(1+\cos\frac{2\theta}{2}\right) + \left(i\sin\frac{2\theta}{2}\right)$$

$$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

or

$$2\cos\frac{\theta}{2}\,\text{cis}\,\frac{\theta}{2}.$$

done well.

Marker's Feedback

Suggested Solutions

Question 7d ii,

$1+z_1 = 2\cos\frac{\alpha}{2}\,\text{cis}\,\frac{\alpha}{2}$; $1+z_2 = 2\cos\frac{\beta}{2}\,\text{cis}\,\frac{\beta}{2}$

$$\left|\frac{z_1(1+z_2)}{1+z_1}\right| = \left|\frac{\text{cis}\,\alpha\left(2\cos\frac{\beta}{2}\,\text{cis}\,\frac{\beta}{2}\right)}{2\cos\frac{\alpha}{2}\,\text{cis}\,\frac{\alpha}{2}}\right|$$

$$= \left|\frac{\text{cis}\left(\alpha+\frac{\beta}{2}\right)\cdot 2\cos\frac{\beta}{2}}{\text{cis}\,\frac{\alpha}{2}\cdot 2\cos\frac{\alpha}{2}}\right|$$

Now:

$$\left|\frac{z_1(1+z_2)}{1+z_1}\right| = \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$$

$$\text{Arg}\left(\frac{z_1(1+z_2)}{1+z_1}\right) = \frac{\alpha+\beta}{2} - \frac{\alpha}{2}$$

$$= \frac{\alpha+\beta}{2}$$

Marker's Feedback

Many found the above questions hard.

Suggested Solutions

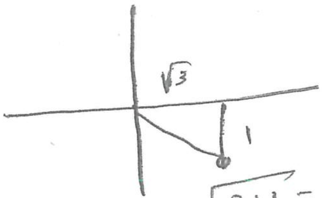
Question 8 a, i,

$$\frac{\sqrt{3}-i}{1-i} \times \frac{(1+i)}{(1+i)}$$

$$= \frac{\sqrt{3}+i\sqrt{3}-i+1}{2}$$

$$= \frac{\sqrt{3}+1}{2} + i\frac{(\sqrt{3}-1)}{2}$$

ii, $\sqrt{3}-i$



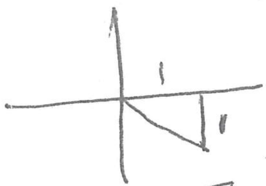
modulus $= \sqrt{3+1} = 2$

$\arg = -\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= -\frac{\pi}{6}$

$\therefore \sqrt{3}-i = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$

$1-i$



modulus $= \sqrt{2}$

$\operatorname{Arg}(1-i) = \tan^{-1}(1)$

$= \frac{-\pi}{4}$

$\therefore 1-i = \sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)$

Marker's Feedback

8 a, i, & ii, Mostly done well. students have shown good understanding of Cartesian form & Mod-Arg form

Suggested Solutions

Question 8 a (iii)

$$\frac{\sqrt{3}-i}{1-i} = \frac{2\,\text{cis}\left(\frac{-\pi}{6}\right)}{\sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)}$$

$$= \sqrt{2}\cdot\text{cis}\left(\frac{\pi}{12}\right) = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

Equating imaginary parts of (i) & (iii)

$$\sqrt{2}\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2}$$

$$\therefore \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Marker's Feedback

Mostly done well.

Suggested Solutions

Question 8b,i,

$p(z) = z^4 - 10z^3 + cz^2 + dz + 169$

coefficients of $p(z)$ are real

$\therefore$ complex roots occur in conjugate pairs

$\therefore$ other two roots are $\bar{\alpha} = a - ib$ and $\bar{\beta} = b - ia$

8b,ii,

Sum of zeros $= -\left(\frac{-10}{1}\right) = 10$

$a + ib + a - ib + b + ia + b - ia = 10$

$2a + 2b = 10$

$a + b = 5$

Product of zeros $= \frac{169}{1} = 169$

$(\alpha\bar{\alpha})(\beta\bar{\beta}) = 169$

$|\alpha|^2 \, |\beta|^2 = 169$

$|\alpha| \, |\beta| = 13$

$\sqrt{a^2+b^2} \cdot \sqrt{a^2+b^2} = 13$

Marker's Feedback

8b,i, 8,ii, Mostly done well. However some students lost marks because of wrong factorisation.

Suggested Solutions

Question 8 b ii, continued

$a^2+b^2=13$

$b=5-a \Rightarrow a^2+(5-a)^2=13$

$\therefore 2a^2-10a+12=0$

$a^2-5a+6=0$

$(a-2)(a-3)=0$

$a=2$ or 3; $b=3$ or 2

8c

$p(z)=z^3+6z-4\sqrt{2}i$

$p'(z)=3z^2+6$

$0=3z^2+6$

$z=\pm\sqrt{2}i$

$p(\sqrt{2}i)=(\sqrt{2}i)^3+6(\sqrt{2}i)-4\sqrt{2}i$

$=-2\sqrt{2}i+6\sqrt{2}i-4\sqrt{2}i$

$=0$

Marker's Feedback

8c. Poorly done! Students to pratise similar style Questions to improve understanding and application.

Suggested Solutions

Question 8 C Continued

$$p(-\sqrt{2}i) = (-\sqrt{2}i)^3 + 6(-\sqrt{2}i) - 4\sqrt{2}i$$

$$= 2\sqrt{2}i - 6\sqrt{2}i - 4\sqrt{2}i$$

$$\neq 0$$

$\therefore \sqrt{2}i$ is a root.

$\therefore \alpha = \sqrt{2}i$

$$(z-\alpha)^2 = z^2 - 2(\sqrt{2}i)z + (\sqrt{2}i)^2$$

$$= z^2 - 2\sqrt{2}zi - 2$$

$$z^3 + 6z - 4\sqrt{2}i = (z^2 - 2\sqrt{2}zi - 2)(z + 2\sqrt{2}i)$$

$$\therefore \beta = -2\sqrt{2}i$$

Marker's Feedback

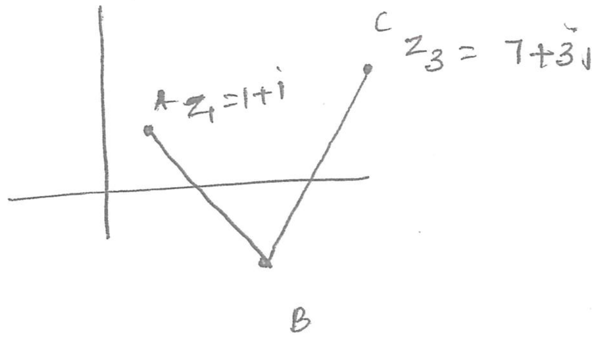
Suggested Solutions

Question 9a

$\overrightarrow{BA} = z_1 - z_2$

$= 1+i-a-bi$

$= 1-a+i(1-b)$



$\overrightarrow{BC} = z_3 - z_2$

$= 7+3i-a-bi$

$= 7-a+i(3-b)$

$i \times \overrightarrow{BC} = \overrightarrow{BA}$

$i \times (7-a+i(3-b)) = 1-a+i(1-b)$

$i(7-a)-(3-b) = 1-a+i(1-b)$

By equating real and imaginary parts

$-(3-b) = 1-a$

$b = 4-a$ and

$(7-a) = (1-b)$

$7-a = 1-(4-a)$

$7-a = -3+a$

$a = 5$

$b = 4-a$

$b = 4-5$

$b = -1$

$z_2 = 5-i$

Marker's Feedback

Mostly done well.

Suggested Solutions

Question 9b, i,

$\therefore |z| = |w|$

$\left|\frac{z}{w}\right| = 1$

$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$

$= \frac{\pi}{3}$

9b ii,

$z = w \operatorname{cis} \frac{\pi}{3}$

$z^3 = w^3 \operatorname{cis} \pi$

$z^3 = -w^3$

$\therefore z^3 + w^3 = 0$

done well.

Marker's Feedback

Suggested Solutions

Question 9 c i,

$$|z| = \left|\frac{1+i}{1-i}\right| = \frac{|1+i|}{|1-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$$|w| = \left|\frac{\sqrt{2}}{1-i}\right| = \frac{|\sqrt{2}|}{|1-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

9 c ii,

$$\arg(z) = \arg\left(\frac{1+i}{1-i}\right) = \arg(1+i) - \arg(1-i)$$
$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$\arg(w) = \arg\left(\frac{\sqrt{2}}{1-i}\right) = \arg(\sqrt{2}) - \arg(1-i)$$
$$= 0 - \left(-\frac{\pi}{4}\right)$$
$$= \frac{\pi}{4}$$

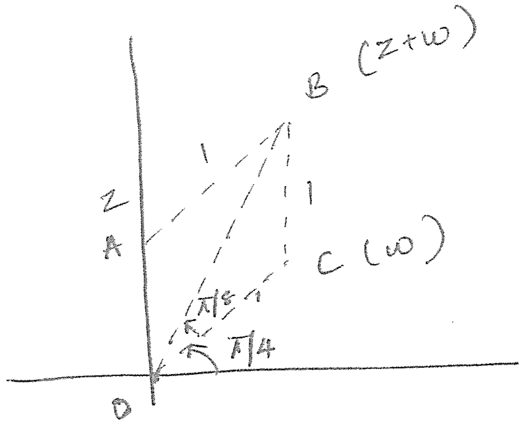
done well

Marker's Feedback

Suggested Solutions

Question 9c(iii)

The Quadrilateral OABC is a parallelogram with equal sides, hence it is a rhombus



OB is the ∠ bisector of ∠AOC

So,

$$\angle AOC = \frac{\pi}{4}$$

$$\angle AOB = \angle BOC = \frac{\pi}{8}$$

So the $\arg(z+w) = \frac{\pi}{4} + \frac{\pi}{8}$

$$= \frac{3\pi}{8}$$

Now,

$$z+w = \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i}$$

Marker's Feedback

Suggested Solutions

Question 9c(iii), continued

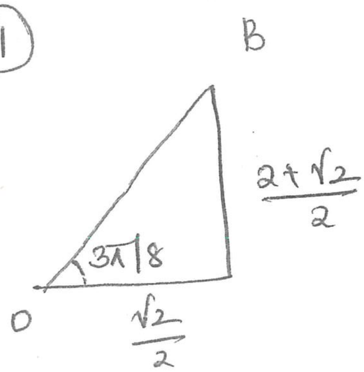
$$= \frac{(1+\sqrt{2})+i}{1-i}$$

$$= \frac{(1+\sqrt{2}+i)(1+i)}{(1-i)(1+i)}$$

$$= \frac{1+\sqrt{2}+(1+\sqrt{2})i+i-1}{2}$$

$$= \frac{\sqrt{2}+(2+\sqrt{2})i}{2}$$

$$= \frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{2}i \quad (1)$$



$$\tan\frac{3\pi}{8} = \frac{\left(\frac{2+\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{2+\sqrt{2}}{\sqrt{2}}$$

$$\tan\frac{3\pi}{8} = \frac{2}{\sqrt{2}} + 1 = 1+\sqrt{2} \text{ as required.}$$

Marker's Feedback

Many didn't get the full mark for this question.

Suggested Solutions

Question 10a i

$z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$z^n + \frac{1}{z^n} = \left(\cos n\theta + i\sin n\theta\right) + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

Marker's Feedback

Very well done!

Suggested Solutions

Question 10a, ii

$$z^5 + \frac{1}{z^5} = 2\cos 5\theta$$

$$\left(z+\frac{1}{z}\right)^5 = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$$

$$(2\cos\theta)^5 = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

$$32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$$

$$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$$

Marker's Feedback

Mostly well done but some students forgot coefficients.

Suggested Solutions

Question 10 b, i,

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$$

$$(\cos 5\theta + i\sin 5\theta) = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5)$$

Now equating real and imaginary parts.

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

and

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

Marker's Feedback

Mostly done well again some students forgot coefficients

Suggested Solutions

Question 10 b (ii)

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$$

$$= \frac{5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cancel{\cos}\cos\theta\sin^4\theta}$$

$\dfrac{\div \cos^5\theta}{\div \cos^5\theta}$ and let $\tan\theta = t$

$$\therefore \tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

$$= \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$ as required.

Marker's Feedback

Very well done, but some students forgot to put it in 't' form but no marks were deducted.

Suggested Solutions

Question 10 b iii)

let $\theta = \frac{\pi}{5}$ So $t = \tan\frac{\pi}{5}$ and $\tan 5\theta = \tan \pi = 0$

So $\frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4} = 0$

$t(t^4 - 10t^2 + 5) = 0$

But $t = \tan\frac{\pi}{5} \neq 0$

$(t^2)^2 - 10t^2 + 5 = 0$

$\therefore t^2 = \frac{10 \pm \sqrt{100 - 20}}{2}$

$t^2 = 5 \pm 2\sqrt{5}$ but $\tan x$ is always increasing, $\tan\frac{\pi}{5} < \tan\frac{\pi}{4}$
$\tan\frac{\pi}{5} < 1$ so $t^2 < 1$

$t = \pm\sqrt{5 - 2\sqrt{5}}$

but $t = \tan\frac{\pi}{5} > 0$

$\therefore \tan\frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$

Marker's Feedback

Some students let $\theta = \frac{\pi}{25}$ because of which they could succeed.
Few students got the wrong roots due to incorrect use of Quadratic formula
Students are advised to reason why $\tan\frac{\pi}{5} = +\sqrt{5-2\sqrt{5}}$
otherwise well done!

Suggested Solutions

Question 10C i,

$$(1+i\tan\theta)^n + (1-i\tan\theta)^n$$

$$\left(\frac{\cos\theta}{\cos\theta} + \frac{i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta}{\cos\theta} - \frac{i\sin\theta}{\cos\theta}\right)^n$$

$$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$$

$$= \frac{\text{cis}\, n\theta + \text{cis}(-n\theta)}{\cos^n\theta}$$

$$= \frac{2\cos n\theta}{\cos^n\theta}$$

Marker's Feedback

Mostly well done.

Suggested Solutions

Question 10 c ii,

let $z = i\tan\theta$

$$(1+i\tan\theta)^2 + (1-i\tan\theta)^2 = \frac{2\cos 2\theta}{\cos^2\theta}$$

$$\therefore 2\cos 2\theta = 0$$

$$\therefore 2\theta = \frac{\pi}{2} \ \& \ \frac{3\pi}{2}$$

$$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\therefore z_1 = i\tan\frac{\pi}{4} \ ; \ z_2 = i\tan\frac{3\pi}{4}$$

$$= i\tan\left(-\frac{\pi}{4}\right)$$

$$\therefore z = \pm i\tan\frac{\pi}{4}.$$

Marker's Feedback

Poorly done. Most students did not know how to use 10 c i, to solve 10 c ii,.

Suggested Solutions

Question 11 a, i,

Roots of $z^5 - 1 = 0$ are

$\text{cis}\frac{2\pi}{5}$; $\text{cis}\frac{4\pi}{5}$; $\text{cis}\frac{6\pi}{5}$; $\text{cis}\frac{8\pi}{5}$

$\omega = \text{cis}\frac{2\pi}{5}$

$\omega^2 = \text{cis}\frac{4\pi}{5}$ → which is a root of $z^5 - 1 = 0$

$\omega^3 = \text{cis}\frac{6\pi}{5}$ "

$\omega^4 = \text{cis}\frac{8\pi}{5}$ "

11 a, ii,

Sum of roots: $\alpha + \beta + \gamma = \frac{-b}{a}$

Roots of $z^5 - 1 = 0$ add upto '0'

$\therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$\omega + \omega^2 + \omega^3 + \omega^4 = -1$

Alternatively as ω is a root of $z^5 - 1 = 0$,

$(\omega - 1)(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$

As $\omega \neq 1$, $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0 \Rightarrow \underline{\omega^4 + \omega^3 + \omega^2 + \omega = -1}$

Marker's Feedback

Generally OK. This was leniently marked.

$(\omega^2)^5 = (\omega^5)^2 = 1$ was also allowed.

In Part (ii) when students used $(\omega - 1)(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$ they often did NOT mention $\omega \neq 1$.

Suggested Solutions

Question 11 b, i,

$$z^5 - 1 = (z-1)\left[z - \text{cis}\frac{2\pi}{5}\right]\left(z - \text{cis}\frac{8\pi}{5}\right)\left[z - \text{cis}\frac{4\pi}{5}\right]\left(z - \text{cis}\frac{6\pi}{5}\right)$$

Now $\left(z - \text{cis}\frac{2\pi}{5}\right)\left(z - \text{cis}\frac{8\pi}{5}\right)$

$$z^2 - z\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right) + 1$$

$\left(\text{as } \cos\frac{8\pi}{5} = \cos\frac{2\pi}{5}\right.$

$$= z^2 - z\left(\cos\frac{2\pi}{5} + \cancel{i\sin\frac{2\pi}{5}} + \cos\frac{2\pi}{5} - \cancel{i\sin\frac{2\pi}{5}}\right) + 1$$

$\left(\because \sin\frac{8\pi}{5} = -\sin\frac{2\pi}{5}\right.$

$$= z^2 - 2z\cos\frac{2\pi}{5} + 1$$

Hence

$$\therefore z^5 - 1 = (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$$

Marker's Feedback

Generally well done. A small minority of students forgot to write the first line $\left(z - \text{cis}\frac{2\pi}{5}\right)\left(z - \text{cis}\frac{4\pi}{5}\right)$ ... etc. $= z^5 - 1$.

Suggested Solutions

Question 11 b ii)

Hence

as $z^5 - 1 = (z-1)(z^4+z^3+z^2+z+1)$

$$(\cancel{z-1})\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right) = (\cancel{z-1})(z^4+z^3+z^2+z+1)$$

$$\therefore \left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right) = z^4+z^3+z^2+z+1$$

11b) iii)

$$z^4+z^3+z^2+z+1 = \left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$$

$$= \left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 + 2z\cos\frac{\pi}{5} + 1\right)$$

$$\left(\text{as } \cos\frac{4\pi}{5} = -\cos\frac{\pi}{5}\right)$$

By dividing both sides by z^2 and rearranging

$$\left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) + 1 = \left(z + \frac{1}{z} - 2\cos\frac{2\pi}{5}\right)\left(z + \frac{1}{z} + 2\cos\frac{\pi}{5}\right)$$

Sub $z = \text{cis}\,\theta$; $z^2 + \frac{1}{z^2} = 2\cos 2\theta$ and $z + \frac{1}{z} = 2\cos\theta$

$$2\cos 2\theta + 2\cos\theta + 1 = \left(2\cos\theta - 2\cos\frac{2\pi}{5}\right)\left(2\cos\theta + 2\cos\frac{\pi}{5}\right)$$

Marker's Feedback

(b)(ii) Well done.

(iii) Students either realised to $\div z^2$ and solved the entire question or could not complete the question if they didn't.

Many students didn't explain that $\cos\frac{\pi}{5} = -\cos\frac{4\pi}{5}$.

Suggested Solutions

Question 11 iv)

$$\left(2\cos\theta - 2\cos\frac{2\pi}{5}\right)\left(2\cos\theta + 2\cos\frac{\pi}{5}\right) = 2\cos 2\theta + 2\cos\theta + 1$$

Substituting $\theta = \frac{\pi}{3}$

$$\left(2\cos\frac{\pi}{3} - 2\cos\frac{2\pi}{5}\right)\left(2\cos\frac{\pi}{3} + 2\cos\frac{2\pi}{5}\right) = 2\cos\frac{2\pi}{3} + 2\cos\frac{\pi}{3} + 1$$

$$\left(1 - 2\cos\frac{2\pi}{5}\right)\left(1 + 2\cos\frac{2\pi}{5}\right) = -2 + 2 + 1$$

$$\left(1 - 2\cos\frac{2\pi}{5}\right)\left(1 + 2\cos\frac{2\pi}{5}\right) = 1$$

Marker's Feedback

(iv) Most students realised to substitute $\theta = \frac{\pi}{3}$ & completed the question well.
A small minority tried the wrong angle & could not complete the question.